A NEW SUBCLASS OF k-UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract: The main object of this paper is to introduce and investigate a subclass $\mathcal{U}(\lambda, \alpha, \beta, k)$ of normalized analytic functions in the open unit disk Δ, which generalizes the familiar class of uniformly convex functions. The various properties and characteristics for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ derived here include (for example) a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

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1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are *univalent* in Δ. Suppose also that, for $0 \leqq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of functions in $\mathcal{A}$ which are, respectively, *starlike of order* α in Δ and *convex of order* α in Δ (see, for example, [11]). Finally, let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions f given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0) \tag{1.2}$$

with *negative* coefficients. Silverman [9] introduced and investigated the following subclasses of the function class $\mathcal{T}$:

$$\mathcal{T}^*(\alpha) := \mathcal{S}^*(\alpha) \cap \mathcal{T} \qquad \text{and} \qquad \mathcal{C}(\alpha) := \mathcal{K}(\alpha) \cap \mathcal{T} \qquad (0 \leqq \alpha < 1). \tag{1.3}$$

Definition 1. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if it satisfies the following inequality:*

$$\Re\left(\frac{zF'(z)}{F(z)}\right) > k\left|\frac{zF'(z)}{F(z)} - 1\right| + \beta \tag{1.4}$$
$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0),$$

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where

$$F(z) := \lambda\alpha z^2 f''(z) + (\lambda - \alpha) z f'(z) + (1 - \lambda + \alpha) f(z). \tag{1.5}$$

The above-defined function class $\mathcal{U}(\lambda, \alpha, \beta, k)$ is of special interest and it contains many well-known as well as new classes of analytic univalent functions. In particular, $\mathcal{U}(\lambda, \alpha, \beta, 0)$ is the class of functions with negative coefficients, which was introduced and studied recently by Kamali and Kadıoğlu [3], and $\mathcal{U}(\lambda, 0, \beta, 0)$ is the function class which was introduced and studied by Srivastava *et al.* [12] (see also Aqlan *et al.* [1]). We note that the class of k-uniformly convex functions was introduced and studied recently by Kanas and Wiśniowska [4]. Subsequently, Kanas and Wiśniowska [5] introduced and studied the class of k-uniformly starlike functions. The various properties of the above two function classes were extensively investigated by Kanas and Srivastava [6]. Furthermore, we have [*cf.* Equation (1.3)]

$$\mathcal{U}(0, 0, \beta, 0) \equiv \mathcal{T}^*(\alpha) \qquad \text{and} \qquad \mathcal{U}(1, 0, \beta, 0) \equiv \mathcal{C}(\alpha). \tag{1.6}$$

We remark here that the classes of k-uniformly starlike functions and k-uniformly convex functions are an extension of the relatively more familiar classes of uniformly starlike functions and uniformly convex functions investigated earlier by (for example) Goodman [2], Rønning [8], and Ma and Minda [7] (see also the more recent contributions on these function classes by Srivastava and Mishra [10]).

In our present investigation of the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$, we obtain a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.

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2. A Characterization Theorem and Resulting Coefficient Estimates

We employ the technique adopted by Aqlan *et al.* [1] to find the coefficient estimates for the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Our main characterization theorem for this function class is stated as Theorem 1 below.

Theorem 1. *A function $f \in \mathcal{T}$ given by (1.2) is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \leqq 1 - \beta \tag{2.1}$$

$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} z^n \quad (n \geqq 2). \tag{2.2}$$

Proof. By Definition 1, $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if the condition (1.4) is satisfied. Since it is easily verified that

$$\Re(\omega) > k|\omega - 1| + \beta \iff \Re\left(\omega(1 + ke^{i\theta}) - ke^{i\theta}\right) > \beta$$
$$(-\pi \leqq \theta < \pi;\ 0 \leqq \beta < 1;\ k \geqq 0),$$

the inequality (1.4) may be rewritten as follows:

$$\Re\left(\frac{zF'(z)}{F(z)}(1 + ke^{i\theta}) - ke^{i\theta}\right) > \beta \tag{2.3}$$

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or, equivalently,

$$\Re\left(\frac{zF'(z)(1+ke^{i\theta})-F(z)ke^{i\theta}}{F(z)}\right)>\beta. \tag{2.4}$$

