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HERMITE-HADAMARD-TYPE INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS

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ABSTRACT. The aim of the present paper is to extend the classical Hermite-Hadamard inequality to the case when the convexity notion is induced by a Chebyshev system.

Key words and phrases: Hermite–Hadamard inequality, generalized convexity, Beckenbach families, Chebyshev systems, Markov–Krein theory.

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1. INTRODUCTION

Let I be a real interval, that is, a nonempty, connected and bounded subset of $\mathbb{R}$. An n-dimensional *Chebyshev system* on I consists of a set of real valued continuous functions $\omega_1, \ldots, \omega_n$ and is determined by the property that each n points of $I \times \mathbb{R}$ with distinct first coordinates can uniquely be interpolated by a linear combination of the functions. More precisely, we have the following

Definition 1.1. *Let $I \subset \mathbb{R}$ be a real interval and $\omega_1, \ldots, \omega_n : I \to \mathbb{R}$ be continuous functions. Denote the column vector whose components are $\omega_1, \ldots, \omega_n$ in turn by $\boldsymbol{\omega}$, that is, $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_n)$. We say that $\boldsymbol{\omega}$ is a* Chebyshev system over I *if, for all elements $x_1 < \cdots < x_n$ of I, the following inequality holds:*

$$\begin{vmatrix} \boldsymbol{\omega}(x_1) & \cdots & \boldsymbol{\omega}(x_n) \end{vmatrix} > 0.$$

In fact, it suffices to assume that the determinant above is nonvanishing whenever the arguments $x_1, \ldots, x_n$ are pairwise distinct points of the domain. Indeed, Bolzano's theorem guarantees that its sign is constant if the arguments are supposed to be in an increasing order, hence the components $\omega_1, \ldots, \omega_n$ can always be rearranged such that $\boldsymbol{\omega}$ fulfills the requirement of the definition. However, considering Chebyshev systems as vectors of functions instead of sets of functions is widely accepted in the technical literature and also turns out to be very convenient in our investigations.

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Without claiming completeness, let us list some important and classical examples of Chebyshev systems. In each example $\boldsymbol{\omega}$ is defined on an arbitrary $I \subset \mathbb{R}$ except for the last one where $I \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$.

- polynomial system: $\boldsymbol{\omega}(x) := (1, x, \dots, x^n)$;
- exponential system: $\boldsymbol{\omega}(x) := (1, \exp x, \dots, \exp nx)$;
- hyperbolic system: $\boldsymbol{\omega}(x) := (1, \cosh x, \sinh x, \dots, \cosh nx, \sinh nx)$;
- trigonometric system: $\boldsymbol{\omega}(x) := (1, \cos x, \sin x, \dots, \cos nx, \sin nx)$.

We make no attempt here to present an exhaustive account of the theory of Chebyshev systems, but only mention that, motivated by some results of A.A. Markov, the first systematic investigation of the geometric theory of Chebyshev systems was done by M. G. Krein. However, let us note that Chebyshev systems play an important role, sometimes indirectly, in numerous fields of mathematics, for example, in the theory of approximation, numerical analysis and the theory of inequalities. The books [16] and [15] contain a rich literature and bibliography of the topics for the interested reader. The notion of convexity can also be extended by applying Chebyshev systems:

Definition 1.2. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a Chebyshev system over the real interval I. A function $f : I \to \mathbb{R}$ is said to be* generalized convex with respect to $\boldsymbol{\omega}$ *if, for all elements $x_0 < \cdots < x_n$ of I, it satisfies the inequality*

$$(-1)^n \begin{vmatrix} f(x_0) & \cdots & f(x_n) \\ \boldsymbol{\omega}(x_0) & \cdots & \boldsymbol{\omega}(x_n) \end{vmatrix} \geq 0.$$

There are other alternatives to express that f is generalized convex with respect to $\boldsymbol{\omega}$, for example, f is *generalized $\boldsymbol{\omega}$-convex* or simply *$\boldsymbol{\omega}$-convex*. If the underlying n-dimensional Chebyshev system can uniquely be identified from the context, we briefly say that f is *generalized n-convex*.

If $\boldsymbol{\omega}$ is the polynomial Chebyshev system, the definition leads to the notion of higher-order monotonicity which was introduced and studied by T. Popoviciu in a sequence of papers [20, 22, 21, 24, 23, 27, 29, 25, 30, 28, 26, 31, 33, 32, 34, 35]. A summary of these results can be found in [36] and [17]. For the sake of uniform terminology, throughout the this paper Popoviciu's setting is called polynomial convexity. That is, a function $f : I \to \mathbb{R}$ is said to be *polynomially n-convex* if, for all elements $x_0 < \cdots < x_n$ of I, it satisfies the inequality

$$(-1)^n \begin{vmatrix} f(x_0) & \dots & f(x_n) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \geq 0.$$

Observe that polynomially 2-convex functions are exactly the "standard" convex ones. The case, when the "generalized" convexity notion is induced by the special two dimensional Chebyshev system $\omega_1(x) := 1$ and $\omega_2(x) := x$, is termed *standard setting* and *standard convexity*, respectively.

The integral average of any standard convex function $f : [a, b] \to \mathbb{R}$ can be estimated from the midpoint and the endpoints of the domain as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

This is the well known Hadamard's inequality ([11]) or, as it is quoted for historical reasons (see [12] and also [18] for interesting remarks), the Hermite–Hadamard-inequality.

The aim of this paper is to verify analogous inequalities for generalized convex functions, that is, to give lower and upper estimations for the integral average of the function using certain base points of the domain. Of course, the base points are supposed to depend only on the underlying Chebyshev system of the induced convexity.

For this purpose, we shall follow an inductive approach since it seems to have more advantages than the deductive one. First of all, it makes the original motivations clear; on the other hand, it allows us to use the most suitable mathematical tools. Hence sophisticated proofs that sometimes occur when using a deductive approach can also be avoided.

SECTION 2 investigates the case of polynomial convexity. The base points of the Hermite–Hadamard-type inequalities turn out to be the zeros of certain orthogonal polynomials. The main tools of the section are based on some methods of numerical analysis, like the Gauss quadrature formula and Hermite-interpolation. A smoothing technique and two theorems of Popoviciu are also crucial.

In SECTION 3 we present Hermite–Hadamard-type inequalities for generalized 2-convex functions. The most important auxiliary result of the proof is a characterization theorem which, in the standard setting, reduces to the well known characterization properties of convex functions. Another theorem of the section establishes a tight relationship between standard and generalized 2-convexity. This result has important regularity consequences and is also essential in verifying Hermite–Hadamard-type inequalities.

The general case is studied in SECTION 4. The main results guarantee only the existence and also the uniqueness of the base points of the Hermite–Hadamard-type inequalities but offer no explicit formulae for determining them. The main tool of the section is the Krein–Markov theory of moment spaces induced by Chebyshev systems. In some special cases (when the dimension of the underlying Chebyshev systems are "small"), an elementary alternative approach is also presented.

SECTION 5 is devoted to showing that, at least in the two dimensional case and requiring weak regularity conditions, Hermite–Hadamard-type inequalities are not merely the consequences of generalized convexity, but they also characterize it.

Specializing the members of Chebyshev systems, several applications and examples are presented for concrete Hermite–Hadamard-type inequalities in both the cases of polynomial convexity and generalized 2-convexity. As a simple consequence, the classical Hermite–Hadamard inequality is among the corollaries in each case as well.

The results of this paper can be found in [3, 4, 5, 6, 7] and [1]. In what follows, we present them without any further references to the mentioned papers.

2. POLYNOMIAL CONVEXITY

The main results of this section state Hermite–Hadamard-type inequalities for polynomially convex functions. Let us recall that a function $f : I \to \mathbb{R}$ is said to be *polynomially n-convex* if, for all elements $x_0 < \cdots < x_n$ of I, it satisfies the inequality

$$(-1)^n \begin{vmatrix} f(x_0) & \dots & f(x_n) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \geq 0.$$

In order to determine the base points and coefficients of the inequalities, Gauss-type quadrature formulae are applied. Then, using the remainder term of the Hermite-interpolation, the main results follow immediately for "sufficiently smooth" functions due to the next two theorems of Popoviciu:

Theorem A. ([17, Theorem 1. p. 387]) *Assume that $f : I \to \mathbb{R}$ is continuous and n times differentiable on the interior of I. Then, f is polynomially n-convex if and only if $f^{(n)} \geq 0$ on the interior of I.*

Theorem B. ([17, Theorem 1. p. 391]) *Assume that $f : I \to \mathbb{R}$ is polynomially n-convex and $n \geq 2$. Then, f is $(n-2)$ times differentiable and $f^{(n-2)}$ is continuous on the interior of I.*

To drop the regularity assumptions, a smoothing technique is developed that guarantees the approximation of polynomially convex functions with smooth polynomially convex ones.

2.1. **Orthogonal polynomials and basic quadrature formulae.** In what follows, ρ denotes a positive, locally integrable function (briefly: *weight function*) on an interval I. The polynomials P and Q are said to be *orthogonal on $[a,b] \subset I$ with respect to the weight function ρ* or simply *ρ-orthogonal on* $[a,b]$ if

$$\langle P, Q \rangle_\rho := \int_a^b PQ\rho = 0.$$

A system of polynomials is called a *ρ-orthogonal polynomial system on* $[a,b] \subset I$ if each member of the system is ρ-orthogonal to the others on $[a,b]$. Define the *moments* of ρ by the formulae

$$\mu_k := \int_a^b x^k \rho(x)dx \qquad (k = 0, 1, 2, \dots).$$

Then, the n^{th} degree member of the ρ-orthogonal polynomial system on $[a,b]$ has the following representation via the moments of ρ:

$$P_n(x) := \begin{vmatrix} 1 & \mu_0 & \cdots & \mu_{n-1} \\ x & \mu_1 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ x^n & \mu_n & \cdots & \mu_{2n-1} \end{vmatrix}.$$

Clearly, it suffices to show that P_n is ρ-orthogonal to the special polynomials $1, x, \dots, x^{n-1}$. Indeed, for $k = 1, \dots, n$, the first and the $(k+1)^{st}$ columns of the determinant $\langle P_n(x), x^{k-1} \rangle_\rho$ are linearly dependent according to the definition of the moments.

In fact, the moments and the orthogonal polynomials depend heavily on the interval $[a,b]$. Therefore, we use the notions $\mu_{k;[a,b]}$ and $P_{n;[a,b]}$ instead of μ_k and P_n above when we want to or have to emphasize the dependence on the underlying interval.

Throughout this section, the following property of the zeros of orthogonal polynomials plays a key role (see [39]). Let P_n denote the n^{th} degree member of the ρ-orthogonal polynomial system on $[a,b]$. Then, P_n has n pairwise distinct zeros $\xi_1 < \dots < \xi_n$ in $]a,b[$.

Let us consider the following

$$\int_a^b f\rho = \sum_{k=1}^n c_k f(\xi_k), \tag{2.1}$$

$$\int_a^b f\rho = c_0 f(a) + \sum_{k=1}^n c_k f(\xi_k), \tag{2.2}$$

$$\int_a^b f\rho = \sum_{k=1}^{n} c_k f(\xi_k) + c_{n+1} f(b), \tag{2.3}$$

$$\int_a^b f\rho = c_0 f(a) + \sum_{k=1}^{n} c_k f(\xi_k) + c_{n+1} f(b). \tag{2.4}$$

Gauss-type quadrature formulae where the coefficients and the base points are to be determined so that (2.1), (2.2), (2.3) and (2.4) are exact when f is a polynomial of degree at most $2n-1$, $2n$, $2n$ and $2n+1$, respectively. The subsequent four theorems investigate these cases.

Theorem 2.1. *Let P_n be the n^{th} degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function ρ. Then* (2.1) *is exact for polynomials f of degree at most $2n-1$ if and only if $\xi_1,\dots,\xi_n$ are the zeros of P_n, and*

$$c_k = \int_a^b \frac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx. \tag{2.5}$$

Furthermore, $\xi_1,\dots,\xi_n$ are pairwise distinct elements of $]a,b[$, and $c_k \geq 0$ for all $k=1,\dots,n$.

This theorem follows easily from well known results in numerical analysis (see [13], [14], [39]). For the sake of completeness, we provide a proof.

