APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS DEFINED BY INFINITE SUM

OVIDIU T. POP
National College "Mihai Eminescu"
5 Mihai Eminescu Street
Satu Mare 440014, Romania
EMail: ovidiutiberiu@yahoo.com

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Abstract: In this paper we start from a class of linear and positive operators defined by infinite sum. We consider the associated GBS operators and we give an approximation of B-continuous and B-differentiable functions with these operators. Through particular cases, we obtain statements verified by the GBS operators of Mirakjan-Favard-Szász, Baskakov and Meyer-König and Zeller.

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1. Introduction

In this section, we recall some notions and results which we will use in this article. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In the following, let X and Y be real intervals.

A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous function in $(x_0, y_0) \in X \times Y$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x,y),(x_0,y_0)] = 0,$$

where

$$\Delta f[(x,y),(x_0,y_0)] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$$

denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous function on $X \times Y$ if and only if it is B-continuous in any point of $X \times Y$.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable function in $(x_0, y_0) \in X \times Y$ if and only if it exists and if the limit is finite

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x,y),(x,y_0)]}{(x-x_0)(y-y_0)}.$$

This limit is called the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable function on $X \times Y$ if and only if it is B-differentiable in any point of $X \times Y$.

The definition of B-continuity and B-differentiability was introduced by K. Bögel in the papers [8] and [9].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if and only if there exists $k > 0$ so that $|\Delta f[(x,y),(s,t)]| \le k$ for any $(x,y),(s,t) \in X \times Y$.

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We shall use the function sets $B(X\times Y)=\{f|f:X\times Y\to\mathbb{R},\ f$ bounded on $X\times Y\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X\times Y)=\{f|f:X\times Y\to\mathbb{R},\ f$ is B-bounded on $X\times Y\}$, $C_b(X\times Y)=\{f|f:X\times Y\to\mathbb{R},\ f$ is B-continuous on $X\times Y\}$ and $D_b(X\times Y)=\{f|f:X\times Y\to\mathbb{R},\ f$ is B-differentiable on $X\times Y\}$.

Let $f\in B_b(X\times Y)$. The function $\omega_{\text{mixed}}(f;\cdot,\cdot):[0,\infty)\times[0,\infty)\to\mathbb{R}$ defined by

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2)=\sup\{\Delta f[(x,y),(s,t)]|:|x-s|\le\delta_1,|y-t|\le\delta_2\}$$

for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ is called the mixed modulus of smoothness.

Theorem 1.1. *Let X and Y be compact real intervals and $f\in B_b(X\times Y)$. Then $\lim\limits_{\delta_1,\delta_2\to 0}\omega_{\mathit{mixed}}(f;\delta_1,\delta_2)=0$ if and only if $f\in C_b(X\times Y)$.*

For any $x\in X$ consider the function $\varphi_x:X\to\mathbb{R}$, defined by $\varphi_x(t)=|t-x|$, for any $t\in X$. For additional information, see the following papers: [1], [3], [15] and [19].

Let $m\in\mathbb{N}$ and the operator $S_m:C_2([0,\infty))\to C([0,\infty))$ defined for any function $f\in C_2([0,\infty))$ by

$$(S_mf)(x)=e^{-mx}\sum_{k=0}^{\infty}\frac{(mx)^k}{k!}f\left(\frac{k}{m}\right), \tag{1.1}$$

for any $x\in[0,\infty)$, where $C_2([0,\infty))=\left\{f\in C([0,\infty)):\lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}\text{ exists and is finite}\right\}$. The operators $(S_m)_{m\ge1}$ are called the Mirakjan-Favard-Szász operators, introduced in 1941 by G. M. Mirakjan in the paper [13].

These operators were intensively studied by J. Favard in 1944 in the paper [11] and O. Szász in the paper [20].

From [18], the following three lemmas result.

