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MEROMORPHIC FUNCTIONS SHARING THREE VALUES WITH WEIGHTS

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ABSTRACT. Using the idea of weighted sharing of values, we prove some uniqueness theorems for meromorphic functions which improve some existing results. Moreover, examples are provided to show that some results in this paper are sharp.

Key words and phrases: Meromorphic functions, Weighted sharing of values, Uniqueness.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper, a meromorphic function means meromorphic in the complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as explained in [3]. We denote by E (respectively, I) a set of finite (respectively, infinite) linear measure, not necessarily the same at each occurence. For any nonconstant meromorphic function $f(z)$, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly for a set E of r of finite linear measure. Let k be a positive integer. We denote by $N_{k)}(r, 1/(f-a))$ the counting function of the zeros of $f-a$ with multiplicity $\leq k$, by $N_{(k}(r, 1/(f-a))$ the counting function of the zeros of $f-a$ with multiplicity $\geq k$, and by $\overline{N}_{k)}(r, 1/(f-a))$ and $\overline{N}_{(k}(r, 1/(f-a))$ the reduced form of $N_{k)}(r, 1/(f-a))$ and $N_{(k}(r, 1/(f-a))$, respectively (see [19]).

Let f and g be two nonconstant meromorphic functions. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. For a complex number a, if the zeros of $f-a$ and $g-a$ coincide in locations and multiplicities, we say that f and g share the value a CM (counting multiplicities)

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and if we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities) (see [2]).

Nevanlinna [10], Ozawa [11], Ueda [12, 13], Brosch [1], Yi [14] – [18], Li [9], Zhang [20], Lahiri [4] – [8], and other authors (see [19]) dealt with the problems of uniqueness of meromorphic functions that share three distinct values. Without loss of generality, we may assume that 0, 1, ∞ are the shared values.

In 1976, Ozawa [11] proved the following result.

Theorem A ([11]). *Let f and g be two entire functions of finite order such that f and g share 0, 1 CM. If $\delta(0, f) > 1/2$, then either $f \equiv g$ or $f \cdot g \equiv 1$.*

In 1983, removing the order restriction in the above result Ueda [12] proved the following theorem.

Theorem B ([12]). *Let f and g be two meromorphic functions sharing 0, 1, and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{N(r, f) + N(r, 1/f)}{T(r, f)} < \frac{1}{2},$$

then either $f \equiv g$ or $f \cdot g \equiv 1$.

In 1998, Yi [17] proved the following theorem, which is an improvement of Theorems A and B.

Theorem C ([17]). *Let f and g be two meromorphic functions sharing 0, 1, and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r, f) + \overline{N}_{1)}(r, 1/f) - (1/2)m(r, 1/(g-1))}{T(r, f)} < \frac{1}{2}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

We now explain the notion of weighted sharing as introduced in [4].

Definition 1.1 ([4]). Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\le k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2001, Lahiri [4] proved the following theorems.

Theorem D ([4]). *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(\infty, 0)$, and $(1, \infty)$. If*

$$\overline{N}_{1)}\left(r, \frac{1}{f}\right) + 4\overline{N}(r, f) < (\lambda + o(1))T(r, f)$$

for $r \in I$ and $0 < \lambda < 1/2$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

Theorem E ([4]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, (∞,∞), and $(1,\infty)$. If*

$$\overline{N}_{1)}\left(r,\frac{1}{f}\right)+\overline{N}_{1)}(r,f)<(\lambda+o(1))T(r,f)$$

for $r\in I$ and $0<\lambda<1/2$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

In 2003, improving Theorems D and E, Yi [18] proved the following results.

Theorem F ([18]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(\infty,0)$, and $(1,5)$. If*

$$\limsup_{r\to\infty}\frac{\overline{N}_{1)}(r,1/f)+3\overline{N}(r,f)-(1/2)m(r,1/(g-1))}{T(r,f)}<\frac{1}{2}$$

for $r\in I$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem G ([18]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(\infty,0)$, and $(1,3)$. If*

$$\limsup_{r\to\infty}\frac{\overline{N}_{1)}(r,1/f)+4\overline{N}(r,f)-(1/2)m(r,1/(g-1))}{T(r,f)}<\frac{1}{2}$$

for $r\in I$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem H ([18]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(\infty,2)$, and $(1,6)$. If*

