

# APPROXIMATION OF $B$ -CONTINUOUS AND $B$ -DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS DEFINED BY INFINITE SUM

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*Abstract:* In this paper we start from a class of linear and positive operators defined by infinite sum. We consider the associated GBS operators and we give an approximation of  $B$ -continuous and  $B$ -differentiable functions with these operators. Through particular cases, we obtain statements verified by the GBS operators of Mirakjan-Favard-Szász, Baskakov and Meyer-König and Zeller.



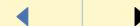
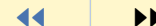
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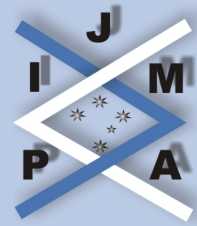
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## 1. Introduction

In this section, we recall some notions and results which we will use in this article. Let  $\mathbb{N}$  be the set of positive integers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

In the following, let  $X$  and  $Y$  be real intervals.

A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -continuous function in  $(x_0, y_0) \in X \times Y$  if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \Delta f[(x, y), (x_0, y_0)] = 0,$$

where

$$\Delta f[(x, y), (x_0, y_0)] = f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)$$

denotes a so-called mixed difference of  $f$ .

A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -continuous function on  $X \times Y$  if and only if it is  $B$ -continuous in any point of  $X \times Y$ .

A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -differentiable function in  $(x_0, y_0) \in X \times Y$  if and only if it exists and if the limit is finite

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{\Delta f[(x, y), (x, y_0)]}{(x - x_0)(y - y_0)}.$$

This limit is called the  $B$ -differential of  $f$  in the point  $(x_0, y_0)$  and is noted by  $D_B f(x_0, y_0)$ .

A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -differentiable function on  $X \times Y$  if and only if it is  $B$ -differentiable in any point of  $X \times Y$ .

The definition of  $B$ -continuity and  $B$ -differentiability was introduced by K. Bögel in the papers [8] and [9].

The function  $f : X \times Y \rightarrow \mathbb{R}$  is  $B$ -bounded on  $X \times Y$  if and only if there exists  $k > 0$  so that  $|\Delta f[(x, y), (s, t)]| \leq k$  for any  $(x, y), (s, t) \in X \times Y$ .



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We shall use the function sets  $B(X \times Y) = \{f|f : X \times Y \rightarrow \mathbb{R}, f \text{ bounded on } X \times Y\}$  with the usual sup-norm  $\|\cdot\|_\infty$ ,  $B_b(X \times Y) = \{f|f : X \times Y \rightarrow \mathbb{R}, f \text{ is } B\text{-bounded on } X \times Y\}$ ,  $C_b(X \times Y) = \{f|f : X \times Y \rightarrow \mathbb{R}, f \text{ is } B\text{-continuous on } X \times Y\}$  and  $D_b(X \times Y) = \{f|f : X \times Y \rightarrow \mathbb{R}, f \text{ is } B\text{-differentiable on } X \times Y\}$ .

Let  $f \in B_b(X \times Y)$ . The function  $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  defined by

$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\{|\Delta f[(x, y), (s, t)]| : |x - s| \leq \delta_1, |y - t| \leq \delta_2\}$$

for any  $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$  is called the mixed modulus of smoothness.

**Theorem 1.1.** *Let  $X$  and  $Y$  be compact real intervals and  $f \in B_b(X \times Y)$ . Then  $\lim_{\delta_1, \delta_2 \rightarrow 0} \omega_{\text{mixed}}(f; \delta_1, \delta_2) = 0$  if and only if  $f \in C_b(X \times Y)$ .*

For any  $x \in X$  consider the function  $\varphi_x : X \rightarrow \mathbb{R}$ , defined by  $\varphi_x(t) = |t - x|$ , for any  $t \in X$ . For additional information, see the following papers: [1], [3], [15] and [19].

Let  $m \in \mathbb{N}$  and the operator  $S_m : C_2([0, \infty)) \rightarrow C([0, \infty))$  defined for any function  $f \in C_2([0, \infty))$  by

$$(1.1) \quad (S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right),$$

for any  $x \in [0, \infty)$ , where  $C_2([0, \infty)) = \{f \in C([0, \infty)) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ exists and is finite}\}$ . The operators  $(S_m)_{m \geq 1}$  are called the Mirakjan-Favard-Szász operators, introduced in 1941 by G. M. Mirakjan in the paper [13].

These operators were intensively studied by J. Favard in 1944 in the paper [11] and O. Szász in the paper [20].

From [18], the following three lemmas result.



