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A NOTE ON SUMS OF POWERS WHICH HAVE A FIXED NUMBER OF PRIME FACTORS

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Abstract

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Abstract

Let us denote by $c_{n,k}$ the sequence of numbers which have in its factorization k prime factors ($k \geq 1$), we obtain in short proofs asymptotic formulas for $c_{n,k}$, $\sum_{i=1}^n c_{i,k}^\alpha$ and $\sum_{c_{i,k} \leq x} c_{i,k}^\alpha$. We generalize the work by T. Sálat y S. Znam when $k = 1$ (see reference [2]).

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Key words: Sums of powers, Numbers with k prime factors.

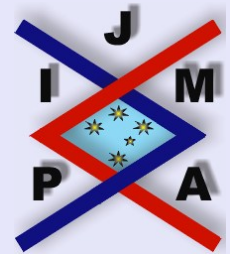
Let $\pi_k(x)$ be the number of these numbers not exceeding x , it was proved by Landau [1] that

$$(1) \quad \lim_{x \rightarrow \infty} \frac{\pi_k(x)}{\frac{x(\log \log x)^{k-1}}{(k-1)! \log x}} = 1$$

Note that if $k = 1$ then $\pi_1(x) = \pi(x)$, $c_{n,1} = p_n$, and equation (1) is the prime number theorem.

Theorem 1. *The following asymptotic formula holds:*

$$(2) \quad c_{n,k} \sim \frac{(k-1)! n \log n}{(\log \log n)^{k-1}}.$$



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Proof. If $k = 1$ the formula is true, since in this case (2) is the prime number theorem $p_n \sim n \log n$. Suppose $k \geq 2$. If we put $x = c_{n,k}$ and substitute into (1) we find that

$$(3) \quad \lim_{n \rightarrow \infty} \frac{(k-1)! n \log c_{n,k}}{c_{n,k} (\log \log c_{n,k})^{k-1}} = 1.$$

Writing

$$(4) \quad c_{n,k} = \frac{(k-1)! n \log n}{(\log \log n)^{k-1}} f(n)$$

and substituting (4) into (3) we obtain

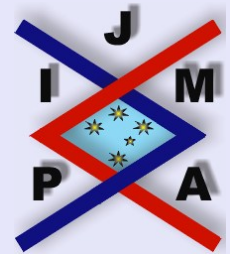
$$(5) \quad \lim_{n \rightarrow \infty} \frac{\log c_{n,k} (\log \log n)^{k-1}}{\log n f(n) (\log \log c_{n,k})^{k-1}} = 1.$$

From equation (1) we find that

$$(6) \quad \lim_{x \rightarrow \infty} \frac{\pi_k(x)}{\pi(x)} = \infty.$$

Assume that the inequalities $c_{n,k} \geq p_n$ have infinitely many solutions, then we have $\pi(c_{n,k}) \geq \pi(p_n) = n = \pi_k(c_{n,k})$, which contradicts (6). Hence for all sufficiently large n we have $c_{n,k} < p_n$. On the other hand, clearly $n \leq c_{n,k}$. Therefore $n \leq c_{n,k} \leq p_n$, that is $\log n \leq \log c_{n,k} \leq \log p_n$, and we find that

$$(7) \quad 1 \leq \frac{\log c_{n,k}}{\log n} \leq \frac{\log p_n}{\log n}.$$



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From (7) and the prime number theorem $p_n \sim n \log n$, we obtain

$$(8) \quad \lim_{n \rightarrow \infty} \frac{\log c_{n,k}}{\log n} = 1.$$

From (5) and (8) we find that

$$(9) \quad \lim_{n \rightarrow \infty} f(n) = 1.$$

To finish, (9) and (4) give (2). The theorem is thus proved. \square

The following proposition is well known, we use it as a lemma

Lemma 2. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$. Then if $\sum_{i=1}^{\infty} b_i$ is divergent, the following limit holds*

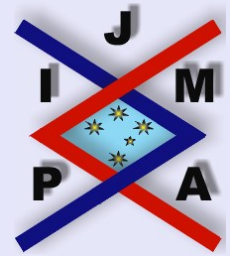
$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} = 1.$$

Theorem 3. *Let $k \geq 1$ and let α be a positive number. The following asymptotic formula holds*

$$(10) \quad \sum_{i=1}^n c_{i,k}^{\alpha} \sim \frac{((k-1)!)^{\alpha} n^{\alpha+1} \log^{\alpha} n}{(\alpha+1) (\log \log n)^{\alpha(k-1)}}.$$

Proof. Let us consider the following two series:

$$\sum_{i=1}^{\infty} c_{i,k}^{\alpha} \quad \text{and} \quad 1 + 2 + \sum_{i=3}^{\infty} \left(\frac{(k-1)! i \log i}{(\log \log i)^{k-1}} \right)^{\alpha}.$$



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Since the function $\left(\frac{(k-1)! t \log t}{(\log \log t)^{k-1}}\right)^\alpha$ is increasing from a certain value of t , we find that

$$(11) \quad 1 + 2 + \sum_{i=3}^n \left(\frac{(k-1)! i \log i}{(\log \log i)^{k-1}}\right)^\alpha = \int_3^n \left(\frac{(k-1)! t \log t}{(\log \log t)^{k-1}}\right)^\alpha dt + O\left(\left(\frac{n \log n}{(\log \log n)^{k-1}}\right)^\alpha\right).$$

On the other hand, from the L'Hospital rule

$$(12) \quad \int_3^n \left(\frac{(k-1)! t \log t}{(\log \log t)^{k-1}}\right)^\alpha dt \sim \frac{((k-1)!)^\alpha n^{\alpha+1} \log^\alpha n}{(\alpha+1) (\log \log n)^{\alpha(k-1)}}.$$

Equation (10) is an immediate consequence of (11), (12) and the lemma.