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Lemma 1.2. *For any $m \in \mathbb{N}$, we have that*

$$(S_m \varphi_x^2)(x) = \frac{x}{m}, \tag{1.2}$$

$$(S_m \varphi_x^4)(x) = \frac{3mx^2 + x}{m^3} \tag{1.3}$$

for any $x \in [0, \infty)$ and

$$(S_m \varphi_x^2)(x) \leq \frac{a}{m}, \tag{1.4}$$

$$(S_m \varphi_x^4)(x) \leq \frac{a(3a+1)}{m^2} \tag{1.5}$$

for any $x \in [0, a]$, where $a > 0$.

Let $m \in \mathbb{N}$ and the operator $V_m : C_2([0, \infty)) \to C([0, \infty))$, defined for any function $f \in C_2([0, \infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x} \right)^k f\left(\frac{k}{m} \right) \tag{1.6}$$

for any $x \in [0, \infty)$.

The operators $(V_m)_{m \geq 1}$ are called Baskakov operators, introduced in 1957 by V. A. Baskakov in the paper [5].

Lemma 1.3. *For any $m \in \mathbb{N}$, we have that*

$$(V_m \varphi_x^2)(x) = \frac{x(1+x)}{m}, \tag{1.7}$$

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$$(1.8)\qquad \left(V_m\varphi_x^4\right)(x)=\frac{3(m+2)x^4+6(m+2)x^3+(3m+7)x^2+x}{m^3}$$

for any $x\in[0,\infty)$ *and*

$$(1.9)\qquad \left(V_m\varphi_x^2\right)(x)\le\frac{a(1+a)}{m},$$

$$(1.10)\qquad \left(V_m\varphi_x^4\right)(x)\le\frac{a(9a^3+18a^2+10a+1)}{m^2}$$

for any $x\in[0,a]$, *where* $a>0$.

W. Meyer-König and K. Zeller have introduced a sequence of linear positive operators in paper [12]. After a slight adjustment, given by E. W. Cheney and A. Sharma in [10], these operators take the form $Z_m : B([0,1))\to C([0,1))$, defined for any function $f\in B([0,1))$ by

$$(1.11)\qquad (Z_mf)(x)=\sum_{k=0}^{\infty}\binom{m+k}{k}(1-x)^{m+1}x^kf\left(\frac{k}{m+k}\right),$$

for any $m\in\mathbb{N}$ and for any $x\in[0,1)$.

These operators are called the Meyer-König and Zeller operators.

In the following we consider $Z_m : C([0,1])\to C([0,1])$, for any $m\in\mathbb{N}$.

Lemma 1.4. *For any* $m\in\mathbb{N}$ *and any* $x\in[0,1]$, *we have that*

$$(1.12)\qquad \left(Z_m\varphi_x^2\right)(x)\le\frac{x(1-x)^2}{m+1}\left(1+\frac{2x}{m+1}\right)$$

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and

$$(1.13) \qquad \left(Z_m \varphi_x^2\right)(x) \leq \frac{2}{m}.$$

The inequality of Corollary 5 from [4], in the condition (1.14) becomes inequality (1.15). Inequality (1.16) is demonstrated in [16].

Theorem 1.5. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Supposing that the operator L has the property*

$$(1.14) \qquad \left(L(\cdot - x)^{2i}(* - y)^{2j}\right)(x,y) = \left(L(\cdot - x)^{2i}\right)(x,y)\left(L(* - y)^{2j}\right)(x,y)$$

for any $(x,y) \in X \times Y$ and any $i,j \in \{1,2\}$, where "$\cdot$" and "$*$" stand for the first and second variable. Then:

(i) *For any function $f \in C_b(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have that*

$$(1.15) \qquad |f(x,y) - (ULf)(x,y)| \leq |f(x,y)||1 - (Le_{00})(x,y)| + \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{mixed}(f;\delta_1,\delta_2).$$

(ii) *For any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x,y) \in X \times Y$ and any*

