journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 3, Article 109, 8 pp.

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AN APPLICATION OF SUBORDINATION ON HARMONIC FUNCTION

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Received 28 June, 2006; accepted 15 May, 2007
Communicated by A. Sofo

ABSTRACT. The purpose of this paper is to obtain sufficient bound estimates for harmonic functions belonging to the classes $S_H^*[A,B]$, $K_H[A,B]$ defined by subordination, and we give some convolution conditions. Finally, we examine the closure properties of the operator D^n on these classes under the generalized Bernardi integral operator.

Key words and phrases: Harmonic, Univalent, Subordination, Convex, Starlike, Close-to-convex.

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 58E20.

1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subset C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|g'(z)| < |h'(z)|$ in D [2].

We denote by S_H the family of functions $f = h + \overline{g}$ which are harmonic univalent and orientation-preserving in the open disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in S_H$, we can express the analytic functions h and g by the following power series expansion:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad g(z) = \sum_{m=1}^{\infty} b_m z^m. \tag{1.1}$$

Note that the family S_H of orientation-preserving, normalized harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero.

Let K, S^*, C, K_H, S_H^* and C_H denote the respective subclasses of S and S_H where the images of $f(u)$ are convex, starlike and close-to-convex.

A function $f(z)$ is subordinate to $F(z)$ in the disk U if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$ for $|z| < 1$. This is written as $f(z) \prec F(z)$.

Let $K[A, B], S^*[A, B]$ denote the subclasses of S defined as follows:

$$S^*[A,B] = \left\{ f \in S,\ \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1 \right\},$$

$$K[A,B] = \left\{ f \in S, \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1 \right\}.$$

We now introduce the following subclasses of harmonic functions in terms of subordination.

Let $f = h + \overline{g} \in S_H$ such that

$$\varphi(z) = \frac{h(z) - g(z)}{1 - b_1}, \tag{1.2}$$

$$\psi(z) = \frac{h(z) - e^{i\theta} g(z)}{1 - e^{i\theta} b_1}, \quad 0 \le \theta < 2\pi, \tag{1.3}$$

and let $-1 \le B < A \le 1$, then we can construct the classes $K_H[A, B],\ S^*_H[A, B]$ using subordination as follows:

$$K_H[A,B] = \left\{ f \in S_H, \frac{(z\psi'(z))'}{\psi'(z)} \prec \frac{1+Az}{1+Bz} \right\},$$

$$S^*_H[A,B] = \left\{ f \in S_H, \frac{z\varphi'(z)}{\varphi(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

Let D^n denote the n-th Ruscheweh derivative of a power series $t(z) = z + \sum_{m=2}^{\infty} t_m z^m$ which is given by

$$\begin{aligned} D^n t &= \frac{z}{(1-z)^{n+1}} * t(z) \\ &= z + \sum_{m=2}^{\infty} C(n,m) t_m z^m, \end{aligned}$$

where

$$C(n,m) = \frac{(n+1)_{m-1}}{(m-1)!} = \frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!}.$$

In [5], the operator D^n was defined on the class of harmonic functions S_H as follows:

$$D^n f = D^n h + \overline{D^n g}.$$

The purpose of this paper is to obtain sufficient bound estimates for harmonic functions belonging to the classes $S^*_H[A, B],\ K_H[A, B]$, and we give some convolution conditions. Finally, we examine the closure properties of the operator D^n on the above classes under the generalized Bernardi integral operator.

2. PRELIMINARY RESULTS

Cluni and Sheil-Small [2] proved the following results:

Lemma 2.1. *If h, g are analytic in U with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each $\epsilon, |\epsilon| = 1$, then $f = h + \overline{g}$ is harmonic close-to-convex.*

Lemma 2.2. *If $f = h + \overline{g}$ is locally univalent in U and $h + \epsilon g$ is convex for some $\epsilon,\ |\epsilon| \le 1$, then f is univalent close-to-convex.*

A domain D is called convex in the direction γ $(0 \le \gamma < \pi)$ if every line parallel to the line through 0 and $e^{i\gamma}$ has a connected intersection with D. Such a domain is close-to-convex. The convex domains are those that are convex in every direction.

