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A NOTE ON SÁNDOR TYPE FUNCTIONS

N. ANITHA

Department of Studies in Mathematics
University of Mysore
Manasagangotri
Mysore 570006, India.

EMail: ani_lohith@yahoo.com

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Abstract

In this paper we introduce the functions G and G_* similar to Sándor's functions which are defined by,

$$G(x) = \min\{m \in \mathbb{N} : x \le e^m\}, \quad x \in [1, \infty),$$

$$G_*(x) = \max\{m \in \mathbb{N} : e^m \le x\}, \quad x \in [e, \infty).$$

We study some interesting properties of G and G_*. The main purpose of this paper is to show that

$$\pi(x) \sim \frac{x}{G_*(x)}$$

where $\pi(x)$ is the number of primes less than or equal to x.

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1. Introduction

In his paper [1], J. Sándor discussed many interesting properties of the functions S and S_* defined by,

$$S(x) = \min\{m \in \mathbb{N} : x \leq m!\}, \quad x \in (1, \infty),$$

and

$$S_*(x) = \max\{m \in \mathbb{N} \colon m! \leq x\}, \quad x \in [1, \infty).$$

He also proved the following theorems:

Theorem 1.1.

$$S_*(x) \sim \frac{\log x}{\log \log x} \quad (x \to \infty).$$

Theorem 1.2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n[S_*(n)]^{\alpha}}$$

is convergent for $\alpha > 1$ *and divergent for* $\alpha \leq 1$.

Now we will define functions $G(x)$ and $G_*(x)$ and discuss their properties. The functions are defined as follows:

$$G(x) = \min\{m \in \mathbb{N} : x \leq e^m\}, \quad x \in [1, \infty),$$

$$G_*(x) = \max\{m \in \mathbb{N} : e^m \leq x\}, \quad x \in [e, \infty).$$

Clearly,

$$G(x) = m + 1, \quad \text{if} \quad x \in [e^m, e^{m+1}) \quad \text{for} \quad m \geq 0.$$

Similarly,

$$G_*(x) = m, \quad \text{if} \quad x \in [e^m, e^{m+1}) \quad \text{for} \quad m \geq 1.$$

It is immediate that

$$G(x) = \begin{cases} G_*(x) + 1, & \text{if} \quad x \in [e^k, e^{k+1}) \quad (k \geq 1) \\ G_*(x), & \text{if} \quad x = e^{k+1} \quad (k \geq 1). \end{cases}$$

Therefore,

$$G_*(x) + 1 \geq G(x) \geq G_*(x).$$

It can be easily verified that the function $G_*(x)$ satisfies the following properties:

1. $G_*(x)$ is surjective and an increasing function.
2. $G_*(x)$ is continuous for all $x \in (e, \infty) \setminus A$, where $A = e^k, k \geq 1$ and since $\lim_{x \nearrow e^k} G_*(x) = k$, $\lim_{x \searrow e^k} G_*(x) = k - 1$ for $k \geq 1$, $G_*(x)$ is continuous from the right at $x = e^k$ $(k \geq 1)$, but it is not continuous from the left.
3. $G_*(x)$ is differentiable on $[e, \infty) \setminus A$, and since

$$\lim_{x \searrow e^k} \frac{G_*(x) - G_*(e^k)}{x - (e^k)} = 0,$$

it has a right derivative at e^k.

4. $G_*(x)$ is Reimann integrable over $[a,b] \subset \mathbb{R}$ for all $a \leq b$. Also

$$\int_{e^k}^{e^l} G_*(x)dx = (e-1)\sum_{m=1}^{l-k}(e^k+m-1)(k+m-1).$$

2. Main Result

The main purpose of this paper is to prove the following theorem:

Theorem 2.1.

$$\pi(x) \sim \frac{x}{G_*(x)}.$$

Proof. To prove our theorem first we will prove that

$$G_*(x) \sim \log x. \tag{2.1}$$

By Stiriling's formula [2] we have

$$n! \sim ce^{-n}n^{n+1/2}$$

i.e.,

$$e^n \sim \frac{cn^{n+1/2}}{n!}$$

Thus,

$$\log e^n \sim \log\left(\frac{cn^{n+1/2}}{n!}\right)$$

and hence,

$$n \sim n + \frac{1}{2}\log n + \log c - \log n!.$$

Also we have,

$$\log(n!) \sim n\log n \Rightarrow n \sim \log n \qquad \text{(cf. [1], Lemma 2).}$$

If $x \geq e$ then $x \in [e^n, e^{n+1})$ for some $n \geq 1$.

Since $G_*(x) = n$ if $x \in [e^n, e^{n+1}),\ n \geq 1$, we have

$$\frac{n}{n+1} \leq \frac{G_*(x)}{\log x} \leq \frac{n}{n}.$$

As

$$\lim_{n\to\infty} \frac{n}{n+1} = 1,$$

we have

$$G_*(x) \sim \log x.$$

From the prime number theorem it follows that

$$\pi(x) \sim \frac{x}{G_*(x)}.$$

$\square$

3. Remark

The following table compares the values of $\pi(x)$ and $\frac{x}{G_*(x)}$:

x	$\pi(x)$	$\frac{x}{G_*(x)}$
10	5	3.3333
100	26	20.00000
1000	169	142.857143
10000	1230	1000
100000	9593	8333.3333
1000000	78499	71428.571429
10000000	664580	588235.294118

Now we prove the following theorem which is similar to Theorem 1.2.

Theorem 3.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n[G_*(n)]^{\alpha}}$$

is convergent for $\alpha > 1$ *and divergent for* $\alpha \leq 1$.

Proof. By (2.1) we have

$$A \log n \leq G_*(n) \leq B \log n$$

where $(A, B \geq 0)$ for $n \geq 1$.

Therefore it is sufficient to study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{\alpha}}.$$

To study the convergence of the above series we use the following result:

If $\phi(x)$ is positive for all positive 'x' and if

$$\lim_{x\to\infty} \phi(x) = 0$$

then the two infinite series

$$\sum_{n=1}^{\infty} \phi(n) \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \phi(a^n)$$

behave alike for any positive integer 'a'.

Therefore the two series

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{\alpha}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a^n}{(a^n)[\log(a^n)]^{\alpha}}$$

behave alike.

However, the second series converges for $\alpha > 1$ and diverges for $\alpha \leq 1$. Hence the theorem is proved. $\square$

References

[1] J. SÁNDOR, On an additive analogue of the function S, *Notes Numb. Th. Discr. Math.*, **7**(2) (2001), 91–95.

[2] W. RUDIN, *Principles of Mathematical Analysis*, Third ed., Mc Graw-Hill Co., Japan, 1976.