ON SOME q-INTEGRAL INEQUALITIES

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Abstract: In this paper, we provide a q-analogue of an open problem posed by Q. A. Ngô et al. in the paper, *Note on an integral inequality*, J. Inequal. Pure and Appl. Math., 7(4)(2006), Art. 120, by using analytic and elementary methods in Quantum Calculus.

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1. Introduction

In [9], Q.A. Ngô et al. studied an interesting integral inequality and proved the following result:

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 tdt, \quad \forall x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

Then, they proposed the following open problem: *Under what condition does the inequality*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x)dx \tag{1.4}$$

hold for α and β?

In view of the interest in this type of inequalities, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 3, 7, 8]). In this paper, we shall discuss a q-analogue of Ngô's problem.

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This paper is organized as follows: In Section 2, we present definitions and facts from the q-calculus necessary for understanding this paper. In Section 3, we discuss a q-analogue of the inequalities given in [9] and [3].

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2. Notations and Preliminaries

Throughout this paper, we will fix $q \in (0, 1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [4] and [6]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if} \quad x \neq 0, \tag{2.1}$$

$(D_q f)(0) = f'(0)$, provided $f'(0)$ exists.

The q-Jackson integral from 0 to a is defined by (see [5])

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.2}$$

provided the sum converges absolutely.

The q-Jackson integral in a generic interval $[a, b]$ is given by (see [5])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.3}$$

We recall that for any function f, we have (see [6])

$$D_q \left(\int_a^x f(t) d_q t \right) = f(x). \tag{2.4}$$

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If F is any anti q-derivative of the function f, namely $D_qF = f$, continuous at $x = 0$, then

$$\int_0^a f(x)d_qx = F(a) - F(0). \tag{2.5}$$

A q-analogue of the integration by parts formula is given by

$$\int_a^b f(x)(D_qg(x))d_qx = f(a)g(a) - f(b)g(b) - \int_a^b (D_qf(x))g(qx)d_qx. \tag{2.6}$$

Finally, we denote

$$[0,1]_q = \{q^k : \quad k = 0, 1, 2, \dots, \infty\}.$$

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3. Main Results

Let us begin with the following useful result:

Lemma 3.1 ([9] General Cauchy inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we always have*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}. \tag{3.1}$$

Theorem 3.2. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$\int_x^1 f^{\beta}(t) d_q t \geq \int_x^1 t^{\beta} d_q t, \quad \forall x \in [0,1]_q. \tag{3.2}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x) d_q x \geq \int_0^1 x^{\alpha} f^{\beta}(x) d_q x, \tag{3.3}$$

holds for all positive real numbers $\alpha > 0$ and $\beta > 0$.

To prove Theorem 3.2, we need the following lemma.

Lemma 3.3. *Under the conditions of Theorem 3.2, we have*

$$\int_0^1 x^{\alpha} f^{\beta}(x) d_q x \geq \frac{1}{[\alpha + \beta + 1]_q}. \tag{3.4}$$

Proof. By using a q-integration by parts, we obtain

$$\begin{aligned} \int_0^1 x^{\alpha-1} \left(\int_x^1 f^{\beta}(t) d_q t \right) d_q x &= \frac{1}{[\alpha]_q} \left[x^{\alpha} \int_x^1 f^{\beta}(t) d_q t \right]_{x=0}^{x=1} + \frac{q^{\alpha}}{[\alpha]_q} \int_0^1 x^{\alpha} f^{\beta}(x) d_q x \\ &= \frac{q^{\alpha}}{[\alpha]_q} \int_0^1 x^{\alpha} f^{\beta}(x) d_q x, \end{aligned}$$

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which yields

$$\int_0^1 x^\alpha f^\beta(x)d_qx = \frac{[\alpha]_q}{q^\alpha}\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)d_qt\right)d_qx. \tag{3.5}$$

On the other hand, from condition (3.2), we get

$$\begin{aligned}\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)d_qt\right)d_qx &\geq \int_0^1 x^{\alpha-1}\left(\int_x^1 t^\beta d_qt\right)d_qx\\ &= \frac{1}{[\beta+1]_q}\int_0^1 (x^{\alpha-1}-x^{\alpha+\beta})d_qx\\ &= \frac{q^\alpha}{[\alpha]_q[\alpha+\beta+1]_q}.\end{aligned}$$

Therefore, from (3.5), we obtain

$$\int_0^1 x^\alpha f^\beta(x)d_qx \geq \frac{1}{[\alpha+\beta+1]_q}. \tag{3.6}$$

□

We now give the proof of Theorem 3.2.

