

ON SOME q -INTEGRAL INEQUALITIES

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Abstract: In this paper, we provide a q -analogue of an open problem posed by Q. A. Ngô et al. in the paper, *Note on an integral inequality*, J. Inequal. Pure and Appl. Math., 7(4)(2006), Art. 120, by using analytic and elementary methods in Quantum Calculus.



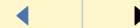
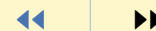
q-Integral Inequalities

Kamel Brahim

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[Title Page](#)

[Contents](#)



Page 1 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

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Contents

1	Introduction	3
2	Notations and Preliminaries	5
3	Main Results	7



q-Integral Inequalities

Kamel Brahim

vol. 9, iss. 4, art. 106, 2008

Title Page

Contents



Page 2 of 14

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 3 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

1. Introduction

In [9], Q.A. Ngô et al. studied an interesting integral inequality and proved the following result:

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0, 1]$ satisfying*

$$(1.1) \quad \int_x^1 f(t)dt \geq \int_x^1 tdt, \quad \forall x \in [0, 1].$$

Then the inequalities

$$(1.2) \quad \int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx$$

and

$$(1.3) \quad \int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx$$

hold for every positive real number $\alpha > 0$.

Then, they proposed the following open problem: *Under what condition does the inequality*

$$(1.4) \quad \int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x)dx$$

hold for α and β ?

In view of the interest in this type of inequalities, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 3, 7, 8]). In this paper, we shall discuss a *q*-analogue of Ngô's problem.

This paper is organized as follows: In Section 2, we present definitions and facts from the q -calculus necessary for understanding this paper. In Section 3, we discuss a q -analogue of the inequalities given in [9] and [3].



q-Integral Inequalities

Kamel Brahim

vol. 9, iss. 4, art. 106, 2008

[Title Page](#)

[Contents](#)



Page 4 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 5 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

2. Notations and Preliminaries

Throughout this paper, we will fix $q \in (0, 1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [4] and [6]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q -derivative $D_q f$ of a function f is given by

$$(2.1) \quad (D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,$$

$(D_q f)(0) = f'(0)$, provided $f'(0)$ exists.

The q -Jackson integral from 0 to a is defined by (see [5])

$$(2.2) \quad \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n,$$

provided the sum converges absolutely.

The q -Jackson integral in a generic interval $[a, b]$ is given by (see [5])

$$(2.3) \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

We recall that for any function f , we have (see [6])

$$(2.4) \quad D_q \left(\int_a^x f(t) d_q t \right) = f(x).$$



If F is any anti q -derivative of the function f , namely $D_q F = f$, continuous at $x = 0$, then

$$(2.5) \quad \int_0^a f(x) d_q x = F(a) - F(0).$$

A q -analogue of the integration by parts formula is given by

$$(2.6) \quad \int_a^b f(x)(D_q g(x)) d_q x = f(a)g(a) - f(b)g(b) - \int_a^b (D_q f(x))g(qx) d_q x.$$

Finally, we denote

$$[0, 1]_q = \{q^k : k = 0, 1, 2, \dots, \infty\}.$$

Title Page

Contents

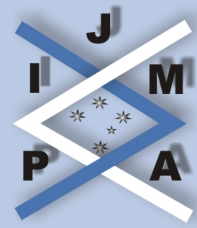


Page 6 of 14

Go Back

Full Screen

Close



Title Page

Contents



Page 7 of 14

Go Back

Full Screen

Close

3. Main Results

Let us begin with the following useful result:

Lemma 3.1 ([9] General Cauchy inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y , we always have*

$$(3.1) \quad \alpha x + \beta y \geq x^\alpha y^\beta.$$

Theorem 3.2. *Let f be a nonnegative function defined on $[0, 1]_q$ satisfying*

$$(3.2) \quad \int_x^1 f^\beta(t) d_q t \geq \int_x^1 t^\beta d_q t, \quad \forall x \in [0, 1]_q.$$

Then the inequality

$$(3.3) \quad \int_0^1 f^{\alpha+\beta}(x) d_q x \geq \int_0^1 x^\alpha f^\beta(x) d_q x,$$

holds for all positive real numbers $\alpha > 0$ and $\beta > 0$.