Now, by setting

$$G(z)=zF'(z)(1+ke^{i\theta})-F(z)ke^{i\theta}, \tag{2.5}$$

the inequality (2.4) becomes equivalent to

$$|G(z)+(1-\beta)F(z)|>|G(z)-(1+\beta)F(z)| \qquad (0\leqq\beta<1),$$

where $F(z)$ and $G(z)$ are defined by (1.5) and (2.5), respectively. We thus observe that

$$\begin{aligned}
&|G(z)+(1-\beta)F(z)| \\
&\quad\geqq |(2-\beta)z|-\left|\sum_{n=2}^{\infty}(n-\beta+1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nz^n\right| \\
&\qquad -\left|ke^{i\theta}\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nz^n\right| \\
&\quad\geqq (2-\beta)|z|-\sum_{n=2}^{\infty}(n-\beta+1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n \\
&\qquad -k\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n \\
&\quad\geqq (2-\beta)|z|-\sum_{n=2}^{\infty}\{(n(k+1)-(k+\beta)+1\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n.
\end{aligned}$$

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Similarly, we obtain

$$|G(z) - (1+\beta)F(z)|$$
$$\leqq \beta|z| + \sum_{n=2}^{\infty}\{(n(k+1)-(k+\beta)-1\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n.$$

Therefore, we have

$$|G(z)+(1-\beta)F(z)| - |G(z)-(1+\beta)F(z)|$$
$$\geqq 2(1-\beta)|z| - 2\sum_{n=2}^{\infty}\{(n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n$$
$$\geqq 0,$$

which implies the inequality (2.1) asserted by Theorem 1.

Conversely, by setting

$$0 \leqq |z| = r < 1,$$

and choosing the values of z on the *positive* real axis, the inequality (2.3) reduces to the following form:

$$\Re\left(\frac{(1-\beta)-\sum_{n=2}^{\infty}\{(n-\beta)-ke^{i\theta}(n-1)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n r^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n r^{n-1}}\right) \geqq 0, \tag{2.6}$$

which, in view of the elementary identity:

$$\Re(-e^{i\theta}) \geqq -|e^{i\theta}| = -1,$$

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becomes

$$(2.7)\quad \Re\left(\frac{(1-\beta)-\sum_{n=2}^{\infty}\{(n-\beta)-k(n-1)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n r^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n r^{n-1}}\right) \geqq 0.$$

Finally, upon letting $r \to 1-$ in (2.7), we get the desired result. □

By taking $\alpha = 0$ and $k = 0$ in Theorem 1, we can deduce the following corollary.

Corollary 1. *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\beta)\{(n-1)\lambda+1\}a_n \leqq 1-\beta.$$

By setting $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 1, we get the following corollary.

Corollary 2 (Silverman [9]). *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{C}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)a_n \leqq 1-\beta.$$

The following coefficient estimates for $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ is an immediate consequence of Theorem 1.

Theorem 2. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ is given by (1.2), then*

$$(2.8)\quad a_n \leqq \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2)$$
$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0).$$

Equality in (2.8) holds true for the function $f(z)$ given by (2.2).

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By taking $\alpha = k = 0$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ if and only if*

$$a_n \leqq \frac{1-\beta}{(n-\beta)\{(n-1)\lambda+1\}} \qquad (n \geqq 2). \tag{2.9}$$

Equality in (2.9) holds true for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{(n-\beta)\{(n-1)\lambda+1\}}\, z^n \qquad (n \geqq 2). \tag{2.10}$$

Lastly, if we set $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 1, we get the following familiar result.

Corollary 4 (Silverman [9]). *Let $f \in \mathcal{T}$ be given by (1.2). Then $f \in \mathcal{C}(\beta)$ if and only if*

$$a_n \leqq \frac{1-\beta}{n(n-\beta)} \qquad (n \geqq 2). \tag{2.11}$$

Equality in (2.11) holds true for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{n(n-\beta)}\, z^n \qquad (n \geqq 2). \tag{2.12}$$

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3. Distortion and Covering Theorems for the Function Class $\mathcal{U}(\lambda, \alpha, \beta, k)$

Theorem 3. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$r - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)} r^2 \leqq |f(z)| \leqq r + \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)} r^2 \qquad (|z| = r < 1). \tag{3.1}$$

Equality in (3) holds true for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)} z^2. \tag{3.2}$$