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**Lemma 1.2.** For any  $m \in \mathbb{N}$ , we have that

$$(1.2) \quad (S_m \varphi_x^2)(x) = \frac{x}{m},$$

$$(1.3) \quad (S_m \varphi_x^4)(x) = \frac{3mx^2 + x}{m^3}$$

for any  $x \in [0, \infty)$  and

$$(1.4) \quad (S_m \varphi_x^2)(x) \leq \frac{a}{m},$$

$$(1.5) \quad (S_m \varphi_x^4)(x) \leq \frac{a(3a + 1)}{m^2}$$

for any  $x \in [0, a]$ , where  $a > 0$ .

Let  $m \in \mathbb{N}$  and the operator  $V_m : C_2([0, \infty)) \rightarrow C([0, \infty))$ , defined for any function  $f \in C_2([0, \infty))$  by

$$(1.6) \quad (V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right)$$

for any  $x \in [0, \infty)$ .

The operators  $(V_m)_{m \geq 1}$  are called Baskakov operators, introduced in 1957 by V. A. Baskakov in the paper [5].

**Lemma 1.3.** For any  $m \in \mathbb{N}$ , we have that

$$(1.7) \quad (V_m \varphi_x^2)(x) = \frac{x(1+x)}{m},$$



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$$(1.8) \quad (V_m \varphi_x^4)(x) = \frac{3(m+2)x^4 + 6(m+2)x^3 + (3m+7)x^2 + x}{m^3}$$

for any  $x \in [0, \infty)$  and

$$(1.9) \quad (V_m \varphi_x^2)(x) \leq \frac{a(1+a)}{m},$$

$$(1.10) \quad (V_m \varphi_x^4)(x) \leq \frac{a(9a^3 + 18a^2 + 10a + 1)}{m^2}$$

for any  $x \in [0, a]$ , where  $a > 0$ .

W. Meyer-König and K. Zeller have introduced a sequence of linear positive operators in paper [12]. After a slight adjustment, given by E. W. Cheney and A. Sharma in [10], these operators take the form  $Z_m : B([0, 1]) \rightarrow C([0, 1])$ , defined for any function  $f \in B([0, 1])$  by

$$(1.11) \quad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any  $m \in \mathbb{N}$  and for any  $x \in [0, 1)$ .

These operators are called the Meyer-König and Zeller operators.

In the following we consider  $Z_m : C([0, 1]) \rightarrow C([0, 1])$ , for any  $m \in \mathbb{N}$ .

**Lemma 1.4.** For any  $m \in \mathbb{N}$  and any  $x \in [0, 1]$ , we have that

$$(1.12) \quad (Z_m \varphi_x^2)(x) \leq \frac{x(1-x)^2}{m+1} \left(1 + \frac{2x}{m+1}\right)$$



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and

$$(1.13) \quad (Z_m \varphi_x^2)(x) \leq \frac{2}{m}.$$

The inequality of Corollary 5 from [4], in the condition (1.14) becomes inequality (1.15). Inequality (1.16) is demonstrated in [16].

**Theorem 1.5.** *Let  $L : C_b(X \times Y) \rightarrow B(X \times Y)$  be a linear positive operator and  $UL : C_b(X \times Y) \rightarrow B(X \times Y)$  the associated GBS operator. Supposing that the operator  $L$  has the property*

$$(1.14) \quad (L(\cdot - x)^{2i}(* - y)^{2j})(x, y) = (L(\cdot - x)^{2i})(x, y) (L(* - y)^{2j})(x, y)$$

for any  $(x, y) \in X \times Y$  and any  $i, j \in \{1, 2\}$ , where "." and "\*" stand for the first and second variable. Then:

(i) For any function  $f \in C_b(X \times Y)$ , any  $(x, y) \in X \times Y$  and any  $\delta_1, \delta_2 > 0$ , we have that

$$(1.15) \quad |f(x, y) - (ULf)(x, y)| \leq |f(x, y)| |1 - (Le_{00})(x, y)| \\ + \left[ (Le_{00})(x, y) + \delta_1^{-1} \sqrt{(L(\cdot - x)^2)(x, y)} + \delta_2^{-1} \sqrt{(L(* - y)^2)(x, y)} \right. \\ \left. + \delta_1^{-1} \delta_2^{-1} \sqrt{(L(\cdot - x)^2)(x, y) (L(* - y)^2)(x, y)} \right] \omega_{mixed}(f; \delta_1, \delta_2).$$

(ii) For any  $f \in D_b(X \times Y)$  with  $D_B f \in B(X \times Y)$ , any  $(x, y) \in X \times Y$  and any

