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NEWER APPLICATIONS OF GENERALIZED MONOTONE SEQUENCES

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ABSTRACT. A particular result of Telyakovskiĭ is extended to the newly defined class of numerical sequences and a specific problem is also highlighted. A further analogous result is also proved.

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1. INTRODUCTION

Recently several papers, see [4], [5] and [6], have dealt with the issue of uniform convergence and boundedness of monotone decreasing sequences. Further results and extensions have also been reported by the author in [6].

In this paper we shall give two further results on boundedness for wider classes of monotone sequences. First we present some theorems which will be useful in the following sections of this paper. In Section 2 we state the main results, in Section 3, we provide definitions and notations and in Section 4 we give detailed proofs of the main theorem and corollary.

In [7] S.A. Telyakovskiĭ proved the following useful theorem.

Theorem 1.1. *If a sequence $\{n_m\}$ of natural numbers $(n_1 = 1 < n_2 < n_3 < \cdots)$ is such that*

$$\sum_{j=m}^{\infty} \frac{1}{n_j} \le \frac{A}{n_m} \tag{1.1}$$

for all $m = 1, 2, \ldots,$ where $A > 1$, then the estimate

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} \frac{\sin kx}{k} \right| \le KA \tag{1.2}$$

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holds for all x, where K is an absolute positive constant.

In [4], the author showed that the sequence $\{k^{-1}\}$ in (1.2) can be replaced by any sequence $\mathbf{c} := \{c_k\}$ which belongs to the class R_0^+BVS.

Recently in [5] and [6] we verified as well that the sequence $\{k^{-1}\}$ can be replaced by sequences which belong to either of the classes $\gamma RBVS$ and $\gamma GBVS$.

More precisely we proved:

Theorem 1.2 (see [6])**.** *Let $\gamma := \{\gamma_n\}$ be a sequence of nonnegative numbers satisfying the condition $\gamma_n = O(n^{-1})$; furthermore let $\alpha := \{\alpha_n\}$ be a similar sequence with the condition $\alpha_n = o(n^{-1})$. If $c := \{c_n\} \in \gamma GBVS$, or belongs to $\alpha GBVS$, furthermore, if the sequence $\{n_m\}$ satisfies (1.1), then the estimates*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx\right| \le K(\mathbf{c}, \{n_m\}), \tag{1.3}$$

or

$$\sum_{j=m}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx\right| = o(1), \qquad m \to \infty, \tag{1.4}$$

hold uniformly in x, respectively.

We note that, in general, (1.3) does not imply (1.4) see the Remark in [5]. We also note that, every quasi geometrically increasing sequence $\{n_m\}$ satisfies the inequality (1.1) (see [3, Lemma 1]).

A consequence of Theorem 1.1 shows that not only series (1.2) but also the Fourier series of any function of bounded variation possesses the property analogous to (1.2) (see [7, Theorem 2]).

Utilizing these results Telyakovskiĭ [7] proved another theorem, which is an interesting variation of a theorem by W.H. Young [8].

This theorem reads as follows.

Theorem 1.3. *If the function $f \in L(0, 2\pi)$ and the function g is of bounded variation on $[0, 2\pi]$, then the estimate*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} (a_k\alpha_k + b_k\beta_k)\right| \le KA\|f\|_L V(g) \tag{1.5}$$

is valid for any sequence $\{n_m\}$ with (1.1), where a_k, b_k and α_k, β_k are the Fourier coefficients of f and g, respectively.

One can see that if we consider the function of bounded variation

$$g(x) := \frac{\pi - x}{2} = \sum_{k=1}^{\infty} \frac{\sin kx}{k}, \qquad 0 < x < 2\pi,$$

then (1.5) reduces to

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} \frac{b_k}{k}\right| \le KA\|f\|_L, \tag{1.6}$$

which strengthens the well-known result by H. Lebesgue [2, p. 102] that the series

$$\sum_{k=1}^{\infty} \frac{b_k}{k}$$

converges for the functions $f \in L(0, 2\pi)$.

These observations are made in [7] as well.

We have recalled (1.6) because one of our aims is to show that the sequence $\{k^{-1}\}$ appearing in (1.6) can be replaced, as was the case in (1.2), by any sequence $\{\beta_k\} \in \gamma GBVS$, if $\gamma_n = O(n^{-1})$.

2. RESULTS

We prove the following assertions.

Theorem 2.1. *If the function $f \in L(0, 2\pi)$ with $\{b_k\}$ Fourier sine coefficients, the sequence $\{n_m\}$ is quasi geometrically increasing, and the sequence $\{\beta_k\}$ belongs to $\gamma GBVS$ or $\alpha GBVS$, where α and γ have the same definition as in Theorem 1.2, then*

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} b_k \beta_k \right| \le K(\{n_m\}, \{\beta_k\}) \|f\|_L, \tag{2.1}$$

or

$$\sum_{j=m}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} b_k \beta_k \right| = o(1), \qquad m \to \infty, \tag{2.2}$$

hold, respectively.