We will make use of the following result which may be found in [2]:

Lemma 2.3. *A function $f = h + \overline{g}$ is harmonic convex if and only if the analytic functions $h(z) - e^{i\gamma} g(z), \quad 0 \le \gamma < 2\pi$, are convex in the direction $\frac{\gamma}{2}$ and f is suitably normalized.*

Necessary and sufficient conditions were found in [2, 1] and [4] for functions to be in K_H, S_H^* and C_H. We now give some sufficient conditions for functions in the classes $S_H^*[A, B]$ and $K_H[A, B]$, but first we need the following results:

Lemma 2.4 ([7])**.** *If $q(z) = z + \sum_{m=2}^{\infty} C_m z^m$ is analytic in U, then q maps onto a starlike domain if $\sum_{m=2}^{\infty} m|C_m| \le 1$ and onto convex domains if $\sum_{m=2}^{\infty} m^2|C_m| \le 1$.*

Lemma 2.5 ([4])**.** *If $f = h + \overline{g}$ with*

$$\sum_{m=2}^{\infty} m|a_m| + \sum_{m=1}^{\infty} m|b_m| \le 1,$$

then $f \in C_H$. The result is sharp.

Lemma 2.6 ([4])**.** *If $f = h + \overline{g}$ with*

$$\sum_{m=2}^{\infty} m^2|a_m| + \sum_{m=1}^{\infty} m^2|b_m| \le 1,$$

then $f \in K_H$. The result is sharp.

Lemma 2.7 ([6])**.** *A function $f(z) \in S$ is in $S^*[A, B]$ if*

$$\sum_{m=2}^{\infty} \{m(1 + A) - (1 + B)\} |a_m| \le A - B,$$

where $-1 \le B < A \le 1$.

Lemma 2.8 ([6])**.** *A function $f(z) \in S$ is in $K[A, B]$ if*

$$\sum_{m=2}^{\infty} m \{m(1 + A) - (1 + B)\} |a_m| \le A - B,$$

where $-1 \le B < A \le 1$.

Lemma 2.9 ([3])**.** *Let h be convex univalent in U with $h(0) = 1$ and $\mathrm{Re}(\lambda h(z) + \mu) > 0$ $(\lambda, \mu \in \mathbb{C})$. If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec h(z) \qquad (z \in U)$$

implies

$$p(z) \prec h(z) \qquad (z \in U).$$

3. MAIN RESULTS

Theorem 3.1. *If*

$$\sum_{m=2}^{\infty}\{m(1+A)-(1+B)\}\,|a_m|+\sum_{m=1}^{\infty}\{m(1+A)-(1+B)\}\,|b_m|\le A-B, \tag{3.1}$$

then $f\in S_H^*[A,B]$. *The result is sharp.*

Proof. From the definition of $S_H^*[A,B]$, we need only to prove that $\varphi(z)\in S^*[A,B]$, where $\phi(z)$ is given by (1.2) such that

$$\phi(z)=z+\sum_{m=2}^{\infty}\left(\frac{a_m-b_m}{1-b_1}\right)\,z^m.$$

Using Lemma 2.7, we have

$$\sum_{m=2}^{\infty}\frac{\{m(1+A)-(1+B)\}}{A-B}\left|\frac{a_m-b_m}{1-b_1}\right|\le\sum_{m=2}^{\infty}\frac{\{m(1+A)-(1+B)\}}{A-B}\left(\frac{|a_m|+|b_m|}{1-|b_1|}\right)$$
$$\le 1$$

if and only if (3.1) holds and hence we have the result.

The harmonic function

$$f(z)=z+\sum_{m=2}^{\infty}\frac{1}{(A-B)\{m(1+A)-(1+B)\}}x_m z^m$$
$$+\sum_{m=1}^{\infty}\frac{1}{(A-B)\{m(1+A)-(1+B)\}}\;\overline{y}_m\overline{z}^m$$
$$\left(\text{where }\sum_{m=2}^{\infty}|x_m|+\sum_{m=1}^{\infty}|y_m|=A-B-1\right)$$

shows that the coefficient bound given by (3.1) is sharp. □

Corollary 3.2. *If* $A=1,\ B=-1$, *then we have the coefficient bound given in* [1] *with a different approach.*

Theorem 3.3. *If* $f=h+\overline{g}$ *with*

$$\sum_{m=2}^{\infty}\{m(1+A)-(1+B)\}|a_m|C(n,m)$$
$$+\sum_{m=1}^{\infty}\{m(1+A)-(1+B)\}|b_m|C(n,m)\le A-B,$$

then $D^nf=H+\overline{G}\in S_H^*[A,B]$. *The function*

$$f(z)=z+\frac{(1+\delta)(A-B)}{\{m(1+A)-(1+B)\}C(n,m)}\overline{z}^m,\quad \delta>0$$

shows that the result is sharp.