Proof. We only prove the second part of the inequality in (3), since the first part can be derived by using similar arguments. Since $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, by using Theorem 1, we find that

$$\begin{aligned}
&(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)\sum_{n=2}^{\infty} a_n \\
&= \sum_{n=2}^{\infty} (2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1) a_n \\
&\leqq \sum_{n=2}^{\infty} \{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} a_n \\
&\leqq 1-\beta,
\end{aligned}$$

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which readily yields the following inequality:

$$\sum_{n=2}^{\infty} a_n \leqq \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)}. \tag{3.3}$$

Moreover, it follows from (1.2) and (3.3) that

$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leqq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leqq r + \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)}\, r^2, \end{aligned}$$

which proves the second part of the inequality in (3). □

Theorem 4. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$\begin{aligned} 1 - \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r &\leqq |f'(z)| \\ &\leqq 1 + \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r \qquad (|z| = r < 1). \end{aligned} \tag{3.4}$$

Equality in (3.4) holds true for the function $f(z)$ given by (3.2).

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Proof. Our proof of Theorem 4 is much akin to that of Theorem 3. Indeed, since $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, it is easily verified from (1.2) that

$$|f'(z)| \leqq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leqq 1 + r \sum_{n=2}^{\infty} n a_n \tag{3.5}$$

and

$$|f'(z)| \geqq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leqq 1 + r \sum_{n=2}^{\infty} n a_n. \tag{3.6}$$

The assertion (3.4) of Theorem 4 would now follow from (3.5) and (3.6) by means of a rather simple consequence of (3.3) given by

$$\sum_{n=2}^{\infty} n a_n \leqq \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha + \lambda - \alpha + 1)}. \tag{3.7}$$

This completes the proof of Theorem 4. □

Theorem 5. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then $f \in \mathcal{T}^*(\delta)$, where*

$$\delta := 1 - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha + \lambda - \alpha) - (1-\beta)}.$$

The result is sharp with the extremal function $f(z)$ given by (3.2).

Proof. It is sufficient to show that (2.1) implies that

$$\sum_{n=2}^{\infty} (n-\delta) a_n \leqq 1 - \delta, \tag{3.8}$$

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that is, that

$$(3.9)\quad \frac{n-\delta}{1-\delta} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta} \qquad (n \geqq 2),$$

Since (3.9) is equivalent to the following inequality:

$$\delta \leqq 1 - \frac{(n-1)(1-\beta)}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}-(1-\beta)} \qquad (n \geqq 2)$$
$$=: \Psi(n),$$

and since

$$\Psi(n) \leqq \Psi(2) \qquad (n \geqq 2),$$

(3.9) holds true for

$$n \geqq 2,\ 0 \leqq \lambda \leqq 1,\ 0 \leqq \alpha \leqq 1,\ 0 \leqq \beta < 1 \quad \text{and} \quad k \geqq 0.$$

This completes the proof of Theorem 5. □

By setting $\alpha = k = 0$ in Theorem 5, we can deduce the following result.

Corollary 5. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$f \in \mathcal{T}^* \left(\frac{\lambda(2-\beta)+\beta}{\lambda(2-\beta)+1} \right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{(\lambda+1)(2-\beta)}\, z^2.$$

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For the choices $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 5, we obtain the following result of Silverman [9].

Corollary 6. *If $f \in \mathcal{C}(\beta)$, then*

$$f \in \mathcal{T}^* \left(\frac{2}{3-\beta} \right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{2(2-\beta)} z^2.$$

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4. Extreme Points of the Function Class $\mathcal{U}(\lambda, \alpha, \beta, k)$

Theorem 6. *Let*

$$f_1(z) = z \quad \textit{and}$$

$$f_n(z) = z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\, z^n \qquad (n \geqq 2).$$

Then $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if it can be represented in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \qquad \left(\mu_n \geqq 0;\ \sum_{n=1}^{\infty} \mu_n = 1\right). \tag{4.1}$$

Proof. Suppose that the function $f(z)$ can be written as in (4.1). Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n \left(z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\, z^n \right) \\ &= z - \sum_{n=2}^{\infty} \mu_n \left(\frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \right) z^n. \end{aligned}$$

Now

$$\begin{aligned} &\sum_{n=2}^{\infty} \mu_n \left(\frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}(1-\beta)}{(1-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \right) \\ &\qquad = \sum_{n=2}^{\infty} \mu_n \\ &\qquad = 1 - \mu_1 \\ &\qquad \leqq 1, \end{aligned}$$

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which implies that $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$.