$$\limsup_{r\to\infty}\frac{\overline{N}_{1)}(r,1/f)+\overline{N}_{1)}(r,f)-(1/2)m(r,1/(g-1))}{T(r,f)}<\frac{1}{2}$$

for $r\in I$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

In this paper, with the aid of the notion of weighted sharing of values, we shall improve the results in Theorems F, G, and H and obtain the following theorems.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(\infty,0)$, and $(1,m)$, where m (≥ 2) is a positive integer or infinity. If*

$$\limsup_{r\to\infty}\frac{\overline{N}_{1)}(r,1/f)+(2(m+1)/(m-1))\overline{N}(r,f)-(1/2)m(r,1/(g-1))}{T(r,f)}<\frac{1}{2} \tag{1.1}$$

for $r\in I$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

The following example shows that in Theorem 1.1 sharing $(0,1)$ cannot be relaxed to sharing $(0,0)$.

Example 1.1. Let $f=(e^z-1)^2$ and $g=e^z-1$. Then f and g share $(0,0)$, (∞,∞), and $(1,\infty)$. Also $\overline{N}_{1)}(r,1/f)\equiv\overline{N}(r,f)\equiv 0$ but neither $f\equiv g$ nor $f\cdot g\equiv 1$.

Corollary 1.2. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(\infty,0)$, and $(1,m)$, where m (≥ 2) is a positive integer or infinity. If*

$$\overline{N}_{1)}(r,1/f)+(2(m+1)/(m-1))\overline{N}(r,f)<(\lambda+o(1))T(r,f) \tag{1.2}$$

for $r\in I$ and $0<\lambda<1/2$, then either $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, (∞,k), and $(1,m)$, where k, m are positive integers or infinity satisfying $(m-1)(km-1) > (1+m)^2$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,1/f)+\overline{N}_{1)}(r,f)-(1/2)m(r,1/(g-1))}{T(r,f)} < \frac{1}{2} \tag{1.3}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

Example 1.1 shows that in Theorem 1.3 sharing $(0,1)$ cannot be relaxed to sharing $(0,0)$, either. Also the following example shows that Theorem 1.3 does not hold when $(m-1)(km-1) = (1+m)^2$.

Example 1.2. Let $f = 4e^z/(1+e^z)^2$ and $g = 2e^z/(1+e^z)$, and $m = k = 0$. Then f and g share $(0,\infty)$, (∞,k), and $(1,m)$. Also $\overline{N}_{1)}(r,1/f) \equiv \overline{N}_{1)}(r,f) \equiv 0$ and $(m-1)(km-1) = (1+m)^2$ but neither $f \equiv g$ nor $f \cdot g \equiv 1$.

It is easily seen from the following examples that the condition (1.3) in Theorem 1.3 is the best possible.

Example 1.3. Let $f = e^{-z}+1$ and $g = e^z + 1$.

Example 1.4. Let $f = e^z/(e^z-1)$ and $g = 1/(1-e^z)$.

Corollary 1.4. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, (∞,k), and $(1,m)$, where k, m are positive integers or infinity satisfying $(m-1)(km-1) > (1+m)^2$. If*

$$\overline{N}_{1)}(r,1/f)+\overline{N}_{1)}(r,f) < (\lambda+o(1))T(r,f) \tag{1.4}$$

for $r \in I$ and $0 < \lambda < 1/2$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

Example 1.5. Let $f = 1/(e^z(1-e^z))$ and $g = e^{2z}/(e^z-1)$.

It is easy to see, from Example 1.5, that the condition (1.4) in Corollary 1.4 is the best possible.

Corollary 1.5. *Theorem 1.3 holds for any one of the following pairs of values of k and m:*

(i) $k = 2, \quad m = 6$,
(ii) $k = 3, \quad m = 4$,
(iii) $k = 4, \quad m = 3$,
(iv) $k = 6, \quad m = 2$.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel. Henceforth we shall denote by H the function

$$\left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.1}$$

Lemma 2.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(\infty,0)$, and $(1,0)$. Then*

$$T(r,f) \le 3T(r,g)+S(r,f), \qquad T(r,g) \le 3T(r,f)+S(r,g),$$
$$S(r,f) = S(r,g) := S(r).$$

Proof. Note that f and g share $(0,0)$, $(\infty,0)$, and $(1,0)$. By the second fundamental theorem, we can easily obtain the conclusion of Lemma 2.1. □

Lemma 2.2 ([18]). *Let H be given by (2.1) and $H \not\equiv 0$. If f and g share $(0,1)$, $(\infty,0)$, and $(1,m)$, where m (≥ 1) is a positive integer or infinity, then*

$$\overline{N}_{1)}\left(r,\frac{1}{f-1}\right) \leq \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right) + N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r), \tag{2.2}$$

where $N_0(r,1/f')$ denotes the counting function corresponding to the zeros of f' that are not zeros of $f(f-1)$, $N_0(r,1/g')$ denotes the counting function corresponding to the zeros of g' that are not zeros of $g(g-1)$.

