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A NEW SUBCLASS OF k -UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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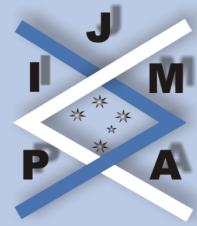
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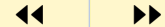
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Abstract:

The main object of this paper is to introduce and investigate a subclass $\mathcal{U}(\lambda, \alpha, \beta, k)$ of normalized analytic functions in the open unit disk Δ , which generalizes the familiar class of uniformly convex functions. The various properties and characteristics for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ derived here include (for example) a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

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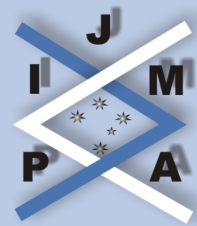
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1. Introduction and Motivation

Let \mathcal{A} denote the class of functions f normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

As usual, we denote by \mathcal{S} the subclass of \mathcal{A} consisting of functions which are *univalent* in Δ . Suppose also that, for $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of functions in \mathcal{A} which are, respectively, *starlike of order α* in Δ and *convex of order α* in Δ (see, for example, [11]). Finally, let \mathcal{T} denote the subclass of \mathcal{S} consisting of functions f given by

$$(1.2) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

with *negative* coefficients. Silverman [9] introduced and investigated the following subclasses of the function class \mathcal{T} :

$$(1.3) \quad \mathcal{T}^*(\alpha) := \mathcal{S}^*(\alpha) \cap \mathcal{T} \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{K}(\alpha) \cap \mathcal{T} \quad (0 \leq \alpha < 1).$$

Definition 1. A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if it satisfies the following inequality:

$$(1.4) \quad \Re \left(\frac{zF'(z)}{F(z)} \right) > k \left| \frac{zF'(z)}{F(z)} - 1 \right| + \beta$$

$(0 \leq \alpha \leq \lambda \leq 1; 0 \leq \beta < 1; k \geq 0),$

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where

$$(1.5) \quad F(z) := \lambda \alpha z^2 f''(z) + (\lambda - \alpha) z f'(z) + (1 - \lambda + \alpha) f(z).$$

The above-defined function class $\mathcal{U}(\lambda, \alpha, \beta, k)$ is of special interest and it contains many well-known as well as new classes of analytic univalent functions. In particular, $\mathcal{U}(\lambda, \alpha, \beta, 0)$ is the class of functions with negative coefficients, which was introduced and studied recently by Kamali and Kadioglu [3], and $\mathcal{U}(\lambda, 0, \beta, 0)$ is the function class which was introduced and studied by Srivastava *et al.* [12] (see also Aqlan *et al.* [1]). We note that the class of k -uniformly convex functions was introduced and studied recently by Kanas and Wiśniowska [4]. Subsequently, Kanas and Wiśniowska [5] introduced and studied the class of k -uniformly starlike functions. The various properties of the above two function classes were extensively investigated by Kanas and Srivastava [6]. Furthermore, we have [cf. Equation (1.3)]

$$(1.6) \quad \mathcal{U}(0, 0, \beta, 0) \equiv \mathcal{T}^*(\alpha) \quad \text{and} \quad \mathcal{U}(1, 0, \beta, 0) \equiv \mathcal{C}(\alpha).$$

We remark here that the classes of k -uniformly starlike functions and k -uniformly convex functions are an extension of the relatively more familiar classes of uniformly starlike functions and uniformly convex functions investigated earlier by (for example) Goodman [2], Rønning [8], and Ma and Minda [7] (see also the more recent contributions on these function classes by Srivastava and Mishra [10]).

In our present investigation of the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$, we obtain a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.



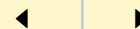
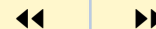
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2. A Characterization Theorem and Resulting Coefficient Estimates

We employ the technique adopted by Aqlan *et al.* [1] to find the coefficient estimates for the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Our main characterization theorem for this function class is stated as Theorem 1 below.