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$\delta_1, \delta_2 > 0$, *we have that*

$$
\begin{aligned}
(1.16)\quad & |f(x,y) - (ULf)(x,y)| \\
& \leq |f(x,y)||1 - (Le_{00})(x,y)| \\
& \quad + 3\|D_B f\|_\infty \sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)} \\
& \quad + \Big[\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)} \\
& \quad + \delta_1^{-1}\sqrt{(L(\cdot - x)^4)(x,y)(L(* - y)^2)(x,y)} \\
& \quad + \delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^4)(x,y)} \\
& \quad + \delta_1^{-1}\delta_2^{-1}(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)\Big]\omega_{mixed}(D_B f; \delta_1, \delta_2).
\end{aligned}
$$

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2. Preliminaries

Let $I, J, K \subset \mathbb{R}$ be intervals, $J \subset K$ and $I \cap J \neq \emptyset$. We consider the sequence of nodes $((x_{m,k})_{k\in\mathbb{N}_0})_{m\geq 1}$ so that $x_{m,k} \in I \cap J$, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and the functions $\varphi_{m,k} : K \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$, for any $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $x \in J$.

Definition 2.1. If $m \in \mathbb{N}$, we define the operator $L^*_m : E(I) \to F(K)$ by

$$(L^*_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) f(x_{m,k}) \tag{2.1}$$

for any function $f \in E(I)$ and any $x \in K$, where $E(I)$ and $F(K)$ are subsets of the set of real functions defined on I, respectively on K.

Proposition 2.2. *The operators* $(L^*_m)_{m\geq 1}$ *are linear and positive on* $E(I \cap J)$.

Proof. The proof follows immediately. □

Definition 2.3. If $m, n \in \mathbb{N}$, the operator $L^*_{m,n} : E(I \times I) \to F(K \times K)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in K \times K$ by

$$(L^*_{m,n} f)(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{m,k}(x) \varphi_{n,j}(y) f(x_{m,k}, x_{n,j}) \tag{2.2}$$

is called the bivariate operator of L^* - type.

Proposition 2.4. *The operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *are linear and positive on* $E[(I \times I) \cap (J \times J)]$.

Proof. The proof follows immediately. □

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Definition 2.5. If $m, n \in \mathbb{N}$, the operator $UL^*_{m,n} : E(I \times I) \to F(K \times K)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in K \times K$ by

$$(2.3) \qquad \left(UL^*_{m,n} f\right)(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{m,k}(x)\varphi_{n,j}(y)\left[f(x_{m,k}, y) + f(x, x_{n,j}) - f(x_{m,k}, x_{n,j})\right]$$

is called a GBS operator of L^* - type.

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3. Main Results

Lemma 3.1. *For any $m,n\in\mathbb{N}$, $i,j\in\mathbb{N}_0$ and $(x,y)\in K\times K$, the identity*

$$\left(L^*_{m,n}(\cdot-x)^{2i}(*-y)^{2j}\right)(x,y)=\left(L^*_m(\cdot-x)^{2i}\right)(x)\left(L^*_n(*-y)^{2j}\right)(y) \tag{3.1}$$

holds.

Proof. We have that

$$\begin{aligned}\left(L^*_{m,n}(\cdot-x)^{2i}(*-y)^{2j}\right)(x,y)&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{m,k}(x)\varphi_{n,j}(y)(x_{m,k}-x)^{2i}(x_{n,j}-y)^{2j}\\&=\sum_{k=0}^{\infty}\varphi_{m,k}(x)(x_{m,k}-x)^{2i}\sum_{j=0}^{\infty}\varphi_{n,j}(y)(x_{n,j}-y)^{2j}\\&=\left(L^*_m(\cdot-x)^{2i}\right)(x)\left(L^*_n(*-y)^{2j}\right)(y),\end{aligned}$$

so (3.1) holds. □

For the operators constructed in this section, we note that $\delta_m(x)=\sqrt{\left(L^*_m\varphi_x^2\right)(x)}$, $\delta_{m,x}=\sqrt{\left(L^*_m\varphi_x^4\right)(x)}$, where $x\in I\cap J$, $m\in\mathbb{N}$, $m\neq 0$.