Corollary 3.4. *If* $A=1,\ B=-1$, *then we have the coefficient bound given in Theorem 3.1,* $\alpha=0$ [5] *with a different approach.*

Theorem 3.5. *If*

$$\sum_{m=2}^{\infty} m\{m(1+A)-(1+B)\}|a_m| + \sum_{m=1}^{\infty} m\{m(1+A)-(1+B)\}|b_m| \le A-B, \tag{3.2}$$

then $f \in K_H[A,B]$. *The result is sharp.*

Proof. From the definition of the class $K_H[A,B]$ and the coefficient bound of $K[A,B]$ given in Lemma 2.8, we have the result. The function

$$f(z) = z + \frac{(1+\delta)(A-B)}{m\{m(1+A)-(1+B)\}}\overline{z}^m, \qquad \delta > 0$$

shows that the upper bound in (3.2) cannot be improved. □

Theorem 3.6. *If* $f = h + \overline{g}$ *with*

$$\sum_{m=2}^{\infty} m\{m(1+A)-(1+B)\}C(n,m)|a_m| + \sum_{m=1}^{\infty} m\{m(1+A)-(1+B)\}C(n,m)|b_m| \le A-B,$$

then $D^n f \in K_H[A,B]$. *The function*

$$f = z + \frac{(1+\delta)(A-B)}{m\{m(1+A)-(1+B)\}C(n,m)}\overline{z}^m, \quad \delta > 0$$

shows that the result is sharp.

Corollary 3.7. *If* $n = 0,\ A = 1, B = -1$, *we have Theorem 3 in* [4] *and if* $A = 1, B = -1$, *we have Theorem 2 in* [5].

In the next two theorems, we give necessary and sufficient convolution conditions for functions in $S_H^*[A,B]$ and $K_H[A,B]$.

Theorem 3.8. *Let* $f = h + \overline{g} \in S_H$. *Then* $f \in S_H^*[A,B]$ *if*

$$h(z) * \left(\frac{z + \frac{(\xi - A)}{A-B}z^2}{(1-z)^2}\right) + \epsilon B\,\overline{g(z)}\left(\frac{\xi\,\overline{z} - \frac{(-1-A\xi)}{A-B}\overline{z}^2}{(1-\overline{z})^2}\right) \ne 0, \quad |\xi| = 1,\ 0 < |z| < 1.$$

Proof. Let $S(z) = \frac{h(z)-g(z)}{1-b_1}$, then $S \in S^*[A,B]$ if and only if

$$\frac{zS'}{S} \prec \frac{1+Az}{1+Bz}$$

or

$$\frac{zS'(z)}{S(z)} \ne \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}, \quad 0 \le \theta < 2\pi,\ z \in U.$$

It follows that

$$\left[zS'(z) - S(z)\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right] \ne 0.$$

Since $zS'(z) = S(z) * \frac{z}{(1-z)^2}$, the above inequality is equivalent to

$$(3.3)\qquad 0 \neq S(z) * \left[\frac{z}{(1-z)^2} - \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\frac{z}{1-z}\right]$$

$$= \frac{1}{\lambda e^{it}}\left\{S(z) * \left[\frac{z + \frac{(-e^{-i\theta}-A)}{(A-B)}z^2}{(-e^{-i\theta}-B)(1-z)^2}\right]\right\}, \quad 1 - b_1 = \lambda e^{it}$$

$$= \frac{1}{\lambda e^{it}}\left\{h(z) * \left(\frac{z + \frac{(-e^{-i\theta}-A)}{A-B}z^2}{(-e^{-i\theta}-B)(1-z)^2}\right) - g(z)\right.$$

$$\left. * \left(\left\{\frac{z + (-e^{-i\theta}-A)z^2}{(A-B)(e^{-i\theta}/B)}\right\}\Big/ (1-z)^2(-B-e^{i\theta})\right)\right\}$$

$$= \frac{1}{\lambda}\left\{h(z) * \left(\frac{z + \frac{(-e^{-i\theta}-A)}{A-B}z^2}{(1-z)^2 e^{it}}\right)\right.$$

$$\left. - g(z) * \left(\frac{Be^{i\theta}z + \frac{B(-e^{-i\theta}-A)e^{i\theta}}{A-B}z^2}{e^{it}(-B-e^{i\theta})(1-z)^2}\right)\right\}.$$