Proof. First assume that $\xi_1,\dots,\xi_n$ are the zeros of the polynomial P_n and, for all $k=1,\dots,n$, denote the primitive Lagrange-interpolation polynomials by $L_k:[a,b]\to\mathbb{R}$. That is,

$$L_k(x) := \begin{cases} \dfrac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)} & \text{if } x \neq \xi_k \\ 1 & \text{if } x = \xi_k. \end{cases}$$

If Q is a polynomial of degree at most $2n-1$, then, using the Euclidian algorithm, Q can be written in the form $Q = PP_n + R$ where $\deg P, \deg R \leq n-1$. The inequality $\deg P \leq n-1$ implies the ρ-orthogonality of P and P_n:

$$\int_a^b PP_n\rho = 0.$$

On the other hand, $\deg R \leq n-1$ yields that R is equal to its Lagrange-interpolation polynomial:

$$R = \sum_{k=1}^{n} R(\xi_k)L_k.$$

Therefore, considering the definition of the coefficients $c_1,\dots,c_n$ in formula (2.5), we obtain that

$$\begin{aligned}\int_a^b Q\rho &= \int_a^b PP_n\rho + \int_a^b R\rho = \sum_{k=1}^{n} R(\xi_k)\int_a^b L_k\rho \\ &= \sum_{k=1}^{n} c_k R(\xi_k) = \sum_{k=1}^{n} c_k\big(P(\xi_k)P_n(\xi_k) + R(\xi_k)\big) = \sum_{k=1}^{n} c_k Q(\xi_k).\end{aligned}$$

That is, the quadrature formula (2.1) is exact for polynomials of degree at most $2n-1$.

Conversely, assume that (2.1) is exact for polynomials of degree at most $2n-1$. Define the polynomial Q by the formula $Q(x) := (x-\xi_1)\cdots(x-\xi_n)$ and let P be a polynomial of degree at most $n-1$. Then, $\deg PQ \leq 2n-1$, and thus

$$\int_a^b PQ\rho = c_1P(\xi_1)Q(\xi_1) + \cdots + c_nP(\xi_n)Q(\xi_n) = 0.$$

Therefore Q is ρ-orthogonal to P. The uniqueness of P_n implies that $P_n = a_nQ$, and $\xi_1, \dots, \xi_n$ are the zeros of P_n. Furthermore, (2.1) is exact if we substitute $f := L_k$ and $f := L_k^2$, respectively. The first substitution gives (2.5), while the second one shows the nonnegativity of c_k. For further details, consult the book [39, p. 44]. □

Theorem 2.2. *Let P_n be the n^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $\rho_a(x) := (x - a)\rho(x)$. Then* (2.2) *is exact for polynomials f of degree at most $2n$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n, and*

$$c_0 = \frac{1}{P_n^2(a)} \int_a^b P_n^2(x)\rho(x)dx, \tag{2.6}$$

$$c_k = \frac{1}{\xi_k - a} \int_a^b \frac{(x-a)P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx. \tag{2.7}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]a, b[$, and $c_k \geq 0$ for all $k = 0, \dots, n$.

Proof. Assume that the quadrature formula (2.2) is exact for polynomials of degree at most $2n$. If P is a polynomial of degree at most $2n - 1$, then

$$\int_a^b P\rho_a = \int_a^b (x-a)P(x)\rho(x)dx = c_1(\xi_1 - a)P(\xi_1) + \dots + c_n(\xi_n - a)P(\xi_n).$$

Applying Theorem 2.1 to the weight function ρ_a and the coefficients

$$c_{a;k} := c_k(\xi_k - a),$$

we get that $\xi_1, \dots, \xi_n$ are the zeros of P_n and, for all $k = 1, \dots, n$, the coefficients $c_{a;k}$ can be computed using formula (2.5). Therefore,

$$c_k(\xi_k - a) = \int_a^b \frac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho_a(x)dx = \int_a^b \frac{(x-a)P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx.$$

Substituting $f := P_n^2$ into (2.1), we obtain that

$$c_0 = \frac{1}{P_n^2(a)} \int_a^b P_n^2\rho.$$

Thus (2.6) and (2.7) are valid, and $c_k \geq 0$ for $k = 0, 1, \dots, n$.

Conversely, assume that $\xi_1, \dots, \xi_n$ are the zeros of the orthogonal polynomial P_n, and the coefficients $c_1, \dots, c_n$ are given by the formula (2.7). Define the coefficient c_0 by $c_0 = \int_a^b \rho - (c_1 + \dots + c_n)$. If P is a polynomial of degree at most $2n$, then there exists a polynomial Q with $\deg Q \leq 2n - 1$ such that

$$P(x) = (x-a)Q(x) + P(a).$$

Indeed, the polynomial $P(x) - P(a)$ vanishes at the point $x = a$, hence it is divisible by $(x - a)$. Applying Theorem 2.1 again to the weight function ρ_a,

$$\int_a^b Q\rho_a = c_{a;1}Q(\xi_1) + \dots + c_{a;n}Q(\xi_n)$$

holds. Thus, using the definition of c_0, the representation of the polynomial P and the quadrature formula above, we have that

$$\begin{aligned}\int_a^b P(x)\rho(x)dx &= \int_a^b \big((x-a)Q(x)+P(a)\big)\rho(x)dx\\ &= \sum_{k=1}^n c_k(\xi_k-a)Q(\xi_k)+\sum_{k=0}^n P(a)c_k\\ &= c_0P(a)+\sum_{k=1}^n c_k\big((\xi_k-a)Q(\xi_k)+P(a)\big)\\ &= c_0P(a)+\sum_{k=1}^n c_kP(\xi_k),\end{aligned}$$

which yields that the quadrature formula (2.2) is exact for polynomials of degree at most $2n$. Therefore, substituting $f := P_n^2$ into (2.2), we get formula (2.6). □

Theorem 2.3. *Let P_n be the n^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $\rho^b(x) := (b-x)\rho(x)$. Then* (2.3) *is exact for polynomials f of degree at most $2n$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n, and*

$$c_k = \frac{1}{b-\xi_k}\int_a^b \frac{(b-x)P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx, \tag{2.8}$$

$$c_{n+1} = \frac{1}{P_n^2(b)}\int_a^b P_n^2(x)\rho(x)dx. \tag{2.9}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]a, b[$, and $c_k \ge 0$ for all $k = 1, \dots, n+1$.

Hint. Applying a similar argument to the previous one to the weight function ρ^b, we obtain the statement of the theorem. □

Theorem 2.4. *Let P_n be the n^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function ρ_a^b. Then* (2.4) *is exact for polynomials f of degree at most $2n+1$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n, and*

$$c_0 = \frac{1}{(b-a)P_n^2(a)}\int_a^b (b-x)P_n^2(x)\rho(x)dx, \tag{2.10}$$

$$c_k = \frac{1}{(b-\xi_k)(\xi_k-a)}\int_a^b \frac{(b-x)(x-a)P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx, \tag{2.11}$$

$$c_{n+1} = \frac{1}{(b-a)P_n^2(b)}\int_a^b (x-a)P_n^2(x)\rho(x)dx. \tag{2.12}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]a, b[$, and $c_k \ge 0$ for all $k = 0, \dots, n+1$.

Proof. Assume that the quadrature formula (2.4) is exact for polynomials of degree at most $2n+1$. If P is a polynomial of degree at most $2n-1$, then

$$\begin{aligned}\int_a^b P\rho_a^b &= \int_a^b (b-x)(x-a)P(x)\rho(x)dx\\ &= c_1(b-\xi_1)(\xi_1-a)P(\xi_1)+\dots+c_n(b-\xi_n)(\xi_n-a)P(\xi_n).\end{aligned}$$

Applying Theorem 2.1 to the weight function ρ_a^b and the coefficients

$$c_{a,b;k} := c_k(b-\xi_k)(\xi_k-a),$$

we get that $\xi_1,\dots,\xi_n$ are the zeros of P_n and, for all $k=1,\dots,n$, the coefficients $c_{a,b;k}$ can be computed using formula (2.5). Therefore,

$$\begin{aligned}c_k(b-\xi_k)(\xi_k-a) &= \int_a^b \frac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho_a^b(x)dx\\ &= \int_a^b \frac{(b-x)(x-a)P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx.\end{aligned}$$

Substituting $f := (b-x)P_n^2(x)$ and $f := (x-a)P_n^2(x)$ into (2.1), we obtain that

$$\begin{aligned}c_0 &= \frac{1}{(b-a)P_n^2(a)}\int_a^b (b-x)P_n^2(x)\rho(x)dx,\\ c_{n+1} &= \frac{1}{(b-a)P_n^2(b)}\int_a^b (x-a)P_n^2(x)\rho(x)dx.\end{aligned}$$

Thus (2.10), (2.11) and (2.12) are valid, furthermore, $c_k \ge 0$ for $k=0,\dots,n+1$.

Conversely, assume that $\xi_1,\dots,\xi_n$ are the zeros of P_n, and the coefficients $c_1,\dots,c_n$ are given by the formula (2.11). Define the coefficients c_0 and c_{n+1} by the equations

$$\begin{aligned}\int_a^b (b-x)\rho(x)dx &= c_0(b-a) + \sum_{k=1}^n c_k(b-\xi_k),\\ \int_a^b (x-a)\rho(x)dx &= \sum_{k=1}^n c_k(\xi_k-a) + c_{n+1}(b-a).\end{aligned}$$

If P is a polynomial of degree at most $2n+1$, then there exists a polynomial Q with $\deg Q \le 2n-1$ such that

$$(b-a)P(x) = (b-x)(x-a)Q(x) + (x-a)P(b) + (b-x)P(a).$$

Indeed, the polynomial $(b-a)P(x)-(x-a)P(b)-(b-x)P(a)$ is divisible by $(b-x)(x-a)$ since $x=a$ and $x=b$ are its zeros. Applying Theorem 2.1 again,

$$\int_a^b Q\rho_a^b = c_{a,b;1}Q(\xi_1)+\cdots+c_{a,b;n}Q(\xi_n)$$

holds. Thus, using the definition of c_0 and c_{n+1}, the representation of the polynomial P and the quadrature formula above, we have that

$$\begin{aligned}&(b-a)\int_a^b P(x)\rho(x)dx\\ &\quad= \int_a^b \big((b-x)(x-a)Q(x)+(x-a)P(b)+(b-x)P(a)\big)\rho(x)dx\\ &\quad= \sum_{k=1}^n c_k(b-\xi_k)(\xi_k-a)Q(\xi_k)\\ &\qquad\quad + P(b)\int_a^b (x-a)\rho(x)dx + P(a)\int_a^b (b-x)\rho(x)dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} c_k(b-\xi_k)(\xi_k-a)Q(\xi_k) \\
&\qquad + c_0(b-a)P(a) + \sum_{k=1}^{n} c_k(b-\xi_k)P(a) \\
&\qquad + \sum_{k=1}^{n} c_k(\xi_k-a)P(b) + c_{n+1}(b-a)P(b) \\
&= \sum_{k=1}^{n} c_k\big((b-\xi_k)(\xi_k-a)Q(\xi_k) + (\xi_k-a)P(b) + (b-\xi_k)P(a)\big) \\
&\qquad + c_0(b-a)P(a) + c_{n+1}(b-a)P(b) \\
&= c_0(b-a)P(a) + \sum_{k=1}^{n} c_k(b-a)P(\xi_k) + c_{n+1}(b-a)P(b),
\end{aligned}
$$

which yields that the quadrature formula (2.4) is exact for polynomials of degree at most $2n+1$. Therefore, substituting $f := (b-x)P_n^2(x)$ and $f := (x-a)P_n^2(x)$ into (2.4), formulae (2.10) and (2.12) follow. □

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, $x_1, \dots, x_n$ be pairwise distinct elements of $[a, b]$, and $1 \le r \le n$ be a fixed integer. We denote the Hermite interpolation polynomial by H, which satisfies the following conditions:

$$
\begin{aligned}
H(x_k) &= f(x_k) \quad (k = 1, \dots, n), \\
H'(x_k) &= f'(x_k) \quad (k = 1, \dots, r).
\end{aligned}
$$

We recall that $\deg H = n + r - 1$. From a well known result, (see [13, Sec. 5.3, pp. 230-231]), for all $x \in [a, b]$ there exists θ such that

$$
f(x) - H(x) = \frac{\omega_n(x)\omega_r(x)}{(n+r)!} f^{(n+r)}(\theta), \tag{2.13}
$$

where

$$
\omega_k(x) = (x - x_1) \cdots (x - x_k).
$$

2.2. **An approximation theorem.** It is well known that there exists a function φ which possesses the following properties:

(i) $\varphi : \mathbb{R} \to \mathbb{R}_+$ is $\mathscr{C}^\infty$, i. e., it is infinitely many times differentiable;
(ii) $\operatorname{supp} \varphi \subset [-1, 1]$;
(iii) $\int_{\mathbb{R}} \varphi = 1$.

Using φ, one can define the function φ_ε for all $\varepsilon > 0$ by the formula

$$
\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\Big(\frac{x}{\varepsilon}\Big) \qquad (x \in \mathbb{R}).
$$

Then, as it can easily be checked, φ_ε satisfies the following conditions:

(i') $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is $\mathscr{C}^\infty$;
(ii') $\operatorname{supp} \varphi_\varepsilon \subset [-\varepsilon, \varepsilon]$;
(iii') $\int_{\mathbb{R}} \varphi_\varepsilon = 1$.

