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A NOTE ON SUMS OF POWERS WHICH HAVE A FIXED NUMBER OF PRIME FACTORS

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ABSTRACT. Let us denote by $c_{n,k}$ the sequence of numbers which have in its factorization k prime factors ($k \geq 1$), we obtain in short proofs asymptotic formulas for $c_{n,k}$, $\sum_{i=1}^{n} c_{i,k}^{\alpha}$ and $\sum_{c_{i,k} \leq x} c_{i,k}^{\alpha}$. We generalize the work by T. Sálat y S. Znam when $k = 1$ (see reference [2]).

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Let $\pi_k(x)$ be the number of these numbers not exceeding x, it was proved by Landau [1] that

$$\lim_{x\to\infty} \frac{\pi_k(x)}{\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}} = 1 \tag{1}$$

Note that if $k = 1$ then $\pi_1(x) = \pi(x)$, $c_{n,1} = p_n$, and equation (1) is the prime number theorem.

Theorem 1. *The following asymptotic formula holds:*

$$c_{n,k} \sim \frac{(k-1)!n\log n}{(\log\log n)^{k-1}}. \tag{2}$$

Proof. If $k = 1$ the formula is true, since in this case (2) is the prime number theorem $p_n \sim n\log n$. Suppose $k \geq 2$. If we put $x = c_{n,k}$ and substitute into (1) we find that

$$\lim_{n\to\infty} \frac{(k-1)!n\log c_{n,k}}{c_{n,k}\ (\log\log c_{n,k})^{k-1}} = 1. \tag{3}$$

Writing

$$c_{n,k} = \frac{(k-1)!n\log n}{(\log\log n)^{k-1}} f(n) \tag{4}$$

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and substituting (4) into (3) we obtain

$$\lim_{n\to\infty} \frac{\log c_{n,k} \left(\log\log n\right)^{k-1}}{\log n\, f(n) \left(\log\log c_{n,k}\right)^{k-1}} = 1. \tag{5}$$

From equation (1) we find that

$$\lim_{x\to\infty} \frac{\pi_k(x)}{\pi(x)} = \infty. \tag{6}$$

Assume that the inequalities $c_{n,k} \geq p_n$ have infinitely many solutions, then we have $\pi(c_{n,k}) \geq \pi(p_n) = n = \pi_k(c_{n,k})$, which contradicts (6). Hence for all sufficiently large n we have $c_{n,k} < p_n$. On the other hand, clearly $n \leq c_{n,k}$. Therefore $n \leq c_{n,k} \leq p_n$, that is $\log n \leq \log c_{n,k} \leq \log p_n$, and we find that

$$1 \leq \frac{\log c_{n,k}}{\log n} \leq \frac{\log p_n}{\log n}. \tag{7}$$

From (7) and the prime number theorem $p_n \sim n \log n$, we obtain

$$\lim_{n\to\infty} \frac{\log c_{n,k}}{\log n} = 1. \tag{8}$$

From (5) and (8) we find that

$$\lim_{n\to\infty} f(n) = 1. \tag{9}$$

To finish, (9) and (4) give (2). The theorem is thus proved. □

The following proposition is well known, we use it as a lemma

Lemma 2. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Then if $\sum_{i=1}^{\infty} b_i$ is divergent, the following limit holds*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = 1.$$

Theorem 3. *Let $k \geq 1$ and let α be a positive number. The following asymptotic formula holds*

$$\sum_{i=1}^{n} c_{i,k}^{\alpha} \sim \frac{\left((k-1)!\right)^{\alpha} n^{\alpha+1} \log^{\alpha} n}{(\alpha+1)\ \left(\log\log n\right)^{\alpha(k-1)}}. \tag{10}$$

Proof. Let us consider the following two series:

$$\sum_{i=1}^{\infty} c_{i,k}^{\alpha} \qquad \text{and} \qquad 1 + 2 + \sum_{i=3}^{\infty} \left(\frac{(k-1)!\ i \log i}{\left(\log\log i\right)^{k-1}} \right)^{\alpha}.$$

Since the function $\left(\frac{(k-1)!\ t \log t}{(\log\log t)^{k-1}} \right)^{\alpha}$ is increasing from a certain value of t, we find that

$$\begin{aligned} 1 + 2 + \sum_{i=3}^{n} \left(\frac{(k-1)!\, i \log i}{\left(\log\log i\right)^{k-1}} \right)^{\alpha} \\ = \int_3^n \left(\frac{(k-1)!\, t \log t}{\left(\log\log t\right)^{k-1}} \right)^{\alpha} dt + O\left(\left(\frac{n \log n}{\left(\log\log n\right)^{k-1}} \right)^{\alpha} \right). \end{aligned} \tag{11}$$

On the other hand, from the L'Hospital rule

$$(12)\qquad \int_3^n \left(\frac{(k-1)!\,t\log t}{(\log\log t)^{k-1}}\right)^{\alpha} dt \sim \frac{((k-1)!)^{\alpha}\, n^{\alpha+1}\log^{\alpha} n}{(\alpha\ +1)\ (\log\log n)^{\alpha\,(k-1)}}.$$

Equation (10) is an immediate consequence of (11), (12) and the lemma.