Now, if $z_1 - z_2 \neq 0$ and $|z_1| \neq |z_2|$, then $z_1 - \epsilon\overline{z}_2 \neq 0, \quad |\epsilon| = 1$, i.e.,

$$= \frac{1}{\lambda(-B-e^{-i\theta})}\left[h(z) * \left(\frac{z + \frac{(-e^{-i\theta}-A)}{A-B}z^2}{(1-z)^2 e^{it}}\right)\right]$$

$$- \epsilon\,\overline{g(z)} * \overline{\left(\frac{Be^{+i\theta}z + \frac{(-1-Ae^{i\theta})B}{A-B}z^2}{e^{it}(-B-e^{i\theta})(1-z)^2}\right)}$$

$$= \frac{1}{\lambda(-B-e^{-i\theta})}\left[h(z) * \left(\frac{z + \frac{(-e^{-i\theta}-A)}{A-B}z^2}{(1-z)^2 e^{it}}\right)\right]$$

$$- \epsilon\,\overline{g(z)} * \left(\frac{(-B)(-e^{-i\theta}\overline{z} + \frac{B(-1-Ae^{-i\theta})}{A-B}\overline{z}^2}{(1-\overline{z})^2 e^{-it}}\right).$$

Since $\arg(1-b_1) = t \neq \pi$, we obtain the result and the proof is thus completed. □

Corollary 3.9. *If $A = 1,\ B - 1$ and $\epsilon = 1$, then we have Theorem 2.6 in* [1] *with a different approach.*

Theorem 3.10. *Let $f = h + \overline{g} \in S_H$. Then $f \in K_H[A,B]$ if and only if*

$$h(z) * \left[\frac{z + \frac{2\xi-A-B}{A-B}z^2}{(1-z)^3}\right] + \epsilon\overline{g(z)} * \left[\frac{\xi\overline{z} - \frac{-2+(A+B)\xi}{A-B}\overline{z}^2}{(1-\overline{z})^3}\right] \neq 0$$

$$|\epsilon| = 1,\ |\xi| = 1, \quad 0 < |z| < 1$$

Proof. Let $\psi(z) = \frac{h(z)-e^{i\gamma}g(z)}{1-e^{i\gamma}b_1}, \quad 0 \leq \gamma < 2\pi$ and $1 - e^{i\gamma}b_1 = \lambda\,e^{it}$, then from (1.3) and (3.3), $z\psi'(z) \in S_H^*[A,B]$ if and only if

$$z\psi'(z) * \left[\frac{z + \frac{(-e^{-i\theta}-A)}{A-B}z^2}{(-e^{-i\theta}-B)(1-z)^2}\right] \neq 0$$

i.e.,

$$
\begin{aligned}
0 &\neq \frac{1}{\lambda e^{it}}\left[zh' * \left\{\frac{z+\frac{(-e^{-i\theta}-A)}{A-B}z^2}{(-e^{i\theta}-B)(1-z)^2}\right\} - \epsilon zg' * \left\{\frac{z+\frac{(-e^{-i\theta}-A)}{A-B}z^2}{(-e^{-i\theta}-B)(1-z)^2}\right\}\right]. \\
&= \frac{1}{\lambda e^{it}}\left[h(z) * \left\{\frac{z+\frac{(-e^{-i\theta}-A)}{A-B}z^2}{(1-z)^2(-e^{-i\theta}-B)}\right\}' - \epsilon g(z) * \left\{\frac{z+\frac{(-e^{-i\theta}-A)}{A-B}z^2}{(1-z)^2(-e^{-i\theta}-B)}\right\}'\right] \\
&= \frac{1}{\lambda e^{it}}\left[h(z) * \left(\frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}z^2}{(1-z)^3(-e^{-i\theta}-B)}\right) - \epsilon g(z) * \left(\frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}z^2}{(1-z)^3(-e^{-i\theta}-B)}\right)\right] \\
&= \frac{1}{\lambda}\left[h(z) * \left(\frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}z^2}{e^{it}(1-z)^3(-e^{-i\theta}-B)}\right) - \epsilon g(z) * \left(\frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}z^2}{e^{it}(1-z)^3(-B-e^{-i\theta})\frac{e^{-i\theta}}{B}}\right)\right] \\
&= \frac{1}{\lambda}\left[h(z) * \frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}z^2}{e^{it}(1-z)^3(-e^{-i\theta}-B)} - \epsilon\overline{g(z)} * \overline{\left(\frac{Be^{i\theta}z+\frac{-2B-(A+B)Be^{i\theta}}{A-B}z^2}{e^{it}(1-z)^3(-B-e^{i\theta})}\right)}\right] \\
&= \frac{1}{\lambda}\left[h(z) * \frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}}{e^{it}(1-z)^3(-e^{-i\theta}-B)} - \epsilon\overline{g(z)} * \left(\frac{(-B)(-e^{-i\theta})\overline{z}+\frac{-2B-(A+B)Be^{-i\theta}}{A-B}\overline{z}^2}{e^{-it}(-B-e^{-i\theta})(1-\overline{z})^3}\right)\right] \\
&= \frac{1}{\lambda}\left[h(z) * \frac{z+\frac{-2e^{-i\theta}-A-B}{A-B}}{e^{it}(1-z)^3(e^{-i\theta}-B)} + \epsilon B\overline{g(z)} * \left(\frac{(-e^{-i\theta})\overline{z}-\frac{-2+(A+B)(-e^{-i\theta})}{A-B}\overline{z}^2}{e^{-it}(-B-e^{-i\theta})(1-\overline{z})^3}\right)\right],
\end{aligned}
$$