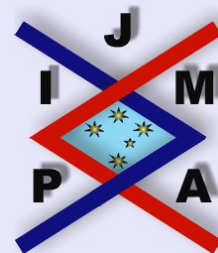
The theorem is thus proved. \square

Theorem 4. Let $k \geq 1$ and let α be a positive number. The following asymptotic formula holds

$$(13) \quad \sum_{c_{i,k} \leq x} c_{i,k}^\alpha \sim \frac{x^{\alpha+1} (\log \log x)^{k-1}}{(\alpha+1) (k-1)! \log x}.$$

Proof. Equation (3) can be written in the form

$$(14) \quad \lim_{n \rightarrow \infty} \frac{n}{\frac{c_{n,k} (\log \log c_{n,k})^{k-1}}{(k-1)! \log c_{n,k}}} = 1.$$



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From (8) we obtain

$$(15) \quad \lim_{n \rightarrow \infty} \frac{\log \log c_{n,k}}{\log \log n} = 1.$$

Substituting (14), (8) and (15) into (10) we find that

$$(16) \quad \sum_{c_{i,k} \leq c_{n,k}} c_{i,k}^\alpha \sim \frac{c_{n,k}^{\alpha+1} (\log \log c_{n,k})^{k-1}}{(\alpha+1)(k-1)! \log c_{n,k}}.$$

Equation (2) gives $c_{n,k} \sim c_{n+1,k}$, therefore

$$(17) \quad \sum_{c_{i,k} \leq c_{n,k}} c_{i,k}^\alpha \sim \frac{c_{n+1,k}^{\alpha+1} (\log \log c_{n+1,k})^{k-1}}{(\alpha+1)(k-1)! \log c_{n+1,k}} \\ \sim \frac{c_{n,k}^{\alpha+1} (\log \log c_{n,k})^{k-1}}{(\alpha+1)(k-1)! \log c_{n,k}}.$$

Since the function

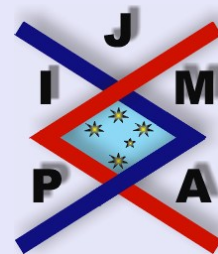
$$\frac{x^{\alpha+1} (\log \log x)^{k-1}}{(\alpha+1)(k-1)! \log x}$$

is increasing from a certain value of x , we have for all n sufficiently large

$$(18) \quad \frac{\sum_{c_{i,k} \leq c_{n,k}} c_{i,k}^\alpha}{\frac{c_{n,k}^{\alpha+1} (\log \log c_{n,k})^{k-1}}{(\alpha+1)(k-1)! \log c_{n,k}}} \leq \frac{\sum_{c_{i,k} \leq x} c_{i,k}^\alpha}{\frac{x^{\alpha+1} (\log \log x)^{k-1}}{(\alpha+1)(k-1)! \log x}} \leq \frac{\sum_{c_{i,k} \leq c_{n,k}} c_{i,k}^\alpha}{\frac{c_{n+1,k}^{\alpha+1} (\log \log c_{n+1,k})^{k-1}}{(\alpha+1)(k-1)! \log c_{n+1,k}}},$$

where $c_{n,k} \leq x < c_{n+1,k}$.

To finish, (17) and (18) give (13). The theorem is proved. \square



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Note. The case $k = 1$ was studied in the reference [2]. In this case (9) and (13) become

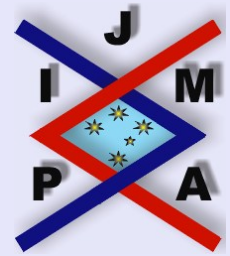
$$\sum_{i=1}^n p_i^\alpha \sim \frac{n^{\alpha+1} \log^\alpha n}{(\alpha + 1)}, \quad \sum_{p_i \leq x} p_i^\alpha \sim \frac{x^{\alpha+1}}{(\alpha + 1) \log x}.$$

Using equation (2) and the lemma, we can prove (as above) other theorems, for example the following:

Theorem 5. *The following asymptotic formulas holds*

$$\sum_{n=1}^{\infty} \frac{1}{c_{n,k}} \sim \frac{(\log \log n)^k}{k!} \quad \text{and} \quad \sum_{c_{n,k} \leq x} \frac{1}{c_{n,k}} \sim \frac{(\log \log x)^k}{k!}.$$

When $k = 1$, this theorem is well known.



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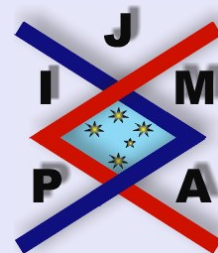
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