Theorem 1. *A function $f \in \mathcal{T}$ given by (1.2) is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if*

$$(2.1) \quad \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \leq 1 - \beta$$

$$(0 \leq \alpha \leq \lambda \leq 1; 0 \leq \beta < 1; k \geq 0).$$

The result is sharp for the function $f(z)$ given by

$$(2.2) \quad f(z) = z - \frac{1 - \beta}{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} z^n \quad (n \geq 2).$$

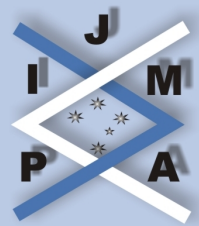
Proof. By Definition 1, $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if the condition (1.4) is satisfied. Since it is easily verified that

$$\Re(\omega) > k|\omega - 1| + \beta \iff \Re(\omega(1 + ke^{i\theta}) - ke^{i\theta}) > \beta$$

$$(-\pi \leq \theta < \pi; 0 \leq \beta < 1; k \geq 0),$$

the inequality (1.4) may be rewritten as follows:

$$(2.3) \quad \Re \left(\frac{zF'(z)}{F(z)} (1 + ke^{i\theta}) - ke^{i\theta} \right) > \beta$$



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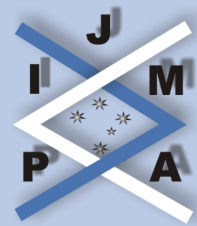
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or, equivalently,

$$(2.4) \quad \Re \left(\frac{zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}}{F(z)} \right) > \beta.$$

Now, by setting

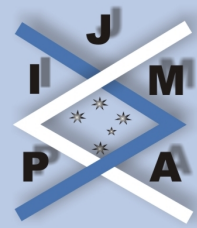
$$(2.5) \quad G(z) = zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta},$$

the inequality (2.4) becomes equivalent to

$$|G(z) + (1 - \beta)F(z)| > |G(z) - (1 + \beta)F(z)| \quad (0 \leq \beta < 1),$$

where $F(z)$ and $G(z)$ are defined by (1.5) and (2.5), respectively. We thus observe that

$$\begin{aligned} & |G(z) + (1 - \beta)F(z)| \\ & \geq |(2 - \beta)z| - \left| \sum_{n=2}^{\infty} (n - \beta + 1) \{ (n - 1)(n\lambda\alpha + \lambda - \alpha) + 1 \} a_n z^n \right| \\ & \quad - \left| ke^{i\theta} \sum_{n=2}^{\infty} (n - 1) \{ (n - 1)(n\lambda\alpha + \lambda - \alpha) + 1 \} a_n z^n \right| \\ & \geq (2 - \beta)|z| - \sum_{n=2}^{\infty} (n - \beta + 1) \{ (n - 1)(n\lambda\alpha + \lambda - \alpha) + 1 \} a_n |z|^n \\ & \quad - k \sum_{n=2}^{\infty} (n - 1) \{ (n - 1)(n\lambda\alpha + \lambda - \alpha) + 1 \} a_n |z|^n \\ & \geq (2 - \beta)|z| - \sum_{n=2}^{\infty} \{ (n(k + 1) - (k + \beta) + 1) \} \{ (n - 1)(n\lambda\alpha + \lambda - \alpha) + 1 \} a_n |z|^n. \end{aligned}$$



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Similarly, we obtain

$$|G(z) - (1 + \beta)F(z)| \leq \beta|z| + \sum_{n=2}^{\infty} \{(n(k+1) - (k + \beta) - 1)\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n|z|^n.$$

Therefore, we have

$$\begin{aligned} &|G(z) + (1 - \beta)F(z)| - |G(z) - (1 + \beta)F(z)| \\ &\geq 2(1 - \beta)|z| - 2 \sum_{n=2}^{\infty} \{(n(k+1) - (k + \beta))\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n|z|^n \\ &\geq 0, \end{aligned}$$

which implies the inequality (2.1) asserted by Theorem 1.