Lemma 2.3 ([18]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(\infty,0)$, and $(1,m)$, where m (≥ 2) is a positive integer or infinity. Then*

$$\overline{N}_{(2}\left(r,\frac{1}{f}\right) \leq \overline{N}(r,f) + \overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right) + S(r), \tag{2.3}$$

$$\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right) \leq \frac{2}{m-1}\overline{N}(r,f) + S(r). \tag{2.4}$$

Lemma 2.4 ([8]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, (∞,k), and $(1,m)$, where k,m are positive integers or infinities satisfying $(m-1)(km-1) > (1+m)^2$. Then*

$$\overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}(r,f) = S(r). \tag{2.5}$$

Lemma 2.5. *Let H be given by (2.1) and $H \not\equiv 0$. If f and g share $(0,1)$, (∞,k), and $(1,m)$, where k, m are positive integers or infinity satisfying $(m-1)(km-1) > (1+m)^2$. Then*

$$\overline{N}_{1)}\left(r,\frac{1}{f-1}\right) \leq \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(k+1}(r,f) + \overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right) + N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r). \tag{2.6}$$

Proof. From the given condition it is clear that $k \geq 2$ and $m \geq 2$. Since f and g share $(1,m)$, it follows that a simple 1-point of f is a simple 1-point of g and conversely. Let z_0 be a simple 1-point of f and g. Then in some neighborhood of z_0 we get $H = (z-z_0)\alpha(z)$, where α is analytic at z_0. Thus

$$\overline{N}_{1)}\left(r,\frac{1}{f-1}\right) \leq N\left(r,\frac{1}{H}\right) \leq N(r,H) + S(r). \tag{2.7}$$

Note that f and g share $(0,1)$, (∞,k), and $(1,m)$. We can deduce by (2.1) that

$$N(r,H) \leq \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(k+1}(r,f) + \overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right) + N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r). \tag{2.8}$$

Combining (2.7) and (2.8), we obtain the conclusion of Lemma 2.5. □

3. PROOFS OF THE THEOREMS AND COROLLARIES

Proof of Theorem 1.1. Note that since f and g share $(1,m)$, we have

$$\begin{aligned}(3.1)\qquad &\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+(m-1)\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)\\ &\le \overline{N}_{1)}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)\\ &\le \overline{N}_{1)}\left(r,\frac{1}{f-1}\right)+T(r,g)-m\left(r,\frac{1}{g-1}\right)+O(1).\end{aligned}$$

By the second fundamental theorem, we obtain

$$(3.2)\qquad T(r,f)\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-1}\right)-N_0\left(r,\frac{1}{f'}\right)+S(r),$$

and

$$(3.3)\qquad T(r,g)\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g-1}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r).$$

Since f and g share $(0,1)$, (∞,k), and $(1,m)$, in view of (3.1) – (3.3) we get

$$\begin{aligned}(3.4)\quad T(r,f)\le 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+\overline{N}_{1)}\left(r,\frac{1}{f-1}\right)-(m-1)\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)\\ -m\left(r,\frac{1}{g-1}\right)-N_0\left(r,\frac{1}{f'}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r).\end{aligned}$$

Let H be given by (2.1). If $H\not\equiv 0$, then by Lemma 2.2 we have

$$\begin{aligned}(3.5)\quad \overline{N}_{1)}\left(r,\frac{1}{f-1}\right)\le \overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)\\ +N_0\left(r,\frac{1}{f'}\right)+N_0\left(r,\frac{1}{g'}\right)+S(r).\end{aligned}$$