Let $I \subset \mathbb{R}$ be a nonempty open interval, $f : I \to \mathbb{R}$ be a continuous function, and choose $\varepsilon > 0$. Denote the convolution of f and φ_ε by f_ε, that is

$$f_\varepsilon(x) := \int_{\mathbb{R}} \bar{f}(y)\varphi_\varepsilon(x-y)dy \qquad (x \in \mathbb{R})$$

where $\bar{f}(y) = f(y)$ if $y \in I$, otherwise $\bar{f}(y) = 0$. Let us recall, that $f_\varepsilon \to f$ uniformly as $\varepsilon \to 0$ on each compact subinterval of I, and f_ε is infinitely many times differentiable on $\mathbb{R}$. These important results can be found for example in [40, p. 549].

Theorem 2.5. *Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a polynomially n-convex continuous function. Then, for all compact subintervals $[a, b] \subset I$, there exists a sequence of polynomially n-convex and $\mathscr{C}^\infty$ functions (f_k) which converges uniformly to f on $[a, b]$.*

Proof. Choose $a, b \in I$ and $\varepsilon_0 > 0$ such that the inclusion $[a - \varepsilon_0, b + \varepsilon_0] \subset I$ holds. We show that the function $\tau_\varepsilon f : [a, b] \to \mathbb{R}$ defined by the formula

$$\tau_\varepsilon f(x) := f(x - \varepsilon)$$

is polynomially n-convex on $[a, b]$ for $0 < \varepsilon < \varepsilon_0$. Let $a \le x_0 < \cdots < x_n \le b$ and $k \le n - 1$ be fixed. By induction, we are going to verify the identity

$$(2.14)\qquad \begin{vmatrix} \tau_\varepsilon f(x_0) & \cdots & \tau_\varepsilon f(x_n) \\ 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{k-1} & \cdots & x_n^{k-1} \\ x_0^k & \cdots & x_n^k \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} \tau_\varepsilon f(x_0) & \cdots & \tau_\varepsilon f(x_n) \\ 1 & \cdots & 1 \\ x_0 - \varepsilon & \cdots & x_n - \varepsilon \\ \vdots & \ddots & \vdots \\ (x_0-\varepsilon)^{k-1} & \cdots & (x_n-\varepsilon)^{k-1} \\ x_0^k & \cdots & x_n^k \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

If $k = 1$, then this equation obviously holds. Assume, for a fixed positive integer $k \le n - 2$, that (2.14) remains true. The binomial theorem implies the identity

$$x^k = \binom{k}{0}\varepsilon^k + \binom{k}{1}\varepsilon^{k-1}(x-\varepsilon) + \cdots + \binom{k}{k}(x-\varepsilon)^k.$$

That is, $(x-\varepsilon)^k$ is the linear combination of the elements $1, x-\varepsilon, \dots, (x-\varepsilon)^k$ and x^k. Therefore, adding the appropriate linear combination of the $2^{nd}, \dots, (k+1)^{st}$ rows to the $(k+2)^{nd}$ row, we arrive at the equation

$$\begin{vmatrix} \tau_\varepsilon f(x_0) & \cdots & \tau_\varepsilon f(x_n) \\ 1 & \cdots & 1 \\ x_0 - \varepsilon & \cdots & x_n - \varepsilon \\ \vdots & \ddots & \vdots \\ (x_0-\varepsilon)^{k-1} & \cdots & (x_n-\varepsilon)^{k-1} \\ x_0^k & \cdots & x_n^k \\ x_0^{k+1} & \cdots & x_n^{k+1} \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} \tau_\varepsilon f(x_0) & \cdots & \tau_\varepsilon f(x_n) \\ 1 & \cdots & 1 \\ x_0 - \varepsilon & \cdots & x_n - \varepsilon \\ \vdots & \ddots & \vdots \\ (x_0-\varepsilon)^{k-1} & \cdots & (x_n-\varepsilon)^{k-1} \\ (x_0-\varepsilon)^k & \cdots & (x_n-\varepsilon)^k \\ x_0^{k+1} & \cdots & x_n^{k+1} \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

Hence formula (2.14) holds for all fixed positive k whenever $1 \le k \le n - 1$. The particular case $k = n - 1$ gives the polynomial n-convexity of $\tau_\varepsilon f$. Applying a change of variables and the previous result, we get that

$$\begin{aligned}
&(-1)^n \begin{vmatrix} f_\varepsilon(x_0) & \cdots & f_\varepsilon(x_n) \\ 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \\
&= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \bar{f}(t)\varphi_\varepsilon(x_0 - t) & \cdots & \bar{f}(t)\varphi_\varepsilon(x_n - t) \\ 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} dt \\
&= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \bar{f}(x_0 - s) & \cdots & \bar{f}(x_n - s) \\ 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \varphi_\varepsilon(s)ds \\
&= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \tau_s f(x_0) & \cdots & \tau_s f(x_n) \\ 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \varphi_\varepsilon(s)ds \geq 0,
\end{aligned}$$

which shows the polynomial n-convexity of f_ε on $[a, b]$ for $0 < \varepsilon < \varepsilon_0$.

To complete the proof, choose a positive integer n_0 such that the relation $\frac{1}{n_0} < \varepsilon_0$ holds. If we define ε_k and f_k by $\varepsilon_k := \frac{1}{n_0+k}$ and $f_k := f_{\varepsilon_k}$ for $k \in \mathbb{N}$, then $0 < \varepsilon_k < \varepsilon_0$, and thus $(f_k)_{k=1}^\infty$ satisfies the requirements of the theorem. □

2.3. **Hermite–Hadamard-type inequalities.** In the sequel, we shall need two additional auxiliary results. The first one investigates the convergence properties of the zeros of orthogonal polynomials.

Lemma 2.1. *Let ρ be a weight function on $[a, b]$, and (a_j) be strictly monotone decreasing, (b_j) be strictly monotone increasing sequences such that $a_j \to a$, $b_j \to b$ and $a_1 < b_1$. Denote the zeros of $P_{m;j}$ by $\xi_{1;j}, \dots, \xi_{m;j}$, where $P_{m;j}$ is the m^{th} degree member of the $\rho|_{[a_j,b_j]}$-orthogonal polynomial system on $[a_j, b_j]$, and denote the zeros of P_m by $\xi_1, \dots, \xi_m$, where P_m is the m^{th} degree member of the ρ-orthogonal polynomial system on $[a, b]$. Then,*

$$\lim_{j\to\infty} \xi_{k;j} = \xi_k \qquad (k = 1, \dots, n).$$

Proof. Observe first that the mapping $(a, b) \mapsto \mu_{k;[a,b]}$ is continuous, therefore $\mu_{k;[a_j,b_j]} \to \mu_{k;[a,b]}$ hence $P_{m;j} \to P_m$ pointwise according to the representation of orthogonal polynomials. Take $\varepsilon > 0$ such that

$$]\xi_k - \varepsilon, \xi_k + \varepsilon[\subset]a, b[,$$
$$]\xi_k - \varepsilon, \xi_k + \varepsilon[\cap]\xi_l - \varepsilon, \xi_l + \varepsilon[= \emptyset \qquad (k \neq l,\ k, l \in \{1, \dots, m\}).$$

The polynomial P_m changes its sign on $]\xi_k - \varepsilon, \xi_k + \varepsilon[$ since it is of degree m and it has m pairwise distinct zeros; therefore, due to the pointwise convergence, $P_{m;j}$ also changes its sign on the same interval up to an index. That is, for sufficiently large j, $\xi_{k;j} \in]\xi_k - \varepsilon, \xi_k + \varepsilon[$. □

The other auxiliary result investigates the one-sided limits of polynomially n-convex functions at the endpoints of the domain. Let us note that its first assertion involves, in fact, two cases according to the parity of the convexity.

Lemma 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a polynomially n-convex function. Then,*

(i) $(-1)^n f(a) \geq \limsup_{t\to a+0}(-1)^n f(t)$;
(ii) $f(b) \geq \limsup_{t\to b-0} f(t)$.

Proof. It suffices to restrict the investigations to the even case of assertion (i) only since the proofs of the other ones are completely the same. For the sake of brevity, we shall use the notation $f_+(a) := \limsup_{t\to a+0} f(t)$. Take the elements $x_0 := a < x_1 := t < \cdots < x_n$ of $[a, b]$. Then, the (even order) polynomial convexity of f implies

$$\begin{vmatrix} f(a) & f(t) & f(x_2) & \dots & f(x_n) \\ 1 & 1 & 1 & \dots & 1 \\ a & t & x_2 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & t^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \geq 0.$$

Therefore, taking the limsup as $t \to a + 0$, we obtain that

$$\begin{vmatrix} f(a) & f_+(a) & f(x_2) & \dots & f(x_n) \\ 1 & 1 & 1 & \dots & 1 \\ a & a & x_2 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & a^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \geq 0.$$

The adjoint determinants of the elements $f(x_2), \dots, f(x_n)$ in the first row are equal to zero since their first and second columns coincide; on the other hand, $f(a)$ and $f_+(a)$ have the same (positive) Vandermonde-type adjoint determinant. Hence, applying the expansion theorem on the first row, we obtain the desired inequality

$$f(a) - f_+(a) \geq 0.$$

□

The main results concern the cases of odd and even order polynomial convexity separately in the subsequent two theorems.

Theorem 2.6. *Let $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Denote the zeros of P_m by $\xi_1, \dots, \xi_m$ where P_m is the m^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $(x - a)\rho(x)$, and denote the zeros of Q_m by $\eta_1, \dots, \eta_m$ where Q_m is the m^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $(b - x)\rho(x)$. Define the coefficients $\alpha_0, \dots, \alpha_m$ and $\beta_1, \dots, \beta_{m+1}$ by the formulae*

$$\alpha_0 := \frac{1}{P_m^2(a)} \int_a^b P_m^2(x)\rho(x)dx,$$
$$\alpha_k := \frac{1}{\xi_k - a} \int_a^b \frac{(x - a)P_m(x)}{(x - \xi_k)P_m'(\xi_k)}\rho(x)dx$$

and

$$\beta_k := \frac{1}{b-\eta_k}\int_a^b \frac{(b-x)Q_m(x)}{(x-\eta_k)Q'_m(\eta_k)}\rho(x)dx,$$
$$\beta_{m+1} := \frac{1}{Q_m^2(b)}\int_a^b Q_m^2(x)\rho(x)dx.$$

If a function $f : [a,b] \to \mathbb{R}$ is polynomially $(2m+1)$-convex, then it satisfies the following Hermite–Hadamard-type inequality

$$\alpha_0 f(a) + \sum_{k=1}^m \alpha_k f(\xi_k) \le \int_a^b f\rho \le \sum_{k=1}^m \beta_k f(\eta_k) + \beta_{m+1} f(b).$$

Proof. First assume that f is $(2m+1)$ times differentiable. Then, according to Theorem A, $f^{(2m+1)} \ge 0$ on $]a,b[$. Let H be the Hermite interpolation polynomial determined by the conditions

$$H(a) = f(a),$$
$$H(\xi_k) = f(\xi_k),$$
$$H'(\xi_k) = f'(\xi_k).$$

By the remainder term (2.13) of the Hermite interpolation, if x is an arbitrary element of $]a,b[$, then there exists $\theta \in]a,b[$ such that

$$f(x) - H(x) = \frac{(x-a)(x-\xi_1)^2\cdots(x-\xi_m)^2}{(2m+1)!} f^{(2m+1)}(\theta).$$

That is, $f\rho \ge H\rho$ on $[a,b]$ due to the nonnegativity of $f^{(2m+1)}$ and the positivity of ρ. On the other hand, H is of degree $2m$, therefore Theorem 2.2 yields that

$$\int_a^b f\rho \ge \int_a^b H\rho = \alpha_0 H(a) + \sum_{k=1}^m \alpha_k H(\xi_k) = \alpha_0 f(a) + \sum_{k=1}^m \alpha_k f(\xi_k).$$

For the general case, let f be an arbitrary polynomially $(2m+1)$-convex function. Without loss of generality we may assume that $m \ge 1$; in this case, f is continuous (see Theorem B). Let (a_j) and (b_j) be sequences fulfilling the requirements of Lemma 2.1. According to Theorem 2.5, there exists a sequence of $\mathscr{C}^\infty$, polynomially $(2m+1)$-convex functions $(f_{i;j})$ such that $f_{i;j} \to f$ uniformly on $[a_j, b_j]$ as $i \to \infty$. Denote the zeros of $P_{m;j}$ by $\xi_{1;j}, \dots, \xi_{m;j}$ where $P_{m;j}$ is the m^{th} degree member of the orthogonal polynomial system on $[a_j, b_j]$ with respect to the weight function $(x-a)\rho(x)$. Define the coefficients $\alpha_{0;j}, \dots, \alpha_{m;j}$ analogously to $\alpha_0, \dots, \alpha_m$ with the help of $P_{m;j}$. Then, $\xi_{k;j} \to \xi_k$ due to Lemma 2.1, and hence $\alpha_{k;j} \to \alpha_k$ as $j \to \infty$. Applying the previous step of the proof on the smooth functions $(f_{i;j})$, it follows that

$$\alpha_{0;j} f_{i;j}(a_j) + \sum_{k=1}^m \alpha_{k;j} f_{i;j}(\xi_{k;j}) \le \int_{a_j}^{b_j} f_{i;j}\rho.$$

Taking the limits $i \to \infty$ and then $j \to \infty$, we get the inequality

$$\alpha_0 \Big(\liminf_{t\to a+0} f(t)\Big) + \sum_{k=1}^m \alpha_k f(\xi_k) \le \int_a^b f\rho.$$

This, together with Lemma 2.2, gives the left hand side inequality to be proved. The proof of the right hand side inequality is analogous, therefore it is omitted. □

The second main result offers Hermite–Hadamard-type inequalities for even-order polynomially convex functions. In this case, the symmetrical structure disappears: the lower estimation involves none of the endpoints, while the upper estimation involves both of them.