To prove Theorem 3.2, we need the following lemma.

Lemma 3.3. *Under the conditions of Theorem 3.2, we have*

$$(3.4) \quad \int_0^1 x^\alpha f^\beta(x) d_q x \geq \frac{1}{[\alpha + \beta + 1]_q}.$$

Proof. By using a q -integration by parts, we obtain

$$\begin{aligned} \int_0^1 x^{\alpha-1} \left(\int_x^1 f^\beta(t) d_q t \right) d_q x &= \frac{1}{[\alpha]_q} \left[x^\alpha \int_x^1 f^\beta(t) d_q t \right]_{x=0}^{x=1} + \frac{q^\alpha}{[\alpha]_q} \int_0^1 x^\alpha f^\beta(x) d_q x \\ &= \frac{q^\alpha}{[\alpha]_q} \int_0^1 x^\alpha f^\beta(x) d_q x, \end{aligned}$$



Title Page

Contents



Page 8 of 14

Go Back

Full Screen

Close

which yields

$$(3.5) \quad \int_0^1 x^\alpha f^\beta(x) d_q x = \frac{[\alpha]_q}{q^\alpha} \int_0^1 x^{\alpha-1} \left(\int_x^1 f^\beta(t) d_q t \right) d_q x.$$

On the other hand, from condition (3.2), we get

$$\begin{aligned} \int_0^1 x^{\alpha-1} \left(\int_x^1 f^\beta(t) d_q t \right) d_q x &\geq \int_0^1 x^{\alpha-1} \left(\int_x^1 t^\beta d_q t \right) d_q x \\ &= \frac{1}{[\beta+1]_q} \int_0^1 (x^{\alpha-1} - x^{\alpha+\beta}) d_q x \\ &= \frac{q^\alpha}{[\alpha]_q [\alpha+\beta+1]_q}. \end{aligned}$$

Therefore, from (3.5), we obtain

$$(3.6) \quad \int_0^1 x^\alpha f^\beta(x) d_q x \geq \frac{1}{[\alpha+\beta+1]_q}.$$

□

We now give the proof of Theorem 3.2.

Proof of Theorem 3.2. Using Lemma 3.1, we obtain

$$(3.7) \quad \frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x),$$

which gives

$$(3.8) \quad \beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \alpha \int_0^1 x^{\alpha+\beta} d_q x \geq (\alpha+\beta) \int_0^1 x^\alpha f^\beta(x) d_q x.$$



[Title Page](#)

[Contents](#)



Page 9 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

Moreover, by using Lemma 3.3, we get

$$(3.9) \quad (\alpha + \beta) \int_0^1 x^\alpha f^\beta(x) d_q x = \alpha \int_0^1 x^\alpha f^\beta(x) d_q x + \beta \int_0^1 x^\alpha f^\beta(x) d_q x \\ \geq \frac{\alpha}{[\alpha + \beta + 1]_q} + \beta \int_0^1 x^\alpha f^\beta(x) d_q x.$$

Then, from relation (3.8), we obtain

$$(3.10) \quad \beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha + \beta + 1]_q} \geq \frac{\alpha}{[\alpha + \beta + 1]_q} + \beta \int_0^1 x^\alpha f^\beta(x) d_q x,$$

which completes the proof. □

Taking $\beta = 1$ in Theorem 3.2, we obtain

Corollary 3.4. *Let f be a nonnegative function defined on $[0, 1]_q$ satisfying*

$$(3.11) \quad \int_x^1 f(t) d_q t \geq \int_x^1 t d_q t, \quad \forall x \in [0, 1]_q.$$

Then the inequality

$$(3.12) \quad \int_0^1 f^{\alpha+1}(x) d_q x \geq \int_0^1 x^\alpha f(x) d_q x$$

holds for every positive real number $\alpha > 0$.