Conversely, we suppose that $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Then, by Theorem 2, we have

$$a_n \leqq \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2).$$

Therefore, we may write

$$\mu_n = \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta}\, a_n \qquad (n \geqq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

with $f_n(z)$ given as in (4.1). This completes the proof of Theorem 6. □

Corollary 7. *The extreme points of the function class $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ are the functions*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\, z^n \qquad (n \geqq 2).$$

For $\alpha = k = 0$ in Corollary 7, we have the following result.

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Corollary 8. *The extreme points of* $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ *are the functions*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1-\beta}{\{n-\beta\}\{(n-1)\lambda+1\}}\, z^n \qquad (n \geqq 2).$$

By setting $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Corollary 7, we obtain the following result obtained by Silverman [9].

Corollary 9. *The extreme points of the class* $\mathcal{C}(\beta)$ *are the functions*

$$f_1(z) = z \qquad \text{and} \qquad f_n(z) = z - \frac{1-\beta}{n(n-\beta)}\, z^n \qquad (n \geqq 2).$$

Theorem 7. *The class* $\mathcal{U}(\lambda, \alpha, \beta, k)$ *is a convex set.*

Proof. Suppose that each of the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0;\ j = 1, 2) \tag{4.2}$$

is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leqq \mu \leqq 1)$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

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with the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} \\
&\qquad \times [\mu a_{n,1}+(1-\mu)a_{n,2}] \\
&\leqq \mu\sum_{n=2}^{\infty}\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_{n,1} \\
&\qquad + (1-\mu)\sum_{n=2}^{\infty}\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_{n,2} \\
&\leqq \mu(1-\beta)+(1-\mu)(1-\beta) \\
&\leqq 1-\beta, \qquad (4.3)
\end{aligned}$$

which implies that $g \in \mathcal{U}(\lambda,\alpha,\beta,k)$. Hence $\mathcal{U}(\lambda,\alpha,\beta,k)$ is indeed a convex set. □

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5. Modified Hadamard Products (or Convolution)

For functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $(f*g)(z)$ is defined, as usual, by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: (g*f)(z). \tag{5.1}$$

On the other hand, for functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2)$$

in the class $\mathcal{T}$, we define the *modified* Hadamard product (or convolution) as follows:

$$(f_1 \bullet f_2)(z) := z - \sum_{n=2}^{\infty} a_{n,1}\, a_{n,2} z^n =: (f_2 \bullet f_1)(z). \tag{5.2}$$

Then we have the following result.

Theorem 8. *If* $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ $(j = 1, 2)$, *then*

$$(f_1 \bullet f_2)(z) \in \mathcal{U}(\lambda, \alpha, \beta, k, \xi),$$

where

$$\xi := \frac{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - 2(1-\beta)^2}{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha + \lambda - \alpha + 1\} - (1-\beta)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ *given as in* (3.2).

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Proof. Since $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ $(j = 1, 2)$, we have

$$\sum_{n=2}^{\infty} \{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_{n,j} \leqq 1-\beta \qquad (j = 1, 2), \tag{5.3}$$

which, in view of the Cauchy-Schwarz inequality, yields

$$\sum_{n=2}^{\infty} \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta} \sqrt{a_{n,1}\, a_{n,2}} \leqq 1. \tag{5.4}$$

We need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\{n(k+1)-(k+\xi)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\xi} a_{n,1}\, a_{n,2} \leqq 1. \tag{5.5}$$

Thus, in light of (5.4) and (5.5), whenever the following inequality:

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{n-\beta}{1-\beta} \qquad (n \geqq 2)$$

holds true, the inequality (5.5) is satisfied. We find from (5.4) that

$$\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2). \tag{5.6}$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left(\frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\right) \leqq \frac{n-\beta}{1-\beta} \qquad (n \geqq 2),$$

or, if

$$\xi \leqq \frac{(n-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - n(1-\beta)^2}{(n-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - (1-\beta)^2} \qquad (n \geqq 2),$$