Theorem 2.7. *Let $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Denote the zeros of P_m by $\xi_1, \dots, \xi_m$ where P_m is the m^{th} degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $\rho(x)$, and denote the zeros of Q_{m-1} by $\eta_1, \dots, \eta_{m-1}$ where Q_{m-1} is the $(m-1)^{st}$ degree member of the orthogonal polynomial system on $[a, b]$ with respect to the weight function $(b-x)(x-a)\rho(x)$. Define the coefficients $\alpha_1, \dots, \alpha_m$ and $\beta_0, \dots, \beta_{m+1}$ by the formulae*

$$\alpha_k := \int_a^b \frac{P_m(x)}{(x-\xi_k)P_m'(\xi_k)}\rho(x)dx$$

and

$$\begin{aligned}
\beta_0 &= \frac{1}{(b-a)Q_{m-1}^2(a)} \int_a^b (b-x)Q_{m-1}^2(x)\rho(x)dx, \\
\beta_k &= \frac{1}{(b-\eta_k)(\xi_k - a)} \int_a^b \frac{(b-x)(x-a)Q_{m-1}(x)}{(x-\eta_k)Q_{m-1}'(\eta_k)}\rho(x)dx, \\
\beta_{m+1} &= \frac{1}{(b-a)Q_{m-1}^2(b)} \int_a^b (x-a)Q_{m-1}^2(x)\rho(x)dx.
\end{aligned}$$

If a function $f : [a, b] \to \mathbb{R}$ is polynomially $(2m)$-convex, then it satisfies the following Hermite–Hadamard-type inequality

$$\sum_{k=1}^{m} \alpha_k f(\xi_k) \le \int_a^b f\rho \le \beta_0 f(a) + \sum_{k=1}^{m-1} \beta_k f(\eta_k) + \beta_m f(b).$$

Proof. First assume that f is $n = 2m$ times differentiable. Then $f^{(2m)} \ge 0$ on $]a, b[$ according to Theorem B. Consider the Hermite interpolation polynomial H that interpolates the function f in the zeros of P_m in the following manner:

$$\begin{aligned}
H(\xi_k) &= f(\xi_k), \\
H'(\xi_k) &= f'(\xi_k).
\end{aligned}$$

By the remainder term (2.13) of the Hermite interpolation, if x is an arbitrary element of $]a, b[$, then there exists $\theta \in]a, b[$ such that

$$f(x) - H(x) = \frac{(x-\xi_1)^2 \cdots (x-\xi_m)^2}{(2m)!} f^{(2m)}(\theta).$$

Hence $f\rho \ge H\rho$ on $[a, b]$ due to the nonnegativity of $f^{(2m)}$ and the positivity of ρ. On the other hand, H is of degree $2m-1$, therefore Theorem 2.1 yields the left hand side of the inequality to be proved:

$$\int_a^b f\rho \ge \int_a^b H\rho = \sum_{k=1}^{m} \alpha_k H(\xi_k) = \sum_{k=1}^{m} \alpha_k f(\xi_k).$$

Now consider the Hermite interpolation polynomial H that interpolates the function f at the zeros of Q_{m-1} and at the endpoints of the domain in the following way:

$$\begin{aligned} H(a) &= f(a), \\ H(\eta_k) &= f(\eta_k), \\ H'(\eta_k) &= f'(\eta_k), \\ H(b) &= f(b). \end{aligned}$$

By the remainder term (2.13) of the Hermite interpolation, if x is an arbitrary element of $]a, b[$, then there exists a $\theta \in]a, b[$ such that

$$f(x) - H(x) = \frac{(x-a)(x-b)(x-\eta_1)^2 \cdots (x-\eta_{m-1})^2}{(2m)!} f^{(2m)}(\theta).$$

The factors of the right hand side are nonnegative except for the factor $(x-b)$ which is negative, hence $f\rho \le H\rho$. On the other hand, H is of degree $2m-1$, therefore Theorem 2.4 yields the right hand side inequality to be proved:

$$\begin{aligned} \int_a^b f\rho \le \int_a^b H\rho &= \beta_0 H(a) + \sum_{k=1}^{m-1} \beta_k H(\eta_k) + \beta_m H(b) \\ &= \beta_0 f(a) + \sum_{k=1}^{m-1} \beta_k f(\eta_k) + \beta_m f(b). \end{aligned}$$

From this point, an analogous argument to the corresponding part of the previous proof gives the statement of the theorem without any differentiability assumptions on the function f. □

Specializing the weight function $\rho \equiv 1$, the roots of the inequalities can be obtained as convex combinations of the endpoints of the domain. The coefficients of the convex combinations are the zeros of certain orthogonal polynomials on $[0, 1]$ in both cases. Observe that interchanging the role of the endpoints in any side of the inequality concerning the odd order case, we obtain the other side of the inequality.

Theorem 2.8. *Let, for $m \ge 0$, the polynomial P_m be defined by the formula*

$$P_m(x) := \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ x & \frac{1}{3} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ x^m & \frac{1}{m+2} & \cdots & \frac{1}{2m+1} \end{vmatrix}.$$

Then, P_m has m pairwise distinct zeros $\lambda_1, \ldots, \lambda_m$ in $]0, 1[$. Define the coefficients $\alpha_0, \ldots, \alpha_m$ by

$$\begin{aligned} \alpha_0 &:= \frac{1}{P_m^2(0)} \int_0^1 P_m^2(x)dx, \\ \alpha_k &:= \frac{1}{\lambda_k} \int_0^1 \frac{xP_m(x)}{(x-\lambda_k)P_m'(\lambda_k)} dx. \end{aligned}$$

If a function $f : [a,b] \to \mathbb{R}$ is polynomially $(2m+1)$-convex, then it satisfies the following Hermite–Hadamard-type inequality

$$\alpha_0 f(a) + \sum_{k=1}^{m} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \sum_{k=1}^{m} \alpha_k f\big(\lambda_k a + (1-\lambda_k)b\big) + \alpha_0 f(b).$$

Proof. Apply Theorem 2.6 in the particular setting when $a := 0$, $b := 1$ and the weight function is $\rho \equiv 1$. Then, as simple calculations show, P_m is exactly the m^{th} degree member of the orthogonal polynomial system on $[0,1]$ with respect to the weight function $\rho(x) = x$ (see the beginning of this section). Therefore, P_m has m pairwise distinct zeros $0 < \lambda_1 < \cdots < \lambda_m < 1$. Moreover, the coefficients $\alpha_0, \ldots, \alpha_m$ have the form above. Define the function $F : [0,1] \to \mathbb{R}$ by the formula

$$F(t) := f\big((1-t)a + tb\big).$$

It is easy to check that F is polynomially $(2m+1)$-convex on the interval $[0,1]$. Hence, applying Theorem 2.6 and the previous observations, it follows that

$$\int_0^1 F(t)dt \ge \alpha_0 F(0) + \sum_{k=1}^{m} \alpha_k F(\lambda_k)$$
$$= \alpha_0 f(a) + \sum_{k=1}^{m} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big).$$

On the other hand, to complete the proof of the left hand side inequality, observe that

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 F(t)dt.$$

For verifying the right hand side one, define the function $\varphi : [a,b] \to \mathbb{R}$ by the formula

$$\varphi(x) := -f(a+b-x).$$

Then, φ is polynomially $(2m+1)$-convex on $[a,b]$. The previous inequality applied on φ gives the upper estimation of the Hermite–Hadamard-type inequality for f. $\square$

Theorem 2.9. *Let, for $m \ge 1$, the polynomials P_m and Q_{m-1} be defined by the formulae*

$$P_m(x) := \begin{vmatrix} 1 & 1 & \cdots & \frac{1}{m} \\ x & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^m & \frac{1}{m+1} & \cdots & \frac{1}{2m} \end{vmatrix},$$

$$Q_{m-1}(x) := \begin{vmatrix} 1 & \frac{1}{2\cdot 3} & \cdots & \frac{1}{m(m+1)} \\ x & \frac{1}{3\cdot 4} & \cdots & \frac{1}{(m+1)(m+2)} \\ \vdots & \vdots & \ddots & \vdots \\ x^{m-1} & \frac{1}{(m+1)(m+2)} & \cdots & \frac{1}{(2m-1)2m} \end{vmatrix}.$$

Then, P_m has m pairwise distinct zeros $\lambda_1, \ldots, \lambda_m$ in $]0,1[$ and Q_{m-1} has $m-1$ pairwise distinct zeros $\mu_1, \ldots, \mu_{m-1}$ in $]0,1[$, respectively. Define the coefficients $\alpha_1, \ldots, \alpha_m$ and $\beta_0, \ldots, \beta_m$ by

$$\alpha_k := \int_0^1 \frac{P_m(x)}{(x-\lambda_k)P_m'(\lambda_k)}dx$$

and

$$\beta_0 := \frac{1}{Q_{m-1}^2(0)} \int_0^1 (1-x) Q_{m-1}^2(x) dx,$$
$$\beta_k := \frac{1}{(1-\mu_k)\mu_k} \int_0^1 \frac{x(1-x)Q_{m-1}(x)}{(x-\mu_k)Q'_{m-1}(\mu_k)} dx,$$
$$\beta_m := \frac{1}{Q_{m-1}^2(1)} \int_0^1 x Q_{m-1}^2(x) dx.$$

If a function $f : [a,b] \to \mathbb{R}$ is polynomially $(2m)$-convex, then it satisfies the following Hermite–Hadamard-type inequality

$$\sum_{k=1}^{m} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \le \frac{1}{b-a} \int_a^b f(x)dx$$
$$\le \beta_0 f(a) + \sum_{k=1}^{m-1} \beta_k f\big((1-\mu_k)a + \mu_k b\big) + \beta_m f(b).$$

Proof. Substitute $a := 0$, $b := 1$ and $\rho \equiv 1$ into Theorem 2.7. Then, P_m is exactly the m^{th} degree member of the orthogonal polynomial system on the interval $[0,1]$ with respect to the weight function $\rho(x) = 1$; similarly, Q_{m-1} is the $(m-1)^{st}$ degree member of the orthogonal polynomial system on the interval $[0,1]$ with respect to the weight function $\rho(x) = (1-x)x$. Therefore, Q_m has m pairwise distinct zeros $0 < \lambda_1 < \cdots < \lambda_m < 1$ and Q_{m-1} has $m-1$ pairwise distinct zeros $0 < \mu_1 < \cdots < \mu_{m-1} < 1$. Moreover, the coefficients $\alpha_1, \dots, \alpha_m$ and $\beta_0, \dots, \beta_m$ have the form above. To complete the proof, apply Theorem 2.7 on the function $F : [0,1] \to \mathbb{R}$ defined by the formula

$$F(t) := f\big((1-t)a + tb\big).$$

□

2.4. **Applications.** In the particular setting when $m = 1$, Theorem 2.8 reduces to the classical Hermite–Hadamard inequality:

Corollary 2.1. *If $f : [a,b] \to \mathbb{R}$ is a polynomially 2-convex (i.e. convex) function, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In the subsequent corollaries we present Hermite–Hadamard-type inequalities in those cases when the zeros of the polynomials in Theorem 2.8 and Theorem 2.9 can explicitly be computed.

Corollary 2.2. *If $f : [a,b] \to \mathbb{R}$ is a polynomially 3-convex function, then the following inequalities hold*

$$\frac{1}{4} f(a) + \frac{3}{4} f\left(\frac{a+2b}{3}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{3}{4} f\left(\frac{2a+b}{3}\right) + \frac{1}{4} f(b).$$

Corollary 2.3. *If $f : [a,b] \to \mathbb{R}$ is a polynomially 4-convex function, then the following inequalities hold*

$$\frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}a+\frac{3-\sqrt{3}}{6}b\right)+\frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}a+\frac{3+\sqrt{3}}{6}b\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b).$$

Corollary 2.4. *If $f : [a,b] \to \mathbb{R}$ is a polynomially 5-convex function, then the following inequalities hold*

$$\begin{aligned}\frac{1}{9}f(a)&+\frac{16+\sqrt{6}}{36}f\left(\frac{4+\sqrt{6}}{10}a+\frac{6-\sqrt{6}}{10}b\right)\\ &+\frac{16-\sqrt{6}}{36}f\left(\frac{4-\sqrt{6}}{10}a+\frac{6+\sqrt{6}}{10}b\right)\\ &\leq \frac{1}{b-a}\int_a^b f(x)dx\\ &\leq \frac{16-\sqrt{6}}{36}f\left(\frac{6+\sqrt{6}}{10}a+\frac{4-\sqrt{6}}{10}b\right)\\ &+\frac{16+\sqrt{6}}{36}f\left(\frac{6-\sqrt{6}}{10}a+\frac{4+\sqrt{6}}{10}b\right)+\frac{1}{9}f(b).\end{aligned}$$

In some other cases analogous statements can be formulated applying Theorem 2.9. For simplicity, instead of writing down these corollaries explicitly, we shall present a list which contains the zeros of P_n (denoted by λ_k), the coefficients α_k for the left hand side inequality, also the zeros of Q_n (denoted by μ_k), and the coefficients β_k for the right hand side inequality, respectively.