Conversely, by setting

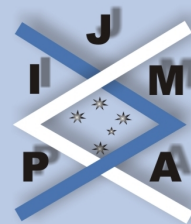
$$0 \leq |z| = r < 1,$$

and choosing the values of z on the *positive* real axis, the inequality (2.3) reduces to the following form:

$$(2.6) \quad \Re \left(\frac{(1-\beta) - \sum_{n=2}^{\infty} \{(n-\beta) - ke^{i\theta}(n-1)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} (n-1) \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n r^{n-1}} \right) \geq 0,$$

which, in view of the elementary identity:

$$\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1,$$



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becomes

$$(2.7) \quad \Re \left(\frac{(1-\beta) - \sum_{n=2}^{\infty} \{(n-\beta) - k(n-1)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} (n-1) \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n r^{n-1}} \right) \geq 0.$$

Finally, upon letting $r \rightarrow 1^-$ in (2.7), we get the desired result. \square

By taking $\alpha = 0$ and $k = 0$ in Theorem 1, we can deduce the following corollary.

Corollary 1. *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ if and only if*

$$\sum_{n=2}^{\infty} (n - \beta) \{(n - 1)\lambda + 1\} a_n \leq 1 - \beta.$$

By setting $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 1, we get the following corollary.

Corollary 2 (Silverman [9]). *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{C}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \beta) a_n \leq 1 - \beta.$$

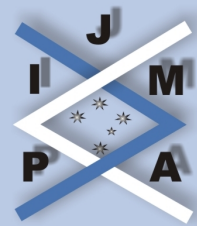
The following coefficient estimates for $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ is an immediate consequence of Theorem 1.

Theorem 2. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ is given by (1.2), then*

$$(2.8) \quad a_n \leq \frac{1 - \beta}{\{n(k + 1) - (k + \beta)\} \{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \quad (n \geq 2)$$

$$(0 \leq \alpha \leq \lambda \leq 1; 0 \leq \beta < 1; k \geq 0).$$

Equality in (2.8) holds true for the function $f(z)$ given by (2.2).



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By taking $\alpha = k = 0$ in Theorem 2, we obtain the following corollary.

Corollary 3. Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ if and only if

$$(2.9) \quad a_n \leq \frac{1 - \beta}{(n - \beta)\{(n - 1)\lambda + 1\}} \quad (n \geq 2).$$

Equality in (2.9) holds true for the function $f(z)$ given by

$$(2.10) \quad f(z) = z - \frac{1 - \beta}{(n - \beta)\{(n - 1)\lambda + 1\}} z^n \quad (n \geq 2).$$

Lastly, if we set $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 1, we get the following familiar result.

Corollary 4 (Silverman [9]). Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{C}(\beta)$ if and only if

$$(2.11) \quad a_n \leq \frac{1 - \beta}{n(n - \beta)} \quad (n \geq 2).$$

Equality in (2.11) holds true for the function $f(z)$ given by

$$(2.12) \quad f(z) = z - \frac{1 - \beta}{n(n - \beta)} z^n \quad (n \geq 2).$$

3. Distortion and Covering Theorems for the Function Class

$$\mathcal{U}(\lambda, \alpha, \beta, k)$$

Theorem 3. If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then

$$(3.1) \quad r - \frac{1 - \beta}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha)} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha)} r^2 \quad (|z| = r < 1).$$

Equality in (3) holds true for the function $f(z)$ given by

$$(3.2) \quad f(z) = z - \frac{1 - \beta}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha)} z^2.$$

Proof. We only prove the second part of the inequality in (3), since the first part can be derived by using similar arguments. Since $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, by using Theorem 1, we find that

$$\begin{aligned} & (2 + k - \beta)(2\lambda\alpha + \lambda - \alpha + 1) \sum_{n=2}^{\infty} a_n \\ &= \sum_{n=2}^{\infty} (2 + k - \beta)(2\lambda\alpha + \lambda - \alpha + 1) a_n \\ &\leq \sum_{n=2}^{\infty} \{n(k + 1) - (k + \beta)\} \{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \\ &\leq 1 - \beta, \end{aligned}$$



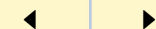
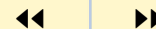
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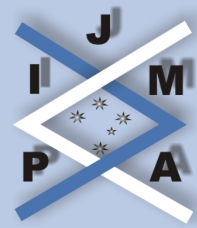
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which readily yields the following inequality:

$$(3.3) \quad \sum_{n=2}^{\infty} a_n \leq \frac{1 - \beta}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha + 1)}.$$