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then (5.4) is satisfied. Setting

$$\Phi(n) := \frac{(n-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}-n(1-\beta)^2}{(n-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}-(1-\beta)^2} \quad (n \geqq 2),$$

we see that $\Phi(n)$ is an *increasing* function for $n \geqq 2$. This implies that

$$\xi \leqq \Phi(2) = \frac{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\}-2(1-\beta)^2}{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\}-(1-\beta)^2}.$$

Finally, by taking each of the functions $f_j(z)$ $(j = 1, 2)$ given as in (3.2), we see that the assertion of Theorem 8 is sharp. □

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6. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 9. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_1(\lambda, \alpha, \beta, \rho, k)$, where*

$$r_1(\lambda, \alpha, \beta, \rho, k)$$
$$:= \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(1-\beta)} \right)^{\frac{1}{n-1}} \qquad (n \geqq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

$$|f'(z)-1| \leqq 1-\rho \qquad \big(0 \leqq \rho < 1;\ |z| < r_1(\lambda, \alpha, \beta, \rho, k)\big).$$

Since

$$|f'(z)-1| = \left| -\sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leqq \sum_{n=2}^{\infty} n a_n |z|^{n-1}, \tag{6.1}$$

we have

$$|f'(z)-1| \leqq 1-\rho \qquad (0 \leqq \rho < 1),$$

if

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\rho} \right) a_n |z|^{n-1} \leqq 1. \tag{6.2}$$

Hence, by Theorem 1, (6.2) will hold true if

$$\left(\frac{n}{1-\rho} \right) |z|^{n-1} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

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that is, if

$$(6.3)\quad |z| \leqq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(1-\beta)}\right)^{\frac{1}{n-1}} \quad (n \geqq 2).$$

The assertion of Theorem 9 would now follow easily from (6.3). □

Theorem 10. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Then $f(z)$ is starlike of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_2(\lambda, \alpha, \beta, \rho, k)$, where*

$$r_2(\lambda, \alpha, \beta, \rho, k)$$
$$:= \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \quad (n \geqq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1-\rho \quad \big(0 \leqq \rho < 1;\ |z| < r_2(\lambda, \alpha, \beta, \rho, k)\big).$$

Since

$$(6.4)\quad \left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}},$$

we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1-\rho \qquad (0 \leqq \rho < 1),$$

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if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1}\leqq 1. \tag{6.5}$$

Hence, by Theorem 1, (6.5) will hold true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1}\leqq\frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

that is, if

$$|z|\leqq\left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}}\quad(n\geqq 2). \tag{6.6}$$

The assertion of Theorem 10 would now follow easily from (6.6). □

Theorem 11. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{U}(\lambda,\alpha,\beta,k)$. Then $f(z)$ is convex of order ρ $(0\leqq\rho<1)$ in $|z|<r_3(\lambda,\alpha,\beta,\rho,k)$, where*

$$r_3(\lambda,\alpha,\beta,\rho,k)$$
$$:=\inf_n\left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}}\quad(n\geqq 2).$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|\leqq 1-\rho\qquad\big(0\leqq\rho<1;\ |z|<r_3(\lambda,\alpha,\beta,\rho,k)\big).$$

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Since

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}, \tag{6.7}$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1-\rho \qquad (0 \leqq \rho < 1),$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z|^{n-1} \leqq 1. \tag{6.8}$$

Hence, by Theorem 1, (6.8) will hold true if

$$\left(\frac{n(n-\rho)}{1-\rho}\right)|z|^{n-1} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

that is, if

$$|z| \leqq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \quad (n \geqq 2). \tag{6.9}$$

Theorem 11 now follows easily from (6.9). □

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7. Hadamard Products and Integral Operators

Theorem 12. *Let $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Suppose also that*

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (0 \leqq g_n \leqq 1). \tag{7.1}$$

Then

$$f * g \in \mathcal{U}(\lambda, \alpha, \beta, k).$$

Proof. Since $0 \leqq g_n \leqq 1 \quad (n \geqq 2)$,

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n g_n$$

$$\leqq \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n$$

$$\leqq 1 - \beta, \tag{7.2}$$

which completes the proof of Theorem 12. □

Corollary 10. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then the function $\mathcal{F}(z)$ defined by*

$$\mathcal{F}(z) := \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt \qquad (c > -1) \tag{7.3}$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.

Proof. Since

$$\mathcal{F}(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right) z^n \qquad \left(0 < \frac{c+1}{c+n} < 1 \right),$$

the result asserted by Corollary 10 follows from Theorem 12. □

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