Case $n = 6$

The zeros of P_3:

$$\frac{5-\sqrt{15}}{10},\quad \frac{1}{2},\quad \frac{5+\sqrt{15}}{10};$$

the corresponding coefficients:

$$\frac{5}{18},\quad \frac{4}{9},\quad \frac{5}{18}.$$

The zeros of Q_2:

$$\frac{5-\sqrt{5}}{10},\quad \frac{5+\sqrt{5}}{10};$$

the corresponding coefficients:

$$\frac{1}{12},\quad \frac{5}{12},\quad \frac{5}{12},\quad \frac{1}{12}.$$

Case $n = 8$

The zeros of P_4:

$$\frac{1}{2}-\frac{\sqrt{525+70\sqrt{30}}}{70},\quad \frac{1}{2}-\frac{\sqrt{525-70\sqrt{30}}}{70},$$
$$\frac{1}{2}+\frac{\sqrt{525-70\sqrt{30}}}{70},\quad \frac{1}{2}+\frac{\sqrt{525+70\sqrt{30}}}{70};$$

the corresponding coefficients:

$$\frac{1}{4}-\frac{\sqrt{30}}{72},\quad \frac{1}{4}+\frac{\sqrt{30}}{72},\quad \frac{1}{4}+\frac{\sqrt{30}}{72},\quad \frac{1}{4}-\frac{\sqrt{30}}{72}.$$

The zeros of Q_3:

$$\frac{1}{2}-\frac{\sqrt{21}}{14},\quad \frac{1}{2},\quad \frac{1}{2}+\frac{\sqrt{21}}{14};$$

the corresponding coefficients:

$$\frac{1}{20},\quad \frac{49}{180},\quad \frac{16}{45},\quad \frac{49}{180},\quad \frac{1}{20}.$$

Case $n=10$

The zeros of P_5:

$$\frac{1}{2}-\frac{\sqrt{245+14\sqrt{70}}}{42},\quad \frac{1}{2}-\frac{\sqrt{245-14\sqrt{70}}}{42},$$
$$\frac{1}{2},\quad \frac{1}{2}+\frac{\sqrt{245-14\sqrt{70}}}{42},\quad \frac{1}{2}+\frac{\sqrt{245+14\sqrt{70}}}{42};$$

the corresponding coefficients:

$$\frac{322-13\sqrt{70}}{1800},\quad \frac{322+13\sqrt{70}}{1800},\quad \frac{64}{225},\quad \frac{322+13\sqrt{70}}{1800},\quad \frac{322-13\sqrt{70}}{1800}.$$

The zeros of Q_4:

$$\frac{1}{2}-\frac{\sqrt{147+42\sqrt{7}}}{42},\quad \frac{1}{2}-\frac{\sqrt{147-42\sqrt{7}}}{42},$$
$$\frac{1}{2}+\frac{\sqrt{147-42\sqrt{7}}}{42},\quad \frac{1}{2}+\frac{\sqrt{147+42\sqrt{7}}}{42};$$

the corresponding coefficients:

$$\frac{1}{30},\quad \frac{14-\sqrt{7}}{60},\quad \frac{14+\sqrt{7}}{60},\quad \frac{14+\sqrt{7}}{60},\quad \frac{14-\sqrt{7}}{60},\quad \frac{1}{30}.$$

Case $n=12$ (right hand side inequality)

The zeros of Q_5:

$$\frac{1}{2}-\frac{\sqrt{495+66\sqrt{15}}}{66},\quad \frac{1}{2}-\frac{\sqrt{495-66\sqrt{15}}}{66},$$
$$\frac{1}{2},\quad \frac{1}{2}+\frac{\sqrt{495-66\sqrt{15}}}{66},\quad \frac{1}{2}+\frac{\sqrt{495+66\sqrt{15}}}{66};$$

the corresponding coefficients:

$$\frac{1}{42},\quad \frac{124-7\sqrt{15}}{700},\quad \frac{124+7\sqrt{15}}{700},\quad \frac{128}{525},$$
$$\frac{124+7\sqrt{15}}{700},\quad \frac{124-7\sqrt{15}}{700},\quad \frac{1}{42}.$$

During the investigations of the higher–order cases above, we can use the symmetry of the zeros of the orthogonal polynomials with respect to $1/2$, and therefore the calculations lead to solving linear or quadratic equations. The first case where "casus irreducibilis" appears is $n=7$; similarly, this is the reason for presenting only the right hand side inequality for polynomially 12-convex functions.

3. GENERALIZED 2-CONVEXITY

In terms of geometry, the Chebyshev property of a two dimensional system can equivalently be formulated: the linear combinations of the members of the system (briefly: *generalized lines*) are continuous; furthermore, any two points of the plain with distinct first coordinates can be connected by a unique generalized line. That is, generalized lines have the most important properties of affine functions. These properties turn out to be so strong that most of the classical results of standard convexity, can be generalized for this setting.

First we investigate some basic properties of generalized lines of two dimensional Chebyshev systems. Then the most important tool of the section, a characterization theorem is proved for generalized 2-convex functions. Two consequences of this theorem, namely the existence of generalized support lines and the property of generalized chords are crucial to verify Hermite–Hadamard-type inequalities. Another result states a tight connection between standard and (ω_1, ω_2)-convexity, and also guarantees the integrability of (ω_1, ω_2)-convex functions. Some classical results of the theory of convex functions, like their representation and stability are also generalized for this setting.

3.1. **Characterizations via generalized lines.** Let us recall that (ω_1, ω_2) is said to be a *Chebyshev system* over an interval I if $\omega_1, \omega_2 : I \to \mathbb{R}$ are continuous functions and, for all elements $x < y$ of I,

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0.$$

Some concrete examples on Chebyshev systems are presented in the last section of the section. Given a Chebyshev system (ω_1, ω_2), a function $f : I \to \mathbb{R}$ is called *generalized convex with respect to* (ω_1, ω_2) or briefly: *generalized* 2*-convex* if, for all elements $x < y < z$ of I, it satisfies the inequality

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geq 0.$$

Clearly, in the standard setting this definition reduces to the notion of (ordinary) convexity. Let (ω_1, ω_2) be a Chebyshev system on an interval I, and denote the set of all linear combinations of the functions ω_1 and ω_2 by $\mathscr{L}(\omega_1, \omega_2)$. We say that a function $\omega : I \to \mathbb{R}$ is a *generalized line* if it belongs to the linear hull $\mathscr{L}(\omega_1, \omega_2)$. The properties of generalized lines play the key role in our further investigations; first we need the following simple but useful ones.

Lemma 3.1. *Let (ω_1, ω_2) be a Chebyshev system over an interval I. Then, two different generalized lines of $\mathscr{L}(\omega_1, \omega_2)$ have at most one common point; moreover, if two different generalized lines have the same value at some $\xi \in I^\circ$, then the difference of the lines is positive on one side of ξ while negative on the other side of ξ. In particular, ω_1 and ω_2 have at most one zero; moreover, if ω_1 (resp., ω_2) vanishes at some $\xi \in I^\circ$, then ω_1 is positive on one side of ξ while negative on the other.*

Proof. Due to the linear structure of $\mathscr{L}(\omega_1, \omega_2)$, without loss of generality we may assume that one of the lines is the constant zero line. Then, the other generalized line ω has the representation $\alpha\omega_1 + \beta\omega_2$, with $\alpha^2 + \beta^2 > 0$.

The first assertion of the theorem is equivalent to the property that ω has at most one zero. To show this, assume indirectly that $\omega(\xi)$ and $\omega(\eta)$ equal zero for $\xi \neq \eta$; that is,

$$\alpha\omega_1(\xi) + \beta\omega_2(\xi) = 0,$$
$$\alpha\omega_1(\eta) + \beta\omega_2(\eta) = 0.$$

By the Chebyshev property of (ω_1, ω_2), the base determinant of the system is nonvanishing, therefore the system has only trivial solutions $\alpha = 0$ and $\beta = 0$ which contradicts the property $\alpha^2 + \beta^2 > 0$.

An equivalent formulation of the second assertion is the following: if $\omega(\xi) = 0$ for some interior point ξ, then $\omega > 0$ on one side of ξ while $\omega < 0$ on the other. If this is not true, then, according to the previous result and Bolzano's theorem, ω is strictly positive (or negative) on both sides of ξ. For simplicity, assume that $\omega(t) > 0$ for $t \neq \xi$. Define the generalized line ω^* by $\omega^* := -\beta\omega_1 + \alpha\omega_2$. Then, (ω, ω^*) is also a Chebyshev system: if $x < y$ are elements of I, then

$$\begin{aligned}\begin{vmatrix} \omega(x) & \omega(\xi) \\ \omega^*(x) & \omega^*(y) \end{vmatrix} &= \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} \cdot \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \\ &= (\alpha^2 + \beta^2) \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0.\end{aligned}$$

Therefore, taking the elements $x < \xi < y$ of I, we arrive at the inequalities

$$0 < \begin{vmatrix} \omega(x) & \omega(\xi) \\ \omega^*(x) & \omega^*(\xi) \end{vmatrix} = \omega(x)\omega^*(\xi),$$

$$0 < \begin{vmatrix} \omega(\xi) & \omega(y) \\ \omega^*(\xi) & \omega^*(y) \end{vmatrix} = -\omega(y)\omega^*(\xi),$$

which yields the contradiction that $\omega^*(\xi)$ is simultaneously positive and negative.

For the last assertion, notice that ω_1, ω_2 and the constant zero functions are special generalized lines and apply the previous part of the theorem. □

The most important property of $\mathscr{L}(\omega_1, \omega_2)$ guarantees the existence of a generalized line "parallel" to the constant zero function, which itself is a generalized line as well (see below). Moreover, as it can also be shown, $\mathscr{L}(\omega_1, \omega_2)$ fulfills the axioms of hyperbolic geometry.

Lemma 3.2. *If (ω_1, ω_2) is a Chebyshev system on an interval I, then there exists $\omega \in \mathscr{L}(\omega_1, \omega_2)$ such that ω is positive on I°.*

Proof. If ω_1 has no zero in I°, then $\omega := \omega_1$ or $\omega := -\omega_1$ (according to the sign of ω_1) will do. Suppose that $\omega_1(\xi) = 0$ for some $\xi \in I^\circ$. Due to Lemma 3.1, without loss of generality we may assume that

$$\begin{aligned} \omega_1(x) &< 0 \qquad (x < \xi,\ x \in I), \\ \omega_1(y) &> 0 \qquad (y > \xi,\ y \in I). \end{aligned}$$

Choose the elements $x < \xi < y$ of I. The Chebyshev property of (ω_1, ω_2) and the negativity of $\omega_1(x)\omega_2(y)$ implies the inequality

$$\frac{\omega_2(y)}{\omega_1(y)} < \frac{\omega_2(x)}{\omega_1(x)}.$$

Hence

$$\alpha := \sup_{y>\xi} \left[\frac{\omega_2(y)}{\omega_1(y)}\right] \le \inf_{x<\xi} \left[\frac{\omega_2(x)}{\omega_1(x)}\right]; \tag{3.1}$$

moreover, both sides are real numbers. We show that the generalized line defined by $\omega := \alpha\omega_1 - \omega_2$ is positive on the interior of I.

First observe that ω takes a positive value at the point ξ. Indeed, by the definition of ω we have $\omega(\xi) := \alpha\omega_1(\xi) - \omega_2(\xi) = -\omega_2(\xi)$; on the other hand, for $y > \xi$, the positivity of $\omega_1(y)$ and the Chebyshev property of (ω_1, ω_2) yields $-\omega_2(\xi) > 0$.

If $y > \xi$, then the definition of α implies

$$\alpha \geq \frac{\omega_2(y)}{\omega_1(y)};$$

multiplying both sides by the positive $\omega_1(y)$ and rearranging the terms we get, $\omega(y) := \alpha\omega_1(y) - \omega_2(y) \geq 0$.

If $x < \xi$, then inequality (3.1) gives that

$$\alpha \leq \frac{\omega_2(x)}{\omega_1(x)};$$

multiplying both sides by the negative $\omega_1(x)$ and rearranging the obtained terms, we arrive at the inequality $\omega(x) := \alpha\omega_1(x) - \omega_2(x) \geq 0$.