Moreover, it follows from (1.2) and (3.3) that

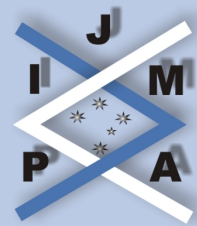
$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + \frac{1 - \beta}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha + 1)} r^2, \end{aligned}$$

which proves the second part of the inequality in (3). □

Theorem 4. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$(3.4) \quad \begin{aligned} 1 - \frac{2(1 - \beta)}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha)} r \\ \leq |f'(z)| \\ \leq 1 + \frac{2(1 - \beta)}{(2 + k - \beta)(2\lambda\alpha + \lambda - \alpha)} r \quad (|z| = r < 1). \end{aligned}$$

Equality in (3.4) holds true for the function $f(z)$ given by (3.2).



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Proof. Our proof of Theorem 4 is much akin to that of Theorem 3. Indeed, since $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, it is easily verified from (1.2) that

$$(3.5) \quad |f'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} na_n$$

and

$$(3.6) \quad |f'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} na_n.$$

The assertion (3.4) of Theorem 4 would now follow from (3.5) and (3.6) by means of a rather simple consequence of (3.3) given by

$$(3.7) \quad \sum_{n=2}^{\infty} na_n \leq \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha + \lambda - \alpha + 1)}.$$

This completes the proof of Theorem 4. □

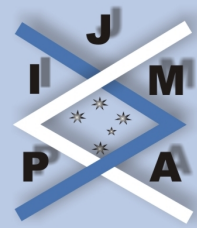
Theorem 5. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then $f \in \mathcal{T}^*(\delta)$, where*

$$\delta := 1 - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha + \lambda - \alpha) - (1-\beta)}.$$

The result is sharp with the extremal function $f(z)$ given by (3.2).

Proof. It is sufficient to show that (2.1) implies that

$$(3.8) \quad \sum_{n=2}^{\infty} (n-\delta)a_n \leq 1-\delta,$$



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that is, that

$$(3.9) \quad \frac{n - \delta}{1 - \delta} \leq \frac{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1 - \beta} \quad (n \geq 2),$$

Since (3.9) is equivalent to the following inequality:

$$\delta \leq 1 - \frac{(n - 1)(1 - \beta)}{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\} - (1 - \beta)} \quad (n \geq 2)$$

$$=: \Psi(n),$$

and since

$$\Psi(n) \leq \Psi(2) \quad (n \geq 2),$$

(3.9) holds true for

$$n \geq 2, \quad 0 \leq \lambda \leq 1, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \beta < 1 \quad \text{and} \quad k \geq 0.$$

This completes the proof of Theorem 5. □

By setting $\alpha = k = 0$ in Theorem 5, we can deduce the following result.

Corollary 5. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$f \in \mathcal{T}^* \left(\frac{\lambda(2 - \beta) + \beta}{\lambda(2 - \beta) + 1} \right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1 - \beta}{(\lambda + 1)(2 - \beta)} z^2.$$

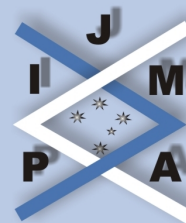
For the choices $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 5, we obtain the following result of Silverman [9].

Corollary 6. *If $f \in \mathcal{C}(\beta)$, then*

$$f \in \mathcal{T}^* \left(\frac{2}{3 - \beta} \right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1 - \beta}{2(2 - \beta)} z^2.$$



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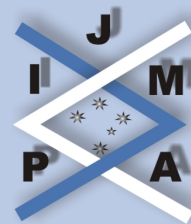
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4. Extreme Points of the Function Class $\mathcal{U}(\lambda, \alpha, \beta, k)$

Theorem 6. *Let*

$$f_1(z) = z \quad \text{and}$$

$$f_n(z) = z - \frac{1 - \beta}{\{n(k+1) - (k + \beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} z^n \quad (n \geq 2).$$

Then $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if it can be represented in the form:

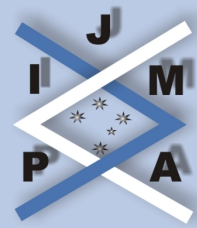
$$(4.1) \quad f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \quad \left(\mu_n \geq 0; \sum_{n=1}^{\infty} \mu_n = 1 \right).$$

Proof. Suppose that the function $f(z)$ can be written as in (4.1). Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n \left(z - \frac{1 - \beta}{\{n(k+1) - (k + \beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} z^n \right) \\ &= z - \sum_{n=2}^{\infty} \mu_n \left(\frac{1 - \beta}{\{n(k+1) - (k + \beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \right) z^n. \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=2}^{\infty} \mu_n \left(\frac{\{n(k+1) - (k + \beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}(1 - \beta)}{(1 - \beta)\{n(k+1) - (k + \beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \right) \\ &= \sum_{n=2}^{\infty} \mu_n \\ &= 1 - \mu_1 \\ &\leq 1, \end{aligned}$$



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which implies that $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$.

Conversely, we suppose that $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Then, by Theorem 2, we have

$$a_n \leq \frac{1 - \beta}{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \quad (n \geq 2).$$

Therefore, we may write

$$\mu_n = \frac{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1 - \beta} a_n \quad (n \geq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

with $f_n(z)$ given as in (4.1). This completes the proof of Theorem 6. \square

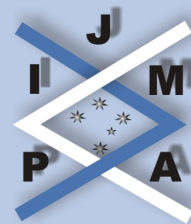
Corollary 7. *The extreme points of the function class $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ are the functions*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{1 - \beta}{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}} z^n \quad (n \geq 2).$$

For $\alpha = k = 0$ in Corollary 7, we have the following result.



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Corollary 8. *The extreme points of $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ are the functions*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1 - \beta}{\{n - \beta\}\{(n - 1)\lambda + 1\}} z^n \quad (n \geq 2).$$

By setting $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Corollary 7, we obtain the following result obtained by Silverman [9].

Corollary 9. *The extreme points of the class $\mathcal{C}(\beta)$ are the functions*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1 - \beta}{n(n - \beta)} z^n \quad (n \geq 2).$$

Theorem 7. *The class $\mathcal{U}(\lambda, \alpha, \beta, k)$ is a convex set.*

Proof. Suppose that each of the functions $f_j(z)$ ($j = 1, 2$) given by

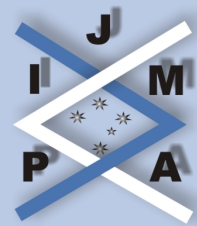
$$(4.2) \quad f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0; j = 1, 2)$$

is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1)$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1 - \mu) a_{n,2}] z^n,$$



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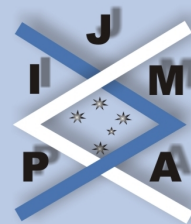
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with the aid of Theorem 1, we have

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} \\
 & \quad \times [\mu a_{n,1} + (1-\mu)a_{n,2}] \\
 & \leq \mu \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_{n,1} \\
 & \quad + (1-\mu) \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_{n,2} \\
 & \leq \mu(1-\beta) + (1-\mu)(1-\beta) \\
 (4.3) \quad & \leq 1-\beta,
 \end{aligned}$$

which implies that $g \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Hence $\mathcal{U}(\lambda, \alpha, \beta, k)$ is indeed a convex set. \square



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5. Modified Hadamard Products (or Convolution)

For functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(5.1) \quad (f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: (g * f)(z).$$

On the other hand, for functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2)$$

in the class \mathcal{T} , we define the *modified* Hadamard product (or convolution) as follows:

$$(5.2) \quad (f_1 \bullet f_2)(z) := z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n =: (f_2 \bullet f_1)(z).$$

Then we have the following result.

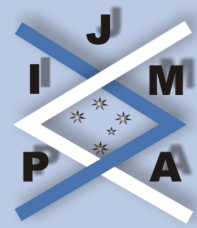
Theorem 8. *If $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ ($j = 1, 2$), then*

$$(f_1 \bullet f_2)(z) \in \mathcal{U}(\lambda, \alpha, \beta, k, \xi),$$

where

$$\xi := \frac{(2 - \beta)\{2 + k - \beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - 2(1 - \beta)^2}{(2 - \beta)\{2 + k - \beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - (1 - \beta)^2}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given as in (3.2).