Substituting (3.5) into (3.4) we derive

$$\begin{aligned}(3.6)\quad T(r,f)&\le 2\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\ &\quad-(2-m)\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)-m\left(r,\frac{1}{g-1}\right)+S(r)\\ &\le 2\overline{N}_{1)}\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+3\overline{N}_{(2}\left(r,\frac{1}{f}\right)-(2-m)\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)\\ &\quad-m\left(r,\frac{1}{g-1}\right)+S(r).\end{aligned}$$

Since f and g share $(0,1)$, $(\infty,0)$, and $(1,m)$, it follows by Lemma 2.3 that

$$(3.7)\qquad \overline{N}_{(2}\left(r,\frac{1}{f}\right)\le \overline{N}(r,f)+\overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)+S(r),$$

$$(3.8)\qquad \overline{N}_{(m+1}\left(r,\frac{1}{f-1}\right)\le \frac{2}{m-1}\overline{N}(r,f)+S(r).$$

Substituting (3.7) and (3.8) into (3.6) we have

$$T(r,f) \leq 2\overline{N}_{1)}\left(r,\frac{1}{f}\right) + \frac{4(m+1)}{m-1}\overline{N}(r,f) - m\left(r,\frac{1}{g-1}\right) + S(r),$$

which contradicts (1.1). Hence $H \equiv 0$ and so

$$\frac{f'}{(f-1)^2} = A\frac{g'}{(g-1)^2}, \tag{3.9}$$

where A is a nonzero constant. Note that f and g share $(0,1)$, $(\infty,0)$, and $(1,m)$. We know from (3.9) that f and g share $(0,\infty)$, (∞,∞), and $(1,\infty)$. Again by Theorem C, we obtain the conclusion of Theorem 1.1. □

Proof of Corollary 1.2. Let

$$T(r,f) = \begin{cases} T(r,f), & \text{for} \quad r \in I_1, \\ T(r,g), & \text{for} \quad r \in I_2, \end{cases} \tag{3.10}$$

where

$$I = I_1 \cup I_2. \tag{3.11}$$

Note that I is a set of infinite linear measure of $(0,\infty)$. We can see by (3.11) that I_1 is a set of infinite linear measure of $(0,\infty)$ or I_2 is a set of infinite linear measure of $(0,\infty)$. Without loss of generality, we assume that I_1 is a set of infinite linear measure of $(0,\infty)$. Then it follows by (1.2) and (3.10) that

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,1/f) + (2(m+1)/(m-1))\overline{N}(r,f)}{T(r,f)} < \frac{1}{2}$$

for $r \in I$. Again by Theorem 1.1, we obtain the conclusion of Corollary 1.2. □

Proof of Theorem 1.3. Since f and g share $(0,1)$, (∞,k), and $(1,m)$, it follows by Lemma 2.4 that

$$\overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}(r,f) = S(r). \tag{3.12}$$

It is easily seen that

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f-1}\right) &+ \overline{N}\left(r,\frac{1}{g-1}\right) \\ &\leq \overline{N}_{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) \\ &\leq \overline{N}_{1)}\left(r,\frac{1}{f-1}\right) + T(r,g) - m\left(r,\frac{1}{g-1}\right) + O(1). \end{aligned} \tag{3.13}$$

Form (3.2), (3.3), (3.12), and (3.13), we obtain

$$\begin{aligned} T(r,f) \leq 2\overline{N}_{1)}\left(r,\frac{1}{f}\right) &+ 2\overline{N}_{1)}(r,f) + \overline{N}_{1)}\left(r,\frac{1}{f-1}\right) \\ &- m\left(r,\frac{1}{g-1}\right) - N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r). \end{aligned} \tag{3.14}$$

Let H be given by (2.1). If $H \not\equiv 0$, then by Lemma 2.5 and (3.12) we get in view of $k \geq 2$ and $m \geq 2$

$$\overline{N}_{1)}\left(r, \frac{1}{f-1}\right) \leq N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) + S(r). \tag{3.15}$$

Substituting (3.15) into (3.14) we have

$$T(r,f) \leq 2\overline{N}_{1)}\left(r, \frac{1}{f}\right) + 2\overline{N}_{1)}(r,f) - m\left(r, \frac{1}{g-1}\right) + S(r),$$

which contradicts (1.3). Hence $H \equiv 0$ and so

$$\frac{f'}{(f-1)^2} = B\frac{g'}{(g-1)^2}, \tag{3.16}$$

where B is a nonzero constant. Note that f and g share $(0,1)$, (∞, k), and $(1, m)$. We can see by (3.16) that f and g share $(0,\infty)$, (∞,∞), and $(1,\infty)$. Again by Theorem C, we obtain the conclusion of Theorem 1.3. □

Proof of Corollary 1.4. Using Theorem 1.3 and proceeding as in the proof of Corollary 1.2, we can prove Corollary 1.4. □

4. FINAL REMARKS

In 2003, Yi [18] proved the following theorem.