Theorem 3.5. *Let f be a nonnegative function defined on $[0, 1]_q$ satisfying*

$$(3.13) \quad \int_x^1 f(t) d_q t \geq \int_x^1 t d_q t, \quad \forall x \in [0, 1]_q.$$



[Title Page](#)

[Contents](#)



Page 10 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

Then the inequality

$$(3.14) \quad \int_0^1 f^{\alpha+1}(x) d_q x \geq \int_0^1 x f^\alpha(x) d_q x$$

holds for every positive real number $\alpha > 0$.

Proof. We have

$$(3.15) \quad \forall x \in [0, 1]_q, \quad (f^\alpha(x) - x^\alpha)(f(x) - x) \geq 0,$$

so

$$(3.16) \quad f^{\alpha+1}(x) + x^{\alpha+1} \geq x^\alpha f(x) + x f^\alpha(x).$$

By integrating with some simple calculations we deduce that

$$(3.17) \quad \int_0^1 f^{\alpha+1}(x) d_q x + \frac{1}{[\alpha + 2]_q} \geq \int_0^1 x^\alpha f(x) d_q x + \int_0^1 x f^\alpha(x) d_q x.$$

Then, from Lemma 3.3 for $\beta = 1$, the result follows. □

Theorem 3.6. Let f be a nonnegative function defined on $[0, 1]_q$ satisfying

$$(3.18) \quad \int_x^1 f(t) d_q t \geq \int_x^1 t d_q t, \quad \forall x \in [0, 1]_q.$$

Then the inequality

$$(3.19) \quad \int_0^1 f^{\alpha+\beta}(x) d_q x \geq \int_0^1 x^\alpha f^\beta(x) d_q x$$

holds for all real numbers $\alpha > 0$ and $\beta \geq 1$.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 14

Go Back

Full Screen

Close

Lemma 3.7. Under the conditions of Theorem 3.6, we have

$$(3.20) \quad \int_0^1 x^\alpha f^\beta(x) d_q x \geq \frac{1}{[\alpha + \beta + 1]_q}$$

for all real numbers $\alpha > 0$ and $\beta \geq 1$.

Proof. Using Lemma 3.1, we obtain

$$(3.21) \quad \frac{1}{\beta} f^\beta(x) + \frac{\beta - 1}{\beta} x^\beta \geq x^{\beta-1} f(x),$$

which implies

$$(3.22) \quad \int_0^1 x^\alpha f^\beta(x) d_q x + (\beta - 1) \int_0^1 x^{\alpha+\beta} d_q x \geq \beta \int_0^1 x^{\alpha+\beta-1} f(x) d_q x.$$

Therefore, from Lemma 3.3, we get

$$(3.23) \quad \int_0^1 x^\alpha f^\beta(x) d_q x + \frac{\beta - 1}{[\alpha + \beta + 1]_q} \geq \frac{\beta}{[\alpha + \beta + 1]_q}.$$

Thus (3.20) is proved. □

We now give the proof of Theorem 3.6.

Proof of Theorem 3.6. By using Lemma 3.1, we obtain

$$(3.24) \quad \frac{\beta}{\alpha + \beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha + \beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x),$$

which implies

$$(3.25) \quad \beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha + \beta + 1]_q} \geq (\alpha + \beta) \int_0^1 x^\alpha f^\beta(x) d_q x.$$



Then, from Lemma 3.7, we obtain

$$(3.26) \quad \beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha + \beta + 1]_q} \geq \frac{\alpha}{[\alpha + \beta + 1]_q} + \beta \int_0^1 x^\alpha f^\beta(x) d_q x,$$

which completes the proof. \square

q-Integral Inequalities

Kamel Brahim

vol. 9, iss. 4, art. 106, 2008

Title Page

Contents



Page 12 of 14

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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q-Integral Inequalities

Kamel Brahim

vol. 9, iss. 4, art. 106, 2008

Title Page

Contents



Page 13 of 14

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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q-Integral Inequalities

Kamel Brahim

vol. 9, iss. 4, art. 106, 2008

Title Page

Contents



Page 14 of 14

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756