To complete the proof, it suffices to show that ω always differs from zero on the interior of the domain. Assume indirectly that $\omega(\eta) := \alpha\omega_1(\eta) - \omega_2(\eta) = 0$ for some $\eta \in I^\circ$. Clearly, $\eta \neq \xi$ since $\omega(\xi) > 0$. Therefore, $\omega_1(\eta) \neq 0$ and α can be expressed explicitly:

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)}.$$

If $\xi < \eta$, choose $y \in I$ such that $\eta < y$ hold. By the positivity of $\omega_1(\eta)\omega_1(y)$ and the Chebyshev property of (ω_1, ω_2),

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)} < \frac{\omega_2(y)}{\omega_1(y)}$$

which contradicts the definition of α. Similarly, if $\xi > \eta$, choose $x \in I$ such that $x < \eta$ is valid. Then, the positivity of $\omega_1(x)\omega_1(\eta)$ and the Chebyshev property of (ω_1, ω_2) imply the inequality

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)} > \frac{\omega_2(x)}{\omega_1(x)},$$

which contradicts (3.1). □

As an important consequence of Lemma 3.2, a Chebyshev system can always be replaced equivalently by a “regular” one. In other words, assuming positivity on the first component of a Chebyshev system, as is required in many further results, is not an essential restriction. Moreover, the next lemma also gives a characterization of pairs of functions to form a Chebyshev system.

Lemma 3.3. *Let (ω_1, ω_2) be a Chebyshev system on an interval $I \subset \mathbb{R}$. Then, there exists a Chebyshev system (ω_1^*, ω_2^*) on I that possesses the following properties:*

(i) *ω_1^* is positive on I°;*
(ii) *ω_2^*/ω_1^* is strictly monotone increasing on I°;*
(iii) *(ω_1, ω_2)-convexity is equivalent to (ω_1^*, ω_2^*)-convexity.*

Conversely, if $\omega_1, \omega_2 : I \to \mathbb{R}$ are continuous functions such that ω_1 is positive and ω_2/ω_1 is strictly monotone increasing, then (ω_1, ω_2) is a Chebyshev system over I.

Proof. Lemma 3.2 guarantees the existence of real constants α and β such that $\alpha\omega_1 + \beta\omega_2 > 0$ holds for all $x \in I^\circ$. Define the functions $\omega_1^*, \omega_2^* : I \to \mathbb{R}$ by the formulae

$$\omega_1^* := \alpha\omega_1 + \beta\omega_2, \qquad \omega_2^* := -\beta\omega_1 + \alpha\omega_2.$$

Choosing the elements $x < y$ of I and applying the product rule of determinants, we get

$$\begin{aligned} \begin{vmatrix} \omega_1^*(x) & \omega_1^*(y) \\ \omega_2^*(x) & \omega_2^*(y) \end{vmatrix} &= \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} \cdot \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \\ &= (\alpha^2+\beta^2) \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0. \end{aligned}$$

Therefore, (ω_1^*, ω_2^*) is also a Chebyshev system over I. Assuming that ω_1^* is positive, as it can easily be checked, the Chebyshev property of (ω_1^*, ω_2^*) yields that the function ω_2^*/ω_1^* is strictly monotone increasing on the interior of I.

Lastly, let $f : I \to \mathbb{R}$ be an arbitrary function and $x < y < z$ be arbitrary elements of I. Then, by the product rule of determinants,

$$\begin{aligned} \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1^*(x) & \omega_1^*(y) & \omega_1^*(z) \\ \omega_2^*(x) & \omega_2^*(y) & \omega_2^*(z) \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{vmatrix} \cdot \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \\ &= (\alpha^2+\beta^2) \cdot \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix}, \end{aligned}$$

which shows that the function f is generalized convex with respect to the Chebyshev system (ω_1, ω_2) if and only if it is generalized convex with respect to the Chebyshev system (ω_1^*, ω_2^*).

The proof of the converse assertion is a simple calculation, therefore it is omitted. $\square$

The following result gives various characterizations of (ω_1, ω_2)-convexity via the monotonicity of the generalized divided difference, the generalized support property and the "local" and the "global" generalized chord properties.

Theorem 3.1. *Let (ω_1, ω_2) be a Chebyshev system over an interval I such that ω_1 is positive on I°. The following statements are equivalent:*

(i) *$f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex;*

(ii) *for all elements $x < y < z$ of I we have that*

$$\frac{\begin{vmatrix} f(y) & f(z) \\ \omega_1(y) & \omega_1(z) \end{vmatrix}}{\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix}} \le \frac{\begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}};$$

(iii) *for all $x_0 \in I^\circ$ there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\begin{aligned} \alpha\omega_1(x_0) + \beta\omega_2(x_0) &= f(x_0), \\ \alpha\omega_1(x) + \beta\omega_2(x) &\le f(x) \qquad (x \in I); \end{aligned}$$

(iv) *for all $n \in \mathbb{N}$, $x_0, x_1, \dots, x_n \in I$ and $\lambda_1, \dots, \lambda_n \ge 0$ satisfying the conditions*

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k \omega_1(x_k) &= \omega_1(x_0), \\ \sum_{k=1}^{n} \lambda_k \omega_2(x_k) &= \omega_2(x_0), \end{aligned}$$

we have that

$$f(x_0) \le \sum_{k=1}^{n} \lambda_k f(x_k);$$

(v) *for all* $x_0, x_1, x_2 \in I$ *and* $\lambda_1, \lambda_2 \geq 0$ *satisfying the conditions*

$$\lambda_1\omega_1(x_1) + \lambda_2\omega_1(x_2) = \omega_1(x_0),$$
$$\lambda_1\omega_2(x_1) + \lambda_2\omega_2(x_2) = \omega_2(x_0),$$

we have that

$$f(x_0) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

(vi) *for all elements* $x < p < y$ *of* I

$$f(p) \leq \alpha\omega_1(p) + \beta\omega_2(p),$$

where the constants α, β *are the solutions of the system of linear equations*

$$f(x) = \alpha\omega_1(x) + \beta\omega_2(x),$$
$$f(y) = \alpha\omega_1(y) + \beta\omega_2(y).$$

Proof. $(i) \Rightarrow (ii)$. Assume indirectly that (ii) is not true. Then, considering the positivity of the denominators, there exist elements $x < y < z$ of I such that the inequality

$$\begin{vmatrix} f(y) & f(z) \\ \omega_1(y) & \omega_1(z) \end{vmatrix} \cdot \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > \begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix} \cdot \begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix}$$

holds or equivalently,

$$f(y)\left(\omega_1(x)\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix} + \omega_1(z)\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}\right)$$
$$> \omega_1(y)\left(f(x)\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix} + f(z)\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}\right).$$

Subtracting

$$f(y)\omega_1(y)\begin{vmatrix} \omega_1(x) & \omega_1(z) \\ \omega_2(x) & \omega_2(z) \end{vmatrix}$$

from both sides and applying the expansion theorem "backwards", we get

$$f(y)\begin{vmatrix} \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} > \omega_1(y)\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix}.$$

The (ω_1, ω_2)-convexity of f implies that the right hand side of the inequality is nonnegative, while the left hand side equals zero, which is a contradiction.

$(ii) \Rightarrow (iii)$. Fix $x_0 \in I^\circ$. Then, for all elements $\xi < x_0 < x$ of I,

$$-\frac{\begin{vmatrix} f(\xi) & f(x_0) \\ \omega_1(\xi) & \omega_1(x_0) \end{vmatrix}}{\begin{vmatrix} \omega_1(\xi) & \omega_1(x_0) \\ \omega_2(\xi) & \omega_2(x_0) \end{vmatrix}} \leq -\frac{\begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix}}$$

holds, therefore

$$\beta := \inf_{x > x_0}\left[-\frac{\begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix}}\right]$$

is a real number. The positivity assumption on ω_1 guarantees that the coefficient α can be chosen such that $\alpha\omega_1(x_0) + \beta\omega_2(x_0) = f(x_0)$ is satisfied. Rewrite the desired inequality $\alpha\omega_1(x) + \beta\omega_2(x) \le f(x)$ in the equivalent form

$$\beta \begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix} + \begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix} \le 0. \tag{3.2}$$

The definition of β guarantees that it is valid if $x_0 < x$. Assume that $x < x_0$ and choose $\xi \in I$ such that $x < x_0 < \xi$ hold. Then, applying (ii), we have the inequality

$$\frac{\begin{vmatrix} f(x_0) & f(\xi) \\ \omega_1(x_0) & \omega_1(\xi) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\xi) \\ \omega_2(x_0) & \omega_2(\xi) \end{vmatrix}} \le \frac{\begin{vmatrix} f(x) & f(x_0) \\ \omega_1(x) & \omega_1(x_0) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(x_0) \\ \omega_2(x) & \omega_2(x_0) \end{vmatrix}}.$$

Observe that the denominator of the right hand side is positive, therefore, after rearranging this inequality, we get

$$-\frac{\begin{vmatrix} f(x_0) & f(\xi) \\ \omega_1(x_0) & \omega_1(\xi) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\xi) \\ \omega_2(x_0) & \omega_2(\xi) \end{vmatrix}} \begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix} + \begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix} \le 0,$$

which, and the choice of β immediately implies (3.2).

$(iii) \Rightarrow (iv)$. First assume that $x_0 = x_1 = \cdots = x_n$. We recall that $\omega_1(x_0)$ and $\omega_2(x_0)$ cannot be equal to zero simultaneously due to Lemma 3.1; therefore one of the conditions gives the identity $\sum_{k=1}^{n} \lambda_k = 1$, and the inequality to be proved trivially holds. If $x_0, x_1, \dots, x_n$ are distinct points of I, then it necessarily follows $x_0 \in I^\circ$. Indeed, if $\inf(I) \in I$ and indirectly $x_0 = \inf(I)$, then we have the inequalities

$$\omega_1(x_0)\omega_2(x_k) - \omega_1(x_k)\omega_2(x_0) \ge 0$$

for all $k = 1, \dots, n$ since (ω_1, ω_2) is a Chebyshev system on I; furthermore, at least one of them is strict. Multiplying the k^{th} inequality by the positive λ_k and summing from 1 to n, we obtain

$$\omega_1(x_0) \sum_{k=1}^{n} \lambda_k \omega_2(x_k) > \omega_2(x_0) \sum_{k=1}^{n} \lambda_k \omega_1(x_k).$$

But, due to the conditions, both sides have the common value $\omega_1(x_0)\omega_2(x_0)$, which is a contradiction. An analogous argument gives that the case $x_0 = \sup(I)$ is also impossible, therefore it follows that $x_0 \in I^\circ$.

Choose $\alpha, \beta \in \mathbb{R}$ so that the relations

$$\begin{aligned} \alpha\omega_1(x_0) + \beta\omega_2(x_0) &= f(x_0), \\ \alpha\omega_1(x) + \beta\omega_2(x) &\le f(x) \qquad (x \in I) \end{aligned}$$

are valid. Then, substituting $x = x_k$ into the last inequality and applying the conditions, we get that

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k f(x_k) &\ge \sum_{k=1}^{n} \lambda_k \alpha\omega_1(x_k) + \sum_{k=1}^{n} \lambda_k \beta\omega_2(x_k) \\ &= \alpha\omega_1(x_0) + \beta\omega_2(x_0) = f(x_0), \end{aligned}$$

which gives the desired implication.

$(iv) \Rightarrow (v)$. Taking the particular case $n = 2$ in (iv), we arrive at (v).

$(v) \Rightarrow (vi)$. According to Cramer's rule, for all elements $x < p < y$ of I, the system of linear equations

$$\lambda_1\omega_1(x) + \lambda_2\omega_1(y) = \omega_1(p),$$
$$\lambda_1\omega_2(x) + \lambda_2\omega_2(y) = \omega_2(p),$$

has unique nonnegative solutions λ_1 and λ_2. Therefore, using the definition of α and β,

$$\begin{aligned} f(p) &\le \lambda_1 f(x) + \lambda_2 f(y) \\ &= \lambda_1\big(\alpha\omega_1(x) + \beta\omega_2(x)\big) + \lambda_2\big(\alpha\omega_1(y) + \beta\omega_2(y)\big) \\ &= \alpha\big(\lambda_1\omega_1(x) + \lambda_2\omega_1(y)\big) + \beta\big(\lambda_1\omega_2(x) + \lambda_2\omega_2(y)\big) \\ &= \alpha\omega_1(p) + \alpha\omega_2(p). \end{aligned}$$

$(vi) \Rightarrow (i)$. Expressing the unknowns α and β with $\omega_j(x), \omega_j(y)$ and $\omega_j(p)$, the inequality $f(p) \le \alpha\omega_1(p) + \beta\omega_2(p)$ can be rewritten into the form

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} f(p) \le \begin{vmatrix} f(x) & f(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \omega_1(p) + \begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix} \omega_2(p)$$

or equivalently

$$0 \le \begin{vmatrix} f(x) & f(p) & f(y) \\ \omega_1(x) & \omega_1(p) & \omega_1(y) \\ \omega_2(x) & \omega_2(p) & \omega_2(y) \end{vmatrix},$$

which completes the proof. $\square$

In the particular setting where $\omega_1(x) := 1$ and $\omega_2(x) := x$, this theorem reduces to the well known characterization properties of standard convex functions. Now the last two assertions coincide: both of them state that the function's graph is under the chord joining the endpoints of the graph. Let us note that in most of the literature, the notion of (standard) convexity is defined exactly by this property (see the last assertion of the obtained corollary).