Theorem I ([18]). *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(\infty,1)$, and $(1,5)$. If*

$$\limsup_{r\to\infty} \frac{3\overline{N}(r,1/f) + \overline{N}_{1)}(r,f) - (1/2)m(r,1/(g-1))}{T(r,f)} < \frac{1}{2}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

From Theorem 1.1 we get the following theorem which is an improvement of Theorem I.

Theorem 4.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(\infty,1)$, and $(1,m)$, where m (≥ 2) is a positive integer or infinity. If*

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,f) + (2(m+1)/(m-1))\overline{N}(r,1/f) - (1/2)m(r,1/(g-1))}{T(r,f)} < \frac{1}{2} \tag{4.1}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

Proof. Let

$$F = \frac{1}{f}, \qquad G = \frac{1}{g}. \tag{4.2}$$

It is easily seen that

$$T(r,f) = T(r,F) + O(1), \tag{4.3}$$

$$m\left(r, \frac{1}{g-1}\right) = m\left(r, \frac{1}{G-1}\right) + O(1). \tag{4.4}$$

From (4.1) – (4.4), we get

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,1/F) + (2(m+1)/(m-1))\overline{N}(r,F) - (1/2)m(r,1/(G-1))}{T(r,F)} < \frac{1}{2} \tag{4.5}$$

for $r \in I$. Note that f and g share $(0,0)$, $(\infty,1)$, and $(1,m)$. From (4.2), we see that F and G share $(0,1)$, $(\infty,0)$, and $(1,m)$. By Theorem 1.1, we get $F \equiv G$ or $F \cdot G \equiv 1$. From this, we deduce that Theorem 4.1 holds. □

In 2003, Yi [18] proved the following result.

Theorem J ([18]). *Let f and g be two nonconstant meromorphic functions sharing $(0,2)$, $(\infty,1)$, and $(1,6)$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,1/f) + \overline{N}_{1)}(r,f) - (1/2)m(r,1/(g-1))}{T(r,f)} < \frac{1}{2}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

Using Theorem 1.3 and proceeding as in the proof of Theorem 4.1, we can prove the following theorem, which is an improvement of Theorem J.

Theorem 4.2. *Let f and g be two nonconstant meromorphic functions sharing $(0,k)$, $(\infty,1)$, and $(1,m)$, where k, m are positive integers or infinity satisfying $(m-1)(km-1) > (1+m)^2$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}_{1)}(r,1/f) + \overline{N}_{1)}(r,f) - (1/2)m(r,1/(g-1))}{T(r,f)} < \frac{1}{2} \tag{4.6}$$

for $r \in I$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

5. APPLICATIONS

In this section, f and g are two nonconstant meromorphic functions.

Definition 5.1. For $S \subset C \cup \{\infty\}$ we define $E_f(S,k)$ as

$$E_f(S,k) = \bigcup_{a\in S} E_k(a,f),$$

where k is a nonnegative integer or infinity.

In 2003, Yi [18] proved the following theorem.

Theorem K ([18]). *Let $S_1 = \{a+b, a+b\omega, \dots, a+b\omega^{n-1}\}$, $S_2 = \{a\}$, and $S_3 = \{\infty\}$, where $n\ (\geq 2)$ is an integer, a and $b\ (\neq 0)$ are constants, and $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. If $E_f(S_1,6) = E_g(S_1,6)$, $E_f(S_2,0) = E_g(S_2,0)$, and $E_f(S_3,1) = E_g(S_3,1)$, then $f - a \equiv t(g-a)$, where $t^n = 1$, or $(f-a)(g-a) \equiv s$, where $s^n = b^{2n}$.*

From Corollary 1.5 we can prove the following theorem.