Then, by taking Lemma 3.1 into account, Theorem 1.5 becomes:

Theorem 3.2.

(i) *For any function $f\in C_b(I\times I)$, any $(x,y)\in(I\times I)\cap(J\times J)$, any $m,n\in\mathbb{N}$, any $\delta_1,\delta_2>0$, we have that*

$$\begin{aligned}&|f(x,y)-(UL^*_{m,n}f)(x,y)|\\&\le|f(x,y)||1-(Le_{00})(x,y)|+\Big((Le_{00})(x,y)+\delta_1^{-1}\delta_m(x)+\delta_2^{-1}\delta_n(y)\\&\qquad+\delta_1^{-1}\delta_2^{-1}\delta_m(x)\delta_n(y)\Big)\omega_{\textit{mixed}}(f;\delta_1,\delta_2).\end{aligned} \tag{3.2}$$

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(ii) *For any function* $f \in D_b(I \times I)$ *with* $D_B f \in B(I \times I)$, *any* $(x,y) \in (I \times I) \cap (J \times J)$, *any* $m,n \in \mathbb{N}$, *any* $\delta_1, \delta_2 > 0$, *we have that*

$$
\begin{aligned}
(3.3)\quad |f(x,y) - (UL^* f)(x,y)| &\le |f(x,y)||1 - (Le_{00})(x,y)| \\
&+ 3\|D_B f\|_\infty \delta_m(x)\delta_n(y) + \big[\delta_m(x)\delta_n(y) + \delta_1^{-1}\delta_{m,x}\delta_n(y) \\
&+ \delta_2^{-1}\delta_m(x)\delta_{n,y} + \delta_1^{-1}\delta_2^{-1}\delta_m^2(x)\delta_n^2(y)\big]\omega_{mixed}(D_B f; \delta_1, \delta_2).
\end{aligned}
$$

In the following, we give examples of operators and of the associated GBS operators.

Application 1. *If* $I = J = K = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(K) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$, $x_{m,k} = \frac{k}{m}$, $x \in [0,\infty)$, $m,k \in \mathbb{N}_0$, $m \neq 0$, *then we obtain the Mirakjan-Favard-Szász operators.*

Theorem 3.3. *Let* $a,b \in \mathbb{R}$, $a > 0$ *and* $b > 0$. *Then:*

(i) *For any function* $f \in C([0,\infty) \times [0,\infty))$, *any* $(x,y) \in [0,a] \times [0,b]$ *and* $m,n \in \mathbb{N}$, *we have that*

$$
\begin{aligned}
(3.4)\quad & |f(x,y) - (US_{m,n} f)(x,y)| \\
& \le \left(1+\sqrt{a}\right)\left(1+\sqrt{b}\right)\omega_{mixed}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

(ii) *For any function* $f \in D_b([0,\infty) \times [0,\infty)) \cap C([0,\infty) \times [0,\infty))$ *with* $D_B f \in$

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$B([0,a]\times[0,b])$*, any* $(x,y)\in[0,a]\times[0,b]$*, any* $m,n\in\mathbb{N}$*, we have that*

$$(3.5)\quad |f(x,y)-(US_{m,n}f)(x,y)|\le\sqrt{ab}\Bigg[3\|D_Bf\|_\infty+\Big(1+\sqrt{3a+1} + \sqrt{3b+1}+\sqrt{ab}\Big)\omega_{mixed}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\Bigg]\frac{1}{\sqrt{mn}}.$$

Proof. It results from Theorem 3.2, by choosing $\delta_1=\frac{1}{\sqrt{m}}$, $\delta_2=\frac{1}{\sqrt{n}}$ and Lemma 1.2. □