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Proof. Since $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ ($j = 1, 2$), we have

$$(5.3) \quad \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_{n,j} \leq 1 - \beta \quad (j = 1, 2),$$

which, in view of the Cauchy-Schwarz inequality, yields

$$(5.4) \quad \sum_{n=2}^{\infty} \frac{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1 - \beta} \sqrt{a_{n,1} a_{n,2}} \leq 1.$$

We need to find the largest ξ such that

$$(5.5) \quad \sum_{n=2}^{\infty} \frac{\{n(k+1) - (k+\xi)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1 - \xi} a_{n,1} a_{n,2} \leq 1.$$

Thus, in light of (5.4) and (5.5), whenever the following inequality:

$$\frac{n - \xi}{1 - \xi} \sqrt{a_{n,1} a_{n,2}} \leq \frac{n - \beta}{1 - \beta} \quad (n \geq 2)$$

holds true, the inequality (5.5) is satisfied. We find from (5.4) that

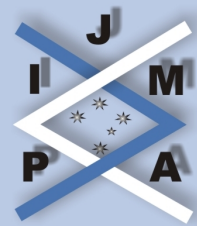
$$(5.6) \quad \sqrt{a_{n,1} a_{n,2}} \leq \frac{1 - \beta}{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \quad (n \geq 2).$$

Thus, if

$$\left(\frac{n - \xi}{1 - \xi} \right) \left(\frac{1 - \beta}{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} \right) \leq \frac{n - \beta}{1 - \beta} \quad (n \geq 2),$$

or, if

$$\xi \leq \frac{(n - \beta) \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} - n(1 - \beta)^2}{(n - \beta) \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} - (1 - \beta)^2} \quad (n \geq 2),$$



then (5.4) is satisfied. Setting

$$\Phi(n) := \frac{(n - \beta)\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\} - n(1 - \beta)^2}{(n - \beta)\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\} - (1 - \beta)^2} \\ (n \geq 2),$$

we see that $\Phi(n)$ is an *increasing* function for $n \geq 2$. This implies that

$$\xi \leq \Phi(2) = \frac{(2 - \beta)\{2 + k - \beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - 2(1 - \beta)^2}{(2 - \beta)\{2 + k - \beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - (1 - \beta)^2}.$$

Finally, by taking each of the functions $f_j(z)$ ($j = 1, 2$) given as in (3.2), we see that the assertion of Theorem 8 is sharp. \square

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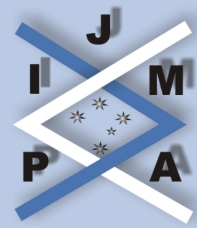
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6. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 9. Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Then $f(z)$ is close-to-convex of order ρ ($0 \leq \rho < 1$) in $|z| < r_1(\lambda, \alpha, \beta, \rho, k)$, where

$$r_1(\lambda, \alpha, \beta, \rho, k) := \inf_n \left(\frac{(1 - \rho)\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{n(1 - \beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

$$|f'(z) - 1| \leq 1 - \rho \quad (0 \leq \rho < 1; |z| < r_1(\lambda, \alpha, \beta, \rho, k)).$$

Since

$$(6.1) \quad |f'(z) - 1| = \left| - \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1},$$

we have

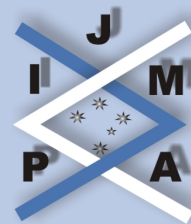
$$|f'(z) - 1| \leq 1 - \rho \quad (0 \leq \rho < 1),$$

if

$$(6.2) \quad \sum_{n=2}^{\infty} \left(\frac{n}{1 - \rho} \right) a_n |z|^{n-1} \leq 1.$$

Hence, by Theorem 1, (6.2) will hold true if

$$\left(\frac{n}{1 - \rho} \right) |z|^{n-1} \leq \frac{\{n(k + 1) - (k + \beta)\}\{(n - 1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1 - \beta},$$