Corollary 3.1. *Let $I \subset \mathbb{R}$ be an interval. The following statements are equivalent:*

(i) *$f : I \to \mathbb{R}$ is convex (in the standard sense);*

(ii) *for all elements $x < y < z$ of I we have that*

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y};$$

(iii) *for all $x_0 \in I^\circ$ there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha + \beta x_0 = f(x_0), \qquad \alpha + \beta x \le f(x) \qquad (x \in I);$$

(iv) *for all $n \in \mathbb{N}$, $x_0, x_1, \dots, x_n \in I$ and $\lambda_1, \dots, \lambda_n \ge 0$ satisfying the conditions*

$$\sum_{k=1}^{n} \lambda_k = 1, \qquad \sum_{k=1}^{n} \lambda_k x_k = x_0,$$

we have that

$$f(x_0) \le \sum_{k=1}^{n} \lambda_k f(x_k);$$

(v) *for all $x_0, x_1, x_2 \in I$ and $\lambda_1, \lambda_2 \ge 0$ satisfying the conditions*

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 x_1 + \lambda_2 x_2 = x_0,$$

we have that

$$f(x_0) \le \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

If the base functions ω_1 and ω_2 are twice differentiable with a positive Wronski determinant, then a twice differentiable function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex if and only if the Wronski determinant of the system (f, ω_1, ω_2) is nonnegative (Bonsall, [2]). This result can also be deduced from Theorem 3.1.

As it is well known, (standard) convex functions are exactly those ones that can be obtained as the pointwise supremum of families of affine functions. As a direct consequence (and also another application) of Theorem 3.1, an analogous statement holds for (ω_1, ω_2)-convex functions.

Corollary 3.2. *Let (ω_1, ω_2) be a Chebyshev system over an open interval I. Then, a function $f : I \to \mathbb{R}$ is generalized convex with respect to (ω_1, ω_2) if and only if*

$$f(x) = \sup\{\, \omega(x) \,|\, \omega \in \mathscr{L}(\omega_1, \omega_2),\ \omega \le f \,\}.$$

Proof. Assertion (iii) of Theorem 3.1 immediately implies the representation above. For the sufficiency part, assertion (v) of Theorem 3.1 is applied. Fix the element x_0 of the open interval I. Take a generalized line $\omega = \alpha\omega_1 + \beta\omega_2$ such that $\omega \le f$, with the elements x_1, x_2 of I and the nonnegative coefficients λ_1, λ_2 that fulfill the conditions

$$\begin{aligned} \lambda_1\omega_1(x_1) + \lambda_2\omega_1(x_2) &= \omega_1(x_0) \\ \lambda_1\omega_2(x_1) + \lambda_2\omega_2(x_2) &= \omega_2(x_0). \end{aligned}$$

Then,

$$\begin{aligned} \lambda_1 f(x_1) + \lambda_2 f(x_2) &\ge \lambda_1\omega(x_1) + \lambda_2\omega(x_2) \\ &= \lambda_1\big(\alpha\omega_1(x_1) + \beta\omega_2(x_1)\big) + \lambda_2\big(\alpha\omega_1(x_2) + \beta\omega_2(x_2)\big) \\ &= \alpha\big(\lambda_1\omega_1(x_1) + \lambda_2\omega_1(x_2)\big) + \beta\big(\lambda_1\omega_2(x_1) + \lambda_2\omega_2(x_2)\big) \\ &= \alpha\omega_1(x_0) + \beta\omega_2(x_0) = \omega(x_0). \end{aligned}$$

That is, $\lambda_1 f(x_1) + \lambda_2 f(x_2) \ge \omega(x_0)$ for all $\omega \le f$, hence, according to the representation, it follows that $\lambda_1 f(x_1) + \lambda_2 f(x_2) \ge f(x_0)$. Therefore f is convex with respect to (ω_1, ω_2). □

3.2. **Connection with standard convexity.** The convexity notion induced by two dimensional Chebyshev systems turns out to be always reducible to standard convexity with the help of a composite function. This connection enables us to generalize many classical results for the case of (ω_1, ω_2)-convexity directly.

Theorem 3.2. *Let (ω_1, ω_2) be a Chebyshev system on an open interval I such that ω_1 is positive. The function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex if and only if the function $g : \omega_2/\omega_1(I) \to \mathbb{R}$ defined by the formula*

$$g := \frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1}$$

is convex in the standard sense.

Proof. In this case the function ω_2/ω_1 is continuous and strictly monotone increasing, according to Lemma 3.3. Therefore, the image of the interval I by the function ω_2/ω_1 is a nonempty open interval. Consider the identity

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix}$$
$$= \omega_1(x)\omega_1(y)\omega_1(z) \begin{vmatrix} (f/\omega_1)(x) & (f/\omega_1)(y) & (f/\omega_1)(z) \\ 1 & 1 & 1 \\ (\omega_2/\omega_1)(x) & (\omega_2/\omega_1)(y) & (\omega_2/\omega_1)(z) \end{vmatrix}$$
$$= \omega_1(x)\omega_1(y)\omega_1(z) \begin{vmatrix} g(u) & g(v) & g(w) \\ 1 & 1 & 1 \\ u & v & w \end{vmatrix}$$

where

$$u = (\omega_2/\omega_1)(x) \qquad v = (\omega_2/\omega_1)(y) \qquad w = (\omega_2/\omega_1)(z).$$

The positivity of ω_1 forces both sides to be simultaneously positive, negative or zero. That is, the function f is (ω_1, ω_2)-convex if and only if the function g is convex in the standard sense. $\square$

Theorem 3.2 yields strong regularity properties for generalized convexity. For example, (ω_1, ω_2)-convex functions defined on compact intervals are integrable, which is essential in formulating the main result of the section.

Theorem 3.3. *Let (ω_1, ω_2) be a Chebyshev system on an interval I. If a function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex, then it is continuous on I°. Moreover, f is bounded on each compact subinterval of I.*

Proof. Without loss of generality we may assume that ω_1 is positive on I°. If the function f is (ω_1, ω_2)-convex on I, then the composite function g in the previous theorem is convex in the standard sense on $J := \omega_2/\omega_1(I)$. Therefore, by the well known regularity properties of convex functions, g is continuous on J°. On the other hand, we have that

$$f = \omega_1 \cdot g \circ \left(\frac{\omega_2}{\omega_1}\right),$$

and the right hand side is continuous on I° whence the continuity of the function f follows.

To prove that f is bounded on the compact subinterval $[a, b]$ of I, we shall apply Theorem 3.1. Take a generalized line which supports f at an arbitrary point $x_0 \in I^\circ$. Then, inequality (iii) implies that f is bounded from below on the *whole* interval I. On the other hand, putting $x := a$ and $y := b$ into (vi), we get that f is also bounded by a certain generalized line from above on $[a, b]$. Hence f is bounded. $\square$

Definition 3.1. *Let (ω_1, ω_2) be a Chebyshev system on an interval I, and $\omega \in \mathscr{L}(\omega_1, \omega_2)$ a generalized line which is positive on I°. A function $f : I \to \mathbb{R}$ is called* generalized ω-convex with respect to (ω_1, ω_2) *if, for all elements $x < y < z$ of I, the following inequality holds:*

$$\begin{vmatrix} f(x) + \omega(x) & f(y) - \omega(y) & f(z) + \omega(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geq 0.$$

Substituting $\omega_1(x) := 1$, $\omega_2(x) := x$ and $\omega := \varepsilon/2$, the definition gives the notion of ε-convexity. By well known results, ε-convexity is stable: every ε-convex function is "close" to a (standard) convex function. As another application of Theorem 3.2, we prove an analogous result for (ω_1, ω_2)-convex functions.

Corollary 3.3. *Let (ω_1, ω_2) be a Chebyshev system on an interval I furthermore $\omega \in \mathscr{L}(\omega_1, \omega_2)$ be a generalized line which is positive on I°. A function $f : I \to \mathbb{R}$ is generalized ω-convex with respect to (ω_1, ω_2) if and only if there exist functions $f, g : I \to \mathbb{R}$ such that g is (ω_1, ω_2)-convex, $\|h\| \le \|\omega\|$, and $f = g + h$.*

Proof. Assume that ω has the representation $\omega = \alpha\omega_1 + \beta\omega_2$ and define the generalized lines ω_1^* and ω_2^* by $\omega_1^* := \alpha\omega_1 + \beta\omega_2$ and $\omega_2^* := -\beta\omega_1 + \alpha\omega_2$, respectively. Then, according to Lemma 3.3, the function ω_2^*/ω_1^* is strictly monotone increasing and the generalized ω-convexity of f is equivalent to the inequality

$$\begin{vmatrix} f(x) + \omega_1^*(x) & f(y) - \omega_1^*(y) & f(z) + \omega_1^*(z) \\ \omega_1^*(x) & \omega_1^*(y) & \omega_1^*(z) \\ \omega_2^*(x) & \omega_2^*(y) & \omega_2^*(z) \end{vmatrix} \ge 0.$$

Dividing both sides by the positive $\omega_1^*(x)\omega_1^*(y)\omega_1^*(z)$, then substituting the arguments $u = (\omega_2^*/\omega_1^*)(x)$, $v = (\omega_2^*/\omega_1^*)(y)$ and $w = (\omega_2^*/\omega_1^*)(z)$, we get the inequality

$$\begin{vmatrix} F(u) + 1 & F(v) - 1 & F(w) + 1 \\ 1 & 1 & 1 \\ u & v & w \end{vmatrix} \ge 0$$

where

$$F := \frac{f}{\omega_1^*} \circ \left(\frac{\omega_2^*}{\omega_1^*}\right)^{-1}.$$

That is, F satisfies the inequality of ε-convexity with $\varepsilon = 1$. Therefore, there exist functions $G, H : I \to \mathbb{R}$ such that G is convex (in the standard sense), $\|H\| \le 1$ and $F = G + H$ or equivalently,

$$f = \omega_1^* \cdot G \circ \left(\frac{\omega_2^*}{\omega_1^*}\right) + \omega_1^* \cdot H \circ \left(\frac{\omega_2^*}{\omega_1^*}\right) =: g + h.$$

Then, Theorem 3.2 and Lemma 3.3 guarantee the (ω_1, ω_2)-convexity of g, while simple calculations imply $\|h\| \le \|\omega\|$. $\square$

3.3. **Hermite–Hadamard-type inequalities.** The main result provides Hermite–Hadamard-type inequalities for generalized 2-convex functions.

Theorem 3.4. *Let (ω_1, ω_2) be a Chebyshev system on an interval $[a, b]$ such that ω_1 is positive on $]a, b[$, furthermore, let $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Define the point ξ and the coefficients c, c_1, c_2 by the formulae*

$$\xi = \left(\frac{\omega_2}{\omega_1}\right)^{-1}\left(\frac{\int_a^b \omega_2\rho}{\int_a^b \omega_1\rho}\right), \qquad c = \frac{\int_a^b \omega_1\rho}{\omega_1(\xi)}$$

and

$$c_1 = \frac{\begin{vmatrix} \int_a^b \omega_1\rho & \omega_1(b) \\ \int_a^b \omega_2\rho & \omega_2(b) \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} \omega_1(a) & \int_a^b \omega_1\rho \\ \omega_2(a) & \int_a^b \omega_2\rho \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}.$$

If $f : [a, b] \to \mathbb{R}$ is an (ω_1, ω_2)-convex function, then the following Hermite–Hadamard-type inequality holds

$$cf(\xi) \le \int_a^b f\rho \le c_1 f(a) + c_2 f(b).$$

Proof. By the definitions of the point ξ and the constant c, we have the formulae

$$\frac{\int_a^b \omega_2\rho}{\int_a^b \omega_1\rho} = \frac{\omega_2(\xi)}{\omega_1(\xi)}$$

and

$$\int_a^b \omega_1\rho = c\omega_1(\xi),$$

which yields the identity

$$\int_a^b \omega_2\rho = c\omega_2(\xi).$$

That is, the left hand side of the Hermite–Hadamard-type inequality to be proved is exact for $f = \omega_1$ and $f = \omega_2$, respectively. Let $f : [a,b] \to \mathbb{R}$ be an arbitrary (ω_1,ω_2)-convex function and choose $\alpha, \beta \in \mathbb{R}$ such that the relations

$$\begin{aligned}\alpha\omega_1(\xi) + \beta\omega_2(\xi) &= f(\xi),\\ \alpha\omega_1(x) + \beta\omega_2(x) &\le f(x),\end{aligned}$$

are satisfied for all $x \in [a,b]$. By Theorem 3.1, such real numbers exist since ξ is an interior point of the domain. Multiplying the last inequality by the positive weight function ρ, we arrive at

$$\int_a^b f\rho \ge \alpha\int_a^b \omega_1\rho + \beta\int_a^b \omega_2\rho = \alpha\big(c\omega_1(\xi)\big) + \beta\big(c\omega_2(\xi)\big) = cf(\xi)$$

which results in the left hand side inequality.