Theorem 3.4. *If* $f\in C([0,\infty)\times[0,\infty))$*, then the convergence*

$$(3.6)\quad \lim_{m,n\to\infty}(US_{m,n}f)(x,y)=f(x,y)$$

is uniform on any compact $[0,a]\times[0,b]$*, where* $a,b>0$*.*

Proof. It results from Theorem 1.1 and Theorem 3.3. □

Application 2. *If* $I=J=K=[0,\infty)$, $E(I)=C_2([0,\infty))$, $F(K)=C([0,\infty))$, $\varphi_{m,k}(x)=(1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$, $x_{m,k}=\frac{k}{m}$, $x\in[0,\infty)$, $m,k\in\mathbb{N}_0$, $m\neq 0$, *then we obtain the Baskakov operators.*

Theorem 3.5. *Let* $a,b\in\mathbb{R}$, $a>0$ *and* $b>0$*. Then:*

(i) *For any function* $f\in C([0,\infty)\times[0,\infty))$*, any* $(x,y)\in[0,a]\times[0,b]$ *and any* $m,n\in\mathbb{N}$*, we have that*

$$(3.7)\quad |f(x,y)-(UV_{m,n}f)(x,y)| \le\left(1+\sqrt{a(1+a)}\right)\left(1+\sqrt{b(1+b)}\right)\omega_{mixed}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

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(ii) *For any function* $f \in D_b([0,\infty)\times[0,\infty)) \cap C([0,\infty)\times[0,\infty))$ *with* $D_B f \in B([0,a]\times[0,b])$, *any* $(x,y)\in[0,a]\times[0,b]$, *any* $m,n\in\mathbb{N}$, *we have that*

$$|f(x,y)-(UV_{m,n}f)(x,y)| \le \sqrt{ab(1+a)(1+b)}\Big\{3\|D_B\|_\infty + \Big[1+\sqrt{9a^3+18a^2+10a+1}+\sqrt{9b^3+18b^2+10b+1} + \sqrt{ab(1+a)(1+b)}\Big]\omega_{mixed}\left(D_B f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\Big\}\frac{1}{\sqrt{mn}}. \tag{3.8}$$

Proof. It results from Theorem 3.2, by choosing $\delta_1 = \frac{1}{\sqrt{m}}$, $\delta_2 = \frac{1}{\sqrt{n}}$ and Lemma 1.3. □

Theorem 3.6. *If* $f\in C([0,\infty)\times[0,\infty))$, *then the convergence*

$$\lim_{m,n\to\infty}(UV_{m,n}f)(x,y) = f(x,y) \tag{3.9}$$

is uniform on any compact $[0,a]\times[0,b]$, *where* $a,b>0$.

Proof. It results from Theorem 1.1 and Theorem 3.5. □

Application 3. *If* $I=J=K=[0,1]$, $E(I)=F(K)=C([0,1])$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$, $x_{m,k}=\frac{k}{m}$, $x\in[0,1]$, $m,k\in\mathbb{N}_0$, $m\neq 0$, *then we obtain the Meyer-König and Zeller operators.*

Theorem 3.7. *For any function* $f\in C([0,1]\times[0,1])$, *any* $(x,y)\in[0,1]\times[0,1]$ *and any* $m,n\in\mathbb{N}$, *we have that*

$$|f(x,y)-(UZ_{m,n}f)(x,y)| \le (3+2\sqrt{2})\omega_{mixed}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right). \tag{3.10}$$

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Proof. It results from Theorem 3.2, by choosing $\delta_1 = \frac{1}{\sqrt{m}}$, $\delta_2 = \frac{1}{\sqrt{n}}$ and Lemma 1.4. □

Theorem 3.8. *If $f \in C([0,1] \times [0,1])$, then the convergence*

$$\lim_{m,n\to\infty} (UZ_{m,n}f)(x,y) = f(x,y) \tag{3.11}$$

is uniform on $[0,1] \times [0,1]$.

Proof. It results from Theorem 1.1 and Theorem 3.7. □

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