and we have the result. □

Corollary 3.11. *If $A = 1, B = -1, \epsilon = -1$, then we have Theorem 2.7 of* [1].

Theorem 3.12. *If $f = h + \overline{g} \in S_H$ with*

$$\sum_{m=2}^{\infty} mC(n,m)|a_m| + \sum_{m=1}^{\infty} mC(n,m)|b_m| \leq 1, \tag{3.4}$$

then $D^n f = H + \overline{G} \in C_H$. The result is sharp.

Proof. The result follows immediately. Using Lemma 2.5, the function

$$f(z) = z + \frac{1+\delta}{mC(n,m)}\overline{z}^m, \quad \delta > 0$$

shows that the upper bound in (3.4) cannot be improved. □

Theorem 3.13. *If $f = h + \overline{g}$ is locally univalent with $\sum_{m=2}^{\infty} m^2 C(n,m)|a_m| \leq 1$, then $D^n f \in C_H$.*

Proof. Take $\epsilon = 0$ in Lemma 2.2 and apply Lemma 2.4. □

Corollary 3.14. *$D^n f = H + \overline{G} \in C_H$ if $|G'(z)| \leq \frac{1}{2}$ and $\sum_{m=2}^{\infty} m^2 C(n,m)|a_m| \leq 1$.*

Proof. The function $D^n f$ is locally univalent if $|H'(z)| > |G'(z)|$ for $z \in U$. Since

$$2\sum_{m=2}^{\infty} mC(n,m)|a_m| \leq \sum_{m=2}^{\infty} m^2 C(n,m)|a_m| \leq 1,$$

we have

$$|H'(z)| > 1 - \sum_{m=2}^{\infty} m|a_m|C(n,m)| \geq \frac{1}{2}.$$

□

Corollary 3.15. *If $h(z) \in K$ and $w(z)$ is analytic with $|w(z)| < 1$, then*

$$f(z) = D^n h(z) + \int_0^z w(t)(D^n h(t))' dt \in C_H.$$

Theorem 3.16. *Let $f = h + \overline{g} \in S_H$. If $D^{n+1} f \in R$, then $D^n f \in R$, where R can be $S_H^*[A, B]$ or $K_H[A, B]$ or C_H.*

Proof. We can prove the result when $R \equiv S_H^*[A, B]$. If $D^{n+1} f \in S_H^*[A, B]$, then $D^{n+1}\left[\frac{h-g}{1-b_1}\right] \in S^*[A, B]$ and $|D^{n+1}h| > |D^{n+1}g|$. Using Lemma 2.9, we have

$$D^n\left[\frac{h-g}{1-b_1}\right] \in S^*[A, B].$$

Since

$$|D^{n+1}h| = \left| z\left(\frac{z}{(1-z)^{n+1}} * h\right)' \right| = \left| z\left\{\frac{1}{z}\frac{z}{(1-z)^{n+1}} * h'\right\}\right|,$$

this implies $|D^n h| > |D^n g|$, or $D^n(h) + \overline{D^n g} \in S_H^*[A, B]$ and we have the result. □

Theorem 3.17. *Let $f = h + \overline{g} \in S_H$ and let $F_c(f) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t) dt$. If $D^n f \in R$, then $D^n F_c(f) \in R$, where R can be $S_H^*[A, B]$ or $K_H[A, B]$ or C_H.*

Proof. If $D^n f \in S_H^*[A, B]$, then $D^n\left(\frac{h-g}{1-b_1}\right) \in S^*[A, B]$. Using Lemma 2.9, we have $D^n F_c(f) \in S^*[A, B]$. That is, $D^n F_c\left(\frac{(h-g)}{1-b_1}\right) \in S^*[A, B]$ or $D^n F_c(h) - D^n F_c(g) \in S^*[A, B]$. Since $|D^n F_c(n)| > |D^n F_c(g)|$, then $D^n F_c(f) \in S_H^*[A, B]$. □

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