Proof of Theorem 3.2. Using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta}x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{3.7}$$

which gives

$$\beta\int_0^1 f^{\alpha+\beta}(x)d_qx + \alpha\int_0^1 x^{\alpha+\beta}d_qx \geq (\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)d_qx. \tag{3.8}$$

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Moreover, by using Lemma 3.3, we get

$$(3.9)\qquad (\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)d_qx = \alpha\int_0^1 x^\alpha f^\beta(x)d_qx + \beta\int_0^1 x^\alpha f^\beta(x)d_qx$$
$$\geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta\int_0^1 x^\alpha f^\beta(x)d_qx.$$

Then, from relation (3.8), we obtain

$$(3.10)\qquad \beta\int_0^1 f^{\alpha+\beta}(x)d_qx + \frac{\alpha}{[\alpha+\beta+1]_q} \geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta\int_0^1 x^\alpha f^\beta(x)d_qx,$$

which completes the proof. □

Taking $\beta = 1$ in Theorem 3.2, we obtain

Corollary 3.4. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$(3.11)\qquad \int_x^1 f(t)d_qt \geq \int_x^1 t d_qt, \quad \forall x \in [0,1]_q.$$

Then the inequality

$$(3.12)\qquad \int_0^1 f^{\alpha+1}(x)d_qx \geq \int_0^1 x^\alpha f(x)d_qx$$

holds for every positive real number $\alpha > 0$.

Theorem 3.5. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$(3.13)\qquad \int_x^1 f(t)d_qt \geq \int_x^1 t d_qt, \quad \forall x \in [0,1]_q.$$

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Then the inequality

$$\int_0^1 f^{\alpha+1}(x)d_qx \geq \int_0^1 xf^{\alpha}(x)d_qx \tag{3.14}$$

holds for every positive real number $\alpha > 0$.

Proof. We have

$$\forall x \in [0,1]_q, \quad (f^{\alpha}(x) - x^{\alpha})(f(x) - x) \geq 0, \tag{3.15}$$

so

$$f^{\alpha+1}(x) + x^{\alpha+1} \geq x^{\alpha}f(x) + xf^{\alpha}(x). \tag{3.16}$$

By integrating with some simple calculations we deduce that

$$\int_0^1 f^{\alpha+1}(x)d_qx + \frac{1}{[\alpha+2]_q} \geq \int_0^1 x^{\alpha}f(x)d_qx + \int_0^1 xf^{\alpha}(x)d_qx. \tag{3.17}$$

Then, from Lemma 3.3 for $\beta = 1$, the result follows. □

Theorem 3.6. *Let* f *be a nonnegative function defined on* $[0,1]_q$ *satisfying*

$$\int_x^1 f(t)d_qt \geq \int_x^1 td_qt, \quad \forall x \in [0,1]_q. \tag{3.18}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)d_qx \geq \int_0^1 x^{\alpha}f^{\beta}(x)d_qx \tag{3.19}$$

holds for all real numbers $\alpha > 0$ *and* $\beta \geq 1$.

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Lemma 3.7. *Under the conditions of Theorem 3.6, we have*

$$\int_0^1 x^\alpha f^\beta(x) d_q x \geq \frac{1}{[\alpha+\beta+1]_q} \tag{3.20}$$

for all real numbers $\alpha > 0$ *and* $\beta \geq 1$.

Proof. Using Lemma 3.1, we obtain

$$\frac{1}{\beta} f^\beta(x) + \frac{\beta-1}{\beta} x^\beta \geq x^{\beta-1} f(x), \tag{3.21}$$

which implies

$$\int_0^1 x^\alpha f^\beta(x) d_q x + (\beta-1)\int_0^1 x^{\alpha+\beta} d_q x \geq \beta \int_0^1 x^{\alpha+\beta-1} f(x) d_q x. \tag{3.22}$$

Therefore, from Lemma 3.3, we get

$$\int_0^1 x^\alpha f^\beta(x) d_q x + \frac{\beta-1}{[\alpha+\beta+1]_q} \geq \frac{\beta}{[\alpha+\beta+1]_q}. \tag{3.23}$$

Thus (3.20) is proved. □

We now give the proof of Theorem 3.6.

Proof of Theorem 3.6. By using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{3.24}$$

which implies

$$\beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha+\beta+1]_q} \geq (\alpha+\beta) \int_0^1 x^\alpha f^\beta(x) d_q x. \tag{3.25}$$

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Then, from Lemma 3.7, we obtain

$$\beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha+\beta+1]_q} \geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta \int_0^1 x^\alpha f^\beta(x) d_q x, \tag{3.26}$$

which completes the proof. □

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