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that is, if

$$(6.3) \quad |z| \leq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(1-\beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

The assertion of Theorem 9 would now follow easily from (6.3). □

Theorem 10. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Then $f(z)$ is starlike of order ρ ($0 \leq \rho < 1$) in $|z| < r_2(\lambda, \alpha, \beta, \rho, k)$, where*

$$r_2(\lambda, \alpha, \beta, \rho, k) := \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{(n-\rho)(1-\beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

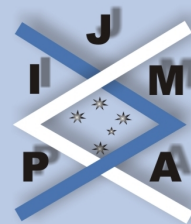
$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (0 \leq \rho < 1; |z| < r_2(\lambda, \alpha, \beta, \rho, k)).$$

Since

$$(6.4) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}},$$

we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (0 \leq \rho < 1),$$



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if

$$(6.5) \quad \sum_{n=2}^{\infty} \left(\frac{n-\rho}{1-\rho} \right) a_n |z|^{n-1} \leq 1.$$

Hence, by Theorem 1, (6.5) will hold true if

$$\left(\frac{n-\rho}{1-\rho} \right) |z|^{n-1} \leq \frac{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1-\beta},$$

that is, if

$$(6.6) \quad |z| \leq \left(\frac{(1-\rho)\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{(n-\rho)(1-\beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

The assertion of Theorem 10 would now follow easily from (6.6). \square

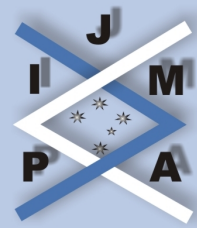
Theorem 11. Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Then $f(z)$ is convex of order ρ ($0 \leq \rho < 1$) in $|z| < r_3(\lambda, \alpha, \beta, \rho, k)$, where

$$r_3(\lambda, \alpha, \beta, \rho, k) := \inf_n \left(\frac{(1-\rho)\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{n(n-\rho)(1-\beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \rho \quad (0 \leq \rho < 1; |z| < r_3(\lambda, \alpha, \beta, \rho, k)).$$



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Since

$$(6.7) \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}},$$

we have

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \rho \quad (0 \leq \rho < 1),$$

if

$$(6.8) \quad \sum_{n=2}^{\infty} \left(\frac{n(n-\rho)}{1-\rho} \right) a_n|z|^{n-1} \leq 1.$$

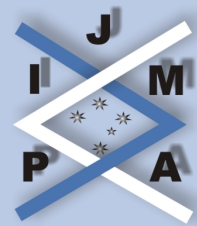
Hence, by Theorem 1, (6.8) will hold true if

$$\left(\frac{n(n-\rho)}{1-\rho} \right) |z|^{n-1} \leq \frac{\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{1-\beta},$$

that is, if

$$(6.9) \quad |z| \leq \left(\frac{(1-\rho)\{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}}{n(n-\rho)(1-\beta)} \right)^{\frac{1}{n-1}} \quad (n \geq 2).$$

Theorem 11 now follows easily from (6.9). □



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7. Hadamard Products and Integral Operators

Theorem 12. Let $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Suppose also that

$$(7.1) \quad g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (0 \leq g_n \leq 1).$$

Then

$$f * g \in \mathcal{U}(\lambda, \alpha, \beta, k).$$

Proof. Since $0 \leq g_n \leq 1$ ($n \geq 2$),

$$(7.2) \quad \begin{aligned} & \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n g_n \\ & \leq \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \\ & \leq 1 - \beta, \end{aligned}$$

which completes the proof of Theorem 12. □

Corollary 10. If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then the function $\mathcal{F}(z)$ defined by

$$(7.3) \quad \mathcal{F}(z) := \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1)$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.

Proof. Since

$$\mathcal{F}(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right) z^n \quad \left(0 < \frac{c+1}{c+n} < 1 \right),$$

the result asserted by Corollary 10 follows from Theorem 12. □

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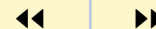
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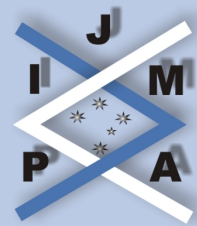
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***k*-Uniformly Convex
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