$\delta_1, \delta_2 > 0$ , we have that

$$\begin{aligned}
 (1.16) \quad & |f(x, y) - (ULf)(x, y)| \\
 & \leq |f(x, y)| |1 - (Le_{00})(x, y)| \\
 & \quad + 3 \|D_B f\|_\infty \sqrt{(L(\cdot - x)^2)(x, y)(L(* - y)^2)(x, y)} \\
 & \quad + \left[ \sqrt{(L(\cdot - x)^2)(x, y)(L(* - y)^2)(x, y)} \right. \\
 & \quad + \delta_1^{-1} \sqrt{(L(\cdot - x)^4)(x, y)(L(* - y)^2)(x, y)} \\
 & \quad + \delta_2^{-1} \sqrt{(L(\cdot - x)^2)(x, y)(L(* - y)^4)(x, y)} \\
 & \quad \left. + \delta_1^{-1} \delta_2^{-1} (L(\cdot - x)^2)(x, y)(L(* - y)^2)(x, y) \right] \omega_{mixed}(D_B f; \delta_1, \delta_2).
 \end{aligned}$$



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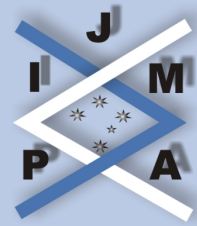
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## 2. Preliminaries

Let  $I, J, K \subset \mathbb{R}$  be intervals,  $J \subset K$  and  $I \cap J \neq \emptyset$ . We consider the sequence of nodes  $((x_{m,k})_{k \in \mathbb{N}_0})_{m \geq 1}$  so that  $x_{m,k} \in I \cap J$ ,  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$  and the functions  $\varphi_{m,k} : K \rightarrow \mathbb{R}$  with the property that  $\varphi_{m,k}(x) \geq 0$ , for any  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$  and  $x \in J$ .

**Definition 2.1.** If  $m \in \mathbb{N}$ , we define the operator  $L_m^* : E(I) \rightarrow F(K)$  by

$$(2.1) \quad (L_m^* f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) f(x_{m,k})$$

for any function  $f \in E(I)$  and any  $x \in K$ , where  $E(I)$  and  $F(K)$  are subsets of the set of real functions defined on  $I$ , respectively on  $K$ .

**Proposition 2.2.** The operators  $(L_m^*)_{m \geq 1}$  are linear and positive on  $E(I \cap J)$ .

*Proof.* The proof follows immediately.  $\square$

**Definition 2.3.** If  $m, n \in \mathbb{N}$ , the operator  $L_{m,n}^* : E(I \times I) \rightarrow F(K \times K)$  defined for any function  $f \in E(I \times I)$  and any  $(x, y) \in K \times K$  by

$$(2.2) \quad (L_{m,n}^* f)(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{m,k}(x) \varphi_{n,j}(y) f(x_{m,k}, x_{n,j})$$

is called the bivariate operator of  $L^*$  - type.

**Proposition 2.4.** The operators  $(L_{m,n}^*)_{m,n \geq 1}$  are linear and positive on  $E[(I \times I) \cap (J \times J)]$ .

*Proof.* The proof follows immediately.  $\square$



**Definition 2.5.** If  $m, n \in \mathbb{N}$ , the operator  $UL_{m,n}^* : E(I \times I) \rightarrow F(K \times K)$  defined for any function  $f \in E(I \times I)$  and any  $(x, y) \in K \times K$  by

$$(2.3) \quad (UL_{m,n}^* f)(x, y) \\ = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{m,k}(x) \varphi_{n,j}(y) [f(x_{m,k}, y) + f(x, x_{n,j}) - f(x_{m,k}, x_{n,j})]$$

is called a GBS operator of  $L^*$  - type.

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### 3. Main Results

**Lemma 3.1.** For any  $m, n \in \mathbb{N}$ ,  $i, j \in \mathbb{N}_0$  and  $(x, y) \in K \times K$ , the identity

$$(3.1) \quad (L_{m,n}^*(\cdot - x)^{2i}(* - y)^{2j})(x, y) = (L_m^*(\cdot - x)^{2i})(x) (L_n^*(\cdot - y)^{2j})(y)$$

holds.