Remark 2.2. It is clear that if a sine series with coefficients $\{\beta_n\} \in \gamma GBVS$ and $\gamma_n = O(n^{-1})$, that is, if the function

$$g(x) := \sum_{k=1}^{\infty} \beta_k \sin kx$$

had a bounded variation, then (2.1) would be a special case of (1.5).

The author is unaware of such a result, or its converse. It is an interesting open question.

Utilizing our result (2.1) and the method of Telyakovskiĭ used in [7] we can also obtain estimates for $E_n(f)_L$ and $\omega_\nu(f, \delta)_L$.

Corollary 2.3. *If $f(x),\ \gamma,\ \{b_k\},\ \{\beta_k\}$ and $\{n_m\}$ are as in Theorem 2.1, then for any n with $n_i \le n < n_{i+1}$ the following estimates*

$$\begin{aligned} \omega_\nu\left(f, \frac{1}{n}\right)_L &\ge K(\nu) E_n(f)_L \\ &\ge K(\nu, \{n_m\}, \{\beta_k\}) \left(\left| \sum_{k=n+1}^{n_{i+1}-1} b_k \beta_k \right| + \sum_{j=i+1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} b_k \beta_k \right| \right) \end{aligned} \tag{2.3}$$

hold.

3. NOTIONS AND NOTATIONS

A *positive* null-sequence $\mathbf{c} := \{c_n\}$ $(c_n \to 0)$ belongs to the family of *sequences of rest bounded variation*, and briefly we write $\mathbf{c} \in R_0^+ BVS$, if

$$\sum_{n=m}^{\infty} |\Delta\, c_n| \leq K c_m, \qquad (\Delta\, c_n = c_n - c_{n+1}),$$

holds for all $m \in \mathbb{N}$, where $K = K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

In this paper we shall use K to designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

Let $\gamma := \{\gamma_n\}$ be a given *positive* sequence. A null-sequence $\mathbf{c}$ of *real numbers* satisfying the inequality

$$\sum_{n=m}^{\infty} |\Delta\, c_n| \leq K\, \gamma_m$$

is said to be a *sequence of* γ *rest bounded variation*, represented by $\mathbf{c} \in \gamma RBVS$.

If γ is a given sequence of *nonnegative* numbers, the terms c_n are real and the inequality

$$\sum_{n=m}^{2m} |\Delta\, c_n| \leq K\, \gamma_m, \qquad m = 1, 2, \dots$$

holds, then we write $\mathbf{c} \in \gamma GBVS$.

The class $\gamma GBVS$ of sequences is wider than any one of the classes $\gamma RBVS$ and $GBVS$. The class $GBVS$ was defined in [1] by Le and Zhou with $\gamma_m := \max\limits_{m \leq n < m+N} |c_n|$, where N is a natural number.

A sequence $\beta := \{\beta_n\}$ of positive numbers is called quasi geometrically increasing (decreasing) if there exist natural numbers μ and $K = K(\beta) \geq 1$ such that for all natural numbers n,

$$\beta_{n+\mu} \geq 2\beta_n \text{ and } \beta_n \leq K\, \beta_{n+1} \quad \left(\beta_{n+\mu} \leq \frac{1}{2}\beta_n \text{ and } \beta_{n+1} \leq K\, \beta_n\right).$$

Let $E_n(f)_L$ denote the best approximation of the function f in the metric L by trigonometric polynomials of order n; and $t_n(f,x)$ be a polynomial of best approximation of $f(x)$ in the metric L by trigonometric polynomials of order n.

Finally denote by $\omega_\nu(f,\delta)_L$ the integral modulus of continuity of order ν of $f \in L$.

4. PROOFS

In this section we detail proofs of Theorem 2.1 and Corollary 2.3.

Proof of Theorem 2.1. It is clear that

$$\sum_{k=n_j}^{n_{j+1}-1} b_k \beta_k = \frac{1}{\pi} \int_0^{2\pi} \sum_{k=n_j}^{n_{j+1}-1} \beta_k \sin kx\, dx.$$

Thus

$$\left| \sum_{k=n_j}^{n_{j+1}-1} b_k \beta_k \right| \leq \frac{1}{\pi} \int_0^{2\pi} |f(x)| \left| \sum_{k=n_j}^{n_{j+1}-1} \beta \sin kx \right| dx.$$

Let us sum up these inequalities and apply the estimate (1.3) with β_k in place of c_k, we get that

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} b_k\beta_k\right| \leq \frac{1}{\pi}\int_0^{2\pi}|f(x)|\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1}\beta_k \sin kx\right| dx$$
$$\leq K(\{\beta_k\},\{n_m\})\|f\|_L,$$

which proves (2.1).

If we sum only from m to infinity and use the assertion (1.4) instead of (1.3), we clearly obtain (2.2).

Herewith Theorem 2.1 is proved. □

Proof of Corollary 2.3. It is easy to see that Jackson's theorem and the estimate (2.1) with $f(x) - t_n(f,x)$ in place of $f(x)$ yield (2.3) immediately. □

An itemized reasoning can be found in [7].

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