The theorem is thus proved. □

Theorem 4. *Let $k \geq 1$ and let α be a positive number. The following asymptotic formula holds*

$$(13)\qquad \sum_{c_{i,k}\,\leq\, x} c_{i,k}^{\alpha} \sim \frac{x^{\alpha+1}(\log\log x)^{k-1}}{(\alpha+1)\,(k-1)!\,\log x}.$$

Proof. Equation (3) can be written in the form

$$(14)\qquad \lim_{n\to\infty} \frac{n}{\frac{c_{n,k}\left(\log\log c_{n,k}\right)^{k-1}}{(k-1)!\,\log c_{n,k}}} = 1.$$

From (8) we obtain

$$(15)\qquad \lim_{n\to\infty} \frac{\log\log c_{n,k}}{\log\log n} = 1.$$

Substituting (14), (8) and (15) into (10) we find that

$$(16)\qquad \sum_{c_{i,k}\,\leq\, c_{n,k}} c_{i,k}^{\alpha} \sim \frac{c_{n,k}^{\alpha+1}(\log\log c_{n,k})^{k-1}}{(\alpha+1)\,(k-1)!\,\log c_{n,k}}.$$

Equation (2) gives $c_{n,k} \sim c_{n+1,k}$, therefore

$$(17)\qquad \sum_{c_{i,k}\,\leq\, c_{n,k}} c_{i,k}^{\alpha} \sim \frac{c_{n+1,k}^{\alpha+1}(\log\log c_{n+1,k})^{k-1}}{(\alpha+1)\,(k-1)!\ \log c_{n+1,k}} \sim \frac{c_{n,k}^{\alpha+1}(\log\log c_{n,k})^{k-1}}{(\alpha+1)\,(k-1)!\,\log c_{n,k}}.$$

Since the function

$$\frac{x^{\alpha+1}(\log\log x)^{k-1}}{(\alpha+1)\,(k-1)!\ \log x}$$

is increasing from a certain value of x, we have for all n sufficiently large

$$(18)\qquad \frac{\sum\limits_{c_{i,k}\,\leq\, c_{n,k}} c_{i,k}^{\alpha}}{\frac{c_{n,k}^{\alpha+1}(\log\log c_{n,k})^{k-1}}{(\alpha+1)\,(k-1)!\ \log c_{n,k}}} \leq \frac{\sum\limits_{c_{i,k}\,\leq\, x} c_{i,k}^{\alpha}}{\frac{x^{\alpha+1}(\log\log x)^{k-1}}{(\alpha+1)\,(k-1)!\ \log x}} \leqslant \frac{\sum\limits_{c_{i,k}\,\leq\, c_{n,k}} c_{i,k}^{\alpha}}{\frac{c_{n+1,k}^{\alpha+1}(\log\log c_{n+1,k})^{k-1}}{(\alpha+1)\,(k-1)!\ \log c_{n+1,k}}},$$

where $c_{n,k} \leq x < c_{n+1,k}$.

To finish, (17) and (18) give (13). The theorem is proved. □

Note. The case $k = 1$ was studied in the reference [2]. In this case (9) and (13) become

$$\sum_{i=1}^{n} p_i^{\alpha} \sim \frac{n^{\alpha+1}\log^{\alpha} n}{(\alpha+1)}, \qquad \sum_{p_i \leq x} p_i^{\alpha} \sim \frac{x^{\alpha+1}}{(\alpha+1)\log x}.$$

Using equation (2) and the lemma, we can prove (as above) other theorems, for example the following:

Theorem 5. *The following asymptotic formulas holds*

$$\sum_{n=1}^{\infty} \frac{1}{c_{n,k}} \sim \frac{(\log\log n)^k}{k\,!} \qquad \textit{and} \qquad \sum_{c_{n,k} \le x} \frac{1}{c_{n,k}} \sim \frac{(\log\log x)^k}{k\,!}.$$

When $k = 1$, this theorem is well known.

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