*Proof.* We have that

$$\begin{aligned} (L_{m,n}^*(\cdot - x)^{2i}(* - y)^{2j})(x, y) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{m,k}(x) \varphi_{n,j}(y) (x_{m,k} - x)^{2i} (x_{n,j} - y)^{2j} \\ &= \sum_{k=0}^{\infty} \varphi_{m,k}(x) (x_{m,k} - x)^{2i} \sum_{j=0}^{\infty} \varphi_{n,j}(y) (x_{n,j} - y)^{2j} \\ &= (L_m^*(\cdot - x)^{2i})(x) (L_n^*(\cdot - y)^{2j})(y), \end{aligned}$$

so (3.1) holds. □

For the operators constructed in this section, we note that  $\delta_m(x) = \sqrt{(L_m^* \varphi_x^2)(x)}$ ,  $\delta_{m,x} = \sqrt{(L_m^* \varphi_x^4)(x)}$ , where  $x \in I \cap J$ ,  $m \in \mathbb{N}$ ,  $m \neq 0$ .

Then, by taking Lemma 3.1 into account, Theorem 1.5 becomes:

#### Theorem 3.2.

(i) For any function  $f \in C_b(I \times I)$ , any  $(x, y) \in (I \times I) \cap (J \times J)$ , any  $m, n \in \mathbb{N}$ , any  $\delta_1, \delta_2 > 0$ , we have that

$$\begin{aligned} (3.2) \quad &|f(x, y) - (UL_{m,n}^* f)(x, y)| \\ &\leq |f(x, y)| |1 - (Le_{00})(x, y)| + ((Le_{00})(x, y) + \delta_1^{-1} \delta_m(x) + \delta_2^{-1} \delta_n(y) \\ &\quad + \delta_1^{-1} \delta_2^{-1} \delta_m(x) \delta_n(y)) \omega_{mixed}(f; \delta_1, \delta_2). \end{aligned}$$



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(ii) For any function  $f \in D_b(I \times I)$  with  $D_B f \in B(I \times I)$ , any  $(x, y) \in (I \times I) \cap (J \times J)$ , any  $m, n \in \mathbb{N}$ , any  $\delta_1, \delta_2 > 0$ , we have that

$$(3.3) \quad |f(x, y) - (UL^* f)(x, y)| \leq |f(x, y)| |1 - (Le_{00})(x, y)| \\ + 3 \|D_B f\|_\infty \delta_m(x) \delta_n(y) + [\delta_m(x) \delta_n(y) + \delta_1^{-1} \delta_{m,x} \delta_n(y) \\ + \delta_2^{-1} \delta_m(x) \delta_{n,y} + \delta_1^{-1} \delta_2^{-1} \delta_m^2(x) \delta_n^2(y)] \omega_{mixed}(D_B f; \delta_1, \delta_2).$$

In the following, we give examples of operators and of the associated GBS operators.

**Application 1.** If  $I = J = K = [0, \infty)$ ,  $E(I) = C_2([0, \infty))$ ,  $F(K) = C([0, \infty))$ ,  $\varphi_{m,k}(x) = e^{-mx} \frac{(mx)^k}{k!}$ ,  $x_{m,k} = \frac{k}{m}$ ,  $x \in [0, \infty)$ ,  $m, k \in \mathbb{N}_0$ ,  $m \neq 0$ , then we obtain the Mirakjan-Favard-Szász operators.

**Theorem 3.3.** Let  $a, b \in \mathbb{R}$ ,  $a > 0$  and  $b > 0$ . Then:

(i) For any function  $f \in C([0, \infty) \times [0, \infty))$ , any  $(x, y) \in [0, a] \times [0, b]$  and  $m, n \in \mathbb{N}$ , we have that

$$(3.4) \quad |f(x, y) - (US_{m,n} f)(x, y)| \\ \leq (1 + \sqrt{a}) (1 + \sqrt{b}) \omega_{mixed} \left( f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right).$$

(ii) For any function  $f \in D_b([0, \infty) \times [0, \infty)) \cap C([0, \infty) \times [0, \infty))$  with  $D_B f \in$



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$B([0, a] \times [0, b])$ , any  $(x, y) \in [0, a] \times [0, b]$ , any  $m, n \in \mathbb{N}$ , we have that

$$(3.5) \quad |f(x, y) - (US_{m,n}f)(x, y)| \leq \sqrt{ab} \left[ 3\|D_B f\|_\infty + \left(1 + \sqrt{3a+1} + \sqrt{3b+1} + \sqrt{ab}\right) \omega_{mixed} \left( D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right) \right] \frac{1}{\sqrt{mn}}.$$

*Proof.* It results from Theorem 3.2, by choosing  $\delta_1 = \frac{1}{\sqrt{m}}$ ,  $\delta_2 = \frac{1}{\sqrt{n}}$  and Lemma 1.2. □

**Theorem 3.4.** If  $f \in C([0, \infty) \times [0, \infty))$ , then the convergence

$$(3.6) \quad \lim_{m,n \rightarrow \infty} (US_{m,n}f)(x, y) = f(x, y)$$

is uniform on any compact  $[0, a] \times [0, b]$ , where  $a, b > 0$ .