Theorem 5.1. *Let S_1, S_2, and S_3 be defined as in Theorem K. If $E_f(S_1,2) = E_g(S_1,2)$, $E_f(S_2,0) = E_g(S_2,0)$, and $E_f(S_3,1) = E_g(S_3,1)$, then $f - a \equiv t(g-a)$, where $t^n = 1$, or $(f-a)(g-a) \equiv s$, where $s^n = b^{2n}$.*

The following example shows that the assumption "$n \geq 2$" in Theorem 5.1 is the best possible.

Example 5.1. Let $f = a + b(1-e^z)^3$ and $g = a + 3b(e^{-z} - e^{-2z})$, and let $S_1 = \{a+b\}$, $S_2 = \{a\}$, and $S_3 = \{\infty\}$, where a and $b\ (\neq 0)$ are constants.

The following example shows that the condition "$E_f(S_3,1) = E_g(S_3,1)$" in Theorem 5.1 is the best possible.

Example 5.2. Let $f = (e^{2z}+1)^2/(2e^z(e^{2z}-1))$ and $g = 2ie^z(e^{2z}+1)/(e^{2z}-1)^2$, and let $S_1 = \{-1, 1\}$, $S_2 = \{0\}$, and $S_3 = \{\infty\}$. Then $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2, 0) = E_g(S_2, 0)$, and $E_f(S_3, 0) = E_g(S_3, 0)$.

Proof of Theorem 5.1. Let $F = ((f-a)/b)^n$ and $G = ((g-a)/b)^n$. Then F and G share $(1, 5)$, $(0, 1)$, and $(\infty, 3)$. Since $\overline{N}_{1)}(r, 1/F) = \overline{N}_{1)}(r, F) = 0$, it follows by (ii) in Corollary 1.5 that $F \equiv G$ or $F \cdot G \equiv 1$. From this, we deduce that Theorem 5.1 holds. □

Similarly, from Corollary 1.5 we can prove the following theorem.

Theorem 5.2. *Let S_1, S_2, and S_3 be defined as in Theorem K. If $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_2, 1) = E_g(S_2, 1)$, and $E_f(S_3, 0) = E_g(S_3, 0)$, then $f - a \equiv t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) \equiv s$, where $s^n = b^{2n}$.*

It is obvious that Theorems 5.1 and 5.2 are improvements of Theorem K.
On the other hand, we can also obtain the following theorems.

Theorem 5.3. *Let $S_1 = \{a + b, a + b\omega, \dots, a + b\omega^{n-1}\}$, $S_2 = \{a\}$, and $S_3 = \{\infty\}$, where n (≥ 3) is an integer, a and b $(\neq 0)$ are constants, and $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. If $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_2, 0) = E_g(S_2, 0)$, and $E_f(S_3, 0) = E_g(S_3, 0)$, then $f - a \equiv t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) \equiv s$, where $s^n = b^{2n}$.*

Proof. Let $F = ((f - a)/b)^n$ and $G = ((g - a)/b)^n$. Note that $n \geq 3$. Then F and G share $(1, 8)$, $(0, 2)$, and $(\infty, 2)$. Since $\overline{N}_{1)}(r, 1/F) = \overline{N}_{1)}(r, F) = 0$, it follows by (i) in Corollary 1.5 that $F \equiv G$ or $F \cdot G \equiv 1$. From this, we deduce that Theorem 5.2 holds. □

Theorem 5.4. *Let $S_1 = \{a + b, a + b\omega, \dots, a + b\omega^{n-1}\}$, $S_2 = \{a\}$, and $S_3 = \{\infty\}$, where n (≥ 3) is an integer, a and b $(\neq 0)$ are constants, and $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. If $E_f(S_1, 1) = E_g(S_1, 1)$, $E_f(S_2, 0) = E_g(S_2, 0)$, and $E_f(S_3, 1) = E_g(S_3, 1)$, then $f - a \equiv t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) \equiv s$, where $s^n = b^{2n}$.*

Proof. Let $F = ((f - a)/b)^n$ and $G = ((g - a)/b)^n$. Note that $n \geq 3$. Then F and G share $(1, 5)$, $(0, 2)$, and $(\infty, 3)$. Since $\overline{N}_{1)}(r, 1/F) = \overline{N}_{1)}(r, F) = 0$, it follows by (ii) in Corollary 1.5 that $F \equiv G$ or $F \cdot G \equiv 1$. From this, we deduce that Theorem 5.3 holds. □

It is easy to see that Example 5.2 also shows that the assumption "$n \geq 3$" in Theorems 5.3 and 5.4 is the best possible.

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