*Proof.* It results from Theorem 1.1 and Theorem 3.3. □

**Application 2.** If  $I = J = K = [0, \infty)$ ,  $E(I) = C_2([0, \infty))$ ,  $F(K) = C([0, \infty))$ ,  $\varphi_{m,k}(x) = (1+x)^{-m} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k$ ,  $x_{m,k} = \frac{k}{m}$ ,  $x \in [0, \infty)$ ,  $m, k \in \mathbb{N}_0$ ,  $m \neq 0$ , then we obtain the Baskakov operators.

**Theorem 3.5.** Let  $a, b \in \mathbb{R}$ ,  $a > 0$  and  $b > 0$ . Then:

(i) For any function  $f \in C([0, \infty) \times [0, \infty))$ , any  $(x, y) \in [0, a] \times [0, b]$  and any  $m, n \in \mathbb{N}$ , we have that

$$(3.7) \quad |f(x, y) - (UV_{m,n}f)(x, y)| \leq \left(1 + \sqrt{a(1+a)}\right) \left(1 + \sqrt{b(1+b)}\right) \omega_{mixed} \left( f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right).$$



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(ii) For any function  $f \in D_b([0, \infty) \times [0, \infty)) \cap C([0, \infty) \times [0, \infty))$  with  $D_B f \in B([0, a] \times [0, b])$ , any  $(x, y) \in [0, a] \times [0, b]$ , any  $m, n \in \mathbb{N}$ , we have that

$$(3.8) \quad |f(x, y) - (UV_{m,n}f)(x, y)| \leq \sqrt{ab(1+a)(1+b)} \left\{ 3\|D_B\|_\infty \right. \\ \left. + \left[ 1 + \sqrt{9a^3 + 18a^2 + 10a + 1} + \sqrt{9b^3 + 18b^2 + 10b + 1} \right. \right. \\ \left. \left. + \sqrt{ab(1+a)(1+b)} \right] \omega_{mixed} \left( D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right) \right\} \frac{1}{\sqrt{mn}}.$$

*Proof.* It results from Theorem 3.2, by choosing  $\delta_1 = \frac{1}{\sqrt{m}}$ ,  $\delta_2 = \frac{1}{\sqrt{n}}$  and Lemma 1.3. □

**Theorem 3.6.** If  $f \in C([0, \infty) \times [0, \infty))$ , then the convergence

$$(3.9) \quad \lim_{m,n \rightarrow \infty} (UV_{m,n}f)(x, y) = f(x, y)$$

is uniform on any compact  $[0, a] \times [0, b]$ , where  $a, b > 0$ .

*Proof.* It results from Theorem 1.1 and Theorem 3.5. □

**Application 3.** If  $I = J = K = [0, 1]$ ,  $E(I) = F(K) = C([0, 1])$ ,  $\varphi_{m,k}(x) = \binom{m+k}{k} (1-x)^{m+1} x^k$ ,  $x_{m,k} = \frac{k}{m}$ ,  $x \in [0, 1]$ ,  $m, k \in \mathbb{N}_0$ ,  $m \neq 0$ , then we obtain the Meyer-König and Zeller operators.

**Theorem 3.7.** For any function  $f \in C([0, 1] \times [0, 1])$ , any  $(x, y) \in [0, 1] \times [0, 1]$  and any  $m, n \in \mathbb{N}$ , we have that

$$(3.10) \quad |f(x, y) - (UZ_{m,n}f)(x, y)| \leq (3 + 2\sqrt{2}) \omega_{mixed} \left( f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right).$$



*Proof.* It results from Theorem 3.2, by choosing  $\delta_1 = \frac{1}{\sqrt{m}}$ ,  $\delta_2 = \frac{1}{\sqrt{n}}$  and Lemma 1.4. □

**Theorem 3.8.** *If  $f \in C([0, 1] \times [0, 1])$ , then the convergence*

$$(3.11) \quad \lim_{m,n \rightarrow \infty} (UZ_{m,n}f)(x, y) = f(x, y)$$

*is uniform on  $[0, 1] \times [0, 1]$ .*

*Proof.* It results from Theorem 1.1 and Theorem 3.7. □

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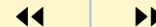
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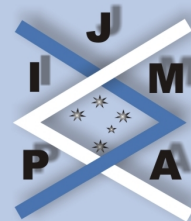
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