To verify the right hand side one, observe first that the coefficients c_1 and c_2 are the solutions of the following system of linear equations

$$\begin{aligned}\int_a^b \omega_1\rho &= c_1\omega_1(a) + c_2\omega_1(b),\\ \int_a^b \omega_2\rho &= c_1\omega_1(a) + c_2\omega_2(b).\end{aligned}$$

In other words, the right hand side of the Hermite–Hadamard-inequality is exact, again, for $f = \omega_1$ and $f = \omega_2$. Let $f : [a,b] \to \mathbb{R}$ be an arbitrary (ω_1,ω_2)-convex function. By Theorem 3.1, if the real numbers α and β are the solutions of the system of linear equations

$$\begin{aligned}f(a) &= \alpha\omega_1(a) + \beta\omega_2(a),\\ f(b) &= \alpha\omega_1(b) + \beta\omega_2(b),\end{aligned}$$

then

$$f(x) \le \alpha\omega_1(x) + \beta\omega_2(x)$$

for all $x \in [a,b]$. Multiplying this inequality by the positive weight function ρ, we get that

$$\begin{aligned}\int_a^b f\rho &\le \alpha\int_a^b \omega_1\rho + \beta\int_a^b \omega_2\rho\\ &= \alpha\big(c_1\omega_1(a) + c_2\omega_1(b)\big) + \beta\big(c_1\omega_2(a) + c_2\omega_2(b)\big)\\ &= c_1\big(\alpha\omega_1(a) + \beta\omega_2(a)\big) + c_2\big(\alpha\omega_1(b) + \beta\omega_2(b)\big) = c_1f(a) + c_2f(b),\end{aligned}$$

thus the proof is complete. $\square$

3.4. **Applications.** Simple calculations show that by setting $\omega_1(x) := 1$, $\omega_2(x) := x$ and $\rho \equiv 1$, Theorem 3.4 reduces to the classical Hermite–Hadamard inequality.

Corollary 3.4. *If $f : [a,b] \to \mathbb{R}$ is a (standard) convex function, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

The subsequent corollaries present further Hermite–Hadamard-type inequalities for generalized convex functions where the underlying Chebyshev systems of the induced convexity are the hyperbolic, trigonometric, exponential and power systems (to see that the pairs (ω_1, ω_2) form a Chebyshev system in each case, consult the converse part of Lemma 3.3).

Corollary 3.5. *If $f : [a,b] \to \mathbb{R}$ is a $(\cosh, \sinh)$-convex function, then*

$$2\sinh\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le \tanh\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

Proof. If $\omega_1 := \cosh$ and $\omega_2 := \sinh$, then ω_1 is positive and $\omega_2/\omega_1 = \tanh$ is strictly monotone increasing; hence, according to Lemma 3.3, (ω_1, ω_2) is a Chebyshev system and $(\omega_2/\omega_1)^{-1} =$ artanh. Applying the addition properties of hyperbolic functions for the identities $b = (b+a)/2 + (b-a)/2$ and $a = (b+a)/2 - (b-a)/2$, the integrals of ω_1 and ω_2 can be written into product form via the formulae

$$\int_a^b \cosh x dx = \sinh(b) - \sinh(a) = 2\cosh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right),$$
$$\int_a^b \sinh x dx = \cosh(b) - \cosh(a) = 2\sinh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right).$$

Therefore,

$$\xi = \text{artanh}\left(\frac{\int_a^b \sinh x dx}{\int_a^b \cosh x dx}\right) = \frac{b+a}{2}$$

and

$$c = \frac{\int_a^b \cosh x dx}{\cosh \xi} = 2\sinh\frac{b+a}{2}.$$

To determine the coefficients of the right hand side, first we calculate the numerator of c_1:

$$\begin{vmatrix} 2\cosh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right) & \cosh b \\ 2\sinh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right) & \sinh b \end{vmatrix}$$
$$= 2\sinh\left(\frac{b-a}{2}\right)\left(\cosh\left(\frac{b+a}{2}\right)\sinh b - \sinh\left(\frac{b+a}{2}\right)\cosh b\right)$$
$$= 2\sinh\left(\frac{b-a}{2}\right)\sinh\left(b - \frac{b+a}{2}\right) = 2\sinh^2\left(\frac{b-a}{2}\right).$$

Similarly, the numerator of the coefficient c_2 can be obtained as follows:

$$\begin{vmatrix} \cosh a & 2\cosh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right) \\ \sinh a & 2\sinh\left(\frac{b+a}{2}\right)\sinh\left(\frac{b-a}{2}\right) \end{vmatrix}$$
$$= 2\sinh\left(\frac{b-a}{2}\right)\left(\sinh\left(\frac{b+a}{2}\right)\cosh a - \cosh\left(\frac{b+a}{2}\right)\sinh a\right)$$
$$= 2\sinh\left(\frac{b-a}{2}\right)\sinh\left(\frac{b+a}{2} - a\right) = 2\sinh^2\left(\frac{b-a}{2}\right).$$

On the other hand, the denominators in both cases coincide and have the common value

$$\begin{vmatrix} \cosh a & \cosh b \\ \sinh a & \sinh b \end{vmatrix} = \sinh(b-a) = 2\sinh\left(\frac{b-a}{2}\right)\cosh\left(\frac{b-a}{2}\right),$$

therefore

$$c_1 = c_2 = \tanh\left(\frac{b-a}{2}\right).$$

□

Replacing the Chebyshev system $(\cosh, \sinh)$ with $(\cos, \sin)$, the obtained Hermite–Hadamard-type inequality is analogous to the previous one due to the similar additional properties of trigonometric and hyperbolic functions.

Corollary 3.6. *If* $f : [a,b] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ *is a* $(\cos, \sin)$*-convex function, then*

$$2\sin\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le \tan\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

Observe that both of the previous two Hermite–Hadamard-type inequalities involve the midpoint of the domain; moreover, dividing by $b-a$ and taking the limit $a \to b$, the coefficient of the left hand sides tends to 1, while the coefficient of the right hand sides tends to $1/2$. Therefore these inequalities can be considered as the “local” version of the Hermite–Hadamard inequality.

We say that a function $f : I \to \mathbb{R}$ is *log-convex* if the composite function $f \circ \log : \exp(I) \to \mathbb{R}$ is convex (in the standard sense). In terms of generalized convexity, log-convex functions are exactly the $(1, \exp)$-convex ones (consult Theorem 3.2). The next corollary gives a Hermite–Hadamard-type inequality for log-convex functions ([9], [10]).

Corollary 3.7. *If* $f : [a,b] \to \mathbb{R}$ *is a* $(1, \exp)$*-convex function, then*

$$\begin{aligned}(b-a)f&\left(\log\frac{\exp(b)-\exp(a)}{b-a}\right)\\ &\le \int_a^b f(x)dx\\ &\le \left(\frac{(b-a)\exp(b)}{\exp(b)-\exp(a)} - 1\right)f(a) + \left(1 - \frac{(b-a)\exp(a)}{\exp(b)-\exp(a)}\right)f(b).\end{aligned}$$

The last corollary concerning the case of “power convexity” also reduces to the classical Hermite–Hadamard inequality on substituting $p = 0$ and $q = 1$:

Corollary 3.8. *If* $p < q,\ p, q \ne -1$ *and* $f : [a,b] \subset]0, \infty[\to \mathbb{R}$ *is an* (x^p, x^q)*-convex function, then*

$$\begin{aligned}\left(\frac{b^{p+1}-a^{p+1}}{p+1}\right)^q &\left(\frac{q+1}{b^{q+1}-a^{q+1}}\right)^p f\left(\sqrt[q-p]{\frac{(p+1)(b^{q+1}-a^{q+1})}{(q+1)(b^{p+1}-a^{p+1})}}\right)\\ &\le \int_a^b f(x)dx\\ &\le \frac{\frac{(b^{p+1}-a^{p+1})b^q}{p+1} - \frac{(b^{q+1}-a^{q+1})b^p}{q+1}}{a^pb^q - a^qb^p} f(a) + \frac{\frac{(b^{q+1}-a^{q+1})a^p}{q+1} - \frac{(b^{p+1}-a^{p+1})a^q}{q+1}}{a^pb^q - a^qb^p} f(b).\end{aligned}$$

The proofs of the last three corollaries need similar calculations as the first one, therefore they are omitted.

4. GENERALIZED CONVEXITY INDUCED BY CHEBYSHEV SYSTEMS

In this section we formulate Hermite–Hadamard-type inequalities for generalized convex functions where the underlying Chebyshev system of the induced convexity is *arbitrary*. The proofs of the main results are based on the Krein–Markov theory of moment spaces induced by Chebyshev systems. According to this theory, the vector integral of a Chebyshev system can uniquely be represented as the linear combination of the values of the system in certain base points of the domain. The number of the points and also the points themselves, depend only on the Chebyshev system and its dimension: it turns out that the cases of odd and even order convexity must be investigated separately. In fact, this is exactly the deeper reason for the analogous phenomenon in the case of polynomial convexity. Once the base points of the representations are determined, its coefficients are obtained as the solutions of a system of linear equations. With the help of the representations and the notion of generalized convexity, the Hermite–Hadamard-type inequalities can be verified using integration and pure linear algebraic methods.

In the previous sections when the basis or the dimension of the studied Chebyshev systems was quite special, the base points of the Hermite–Hadamard-type inequalities could be explicitly given. Unfortunately, under the present general circumstances, we can guarantee only the *existence* (and the uniqueness) of the base points, but *cannot give any explicit formulae for them*.

Lastly, motivated by Rolle's mean-value theorem, an alternative and elementary approach is presented for the cases when the Hermite–Hadamard-type inequalities involve at most one interior base point of the domain. Some examples are also presented of these particular cases.

4.1. **Characterizations and regularity properties.** Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a Chebyshev system over an interval I and denote the set of all linear combinations of its members by $\mathscr{L}(\omega_1, \dots, \omega_n)$. A function is called *generalized polynomial* (belonging to the system in question) if it is the element of the linear span $\mathscr{L}(\omega_1, \dots, \omega_n)$. In terms of generalized polynomials, generalized convexity can be characterized in a geometrical manner. Namely, a function is generalized convex if and only if it intersects its generalized polynomial that interpolates the function in any prescribed points alternately. (The number of the points depends on the dimension of the underlying Chebyshev system.) More precisely, we have the following

Theorem 4.1. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a Chebyshev system over an interval I. Then, for a function $f : I \to \mathbb{R}$, the following statements are equivalent:*

(i) *f is generalized convex with respect to $\boldsymbol{\omega}$;*

(ii) *for all $y_1 < \dots < y_n$ in I, the generalized polynomial ω of $\omega_1, \dots, \omega_n$ determined uniquely by the interpolation conditions*

$$f(y_k) = \omega(y_k) \qquad (k = 1, \dots, n)$$

satisfies the inequalities

$$(-1)^{n+k}(f(y) - \omega(y)) \geq 0 \qquad (y_k < y < y_{k+1},\ k = 0, \dots, n)$$

under the conventions $y_0 := \inf I$ and $y_{n+1} := \sup I$;

(iii) *keeping the previous notations and settings, for fixed $k \in \{0, \dots, n\}$, the following inequality holds*

$$(-1)^{n+k}(f(y) - \omega(y)) \geq 0 \quad (y_k \leq y \leq y_{k+1}).$$

Proof. First of all, in order to simplify the proof, two useful formulas are derived. Denote the $n - 1$ tuple obtained by dropping the k^{th} component of $\boldsymbol{\omega}$ by $\boldsymbol{\omega}_k$, and define the determinants

$D_0, D_1, \ldots, D_n$, and the generalized polynomial ω of $\omega_1, \ldots, \omega_n$ by

$$D_0 := \begin{vmatrix} \boldsymbol{\omega}(y_1) & \cdots & \boldsymbol{\omega}(y_n) \end{vmatrix}$$
$$D_k := \begin{vmatrix} f(y_1) & \cdots & f(y_n) \\ \boldsymbol{\omega}_k(y_1) & \cdots & \boldsymbol{\omega}_k(y_n) \end{vmatrix}$$
$$\omega := \sum_{k=1}^{n} \frac{(-1)^{k+1} D_k}{D_0} \omega_k.$$

Due to the Chebyshev property of $\boldsymbol{\omega}$, the determinant D_0 is positive, hence the definition of ω is correct. Fix $y \in I$. Applying the expansion theorem to the first column of the following determinant, we get the identity

$$\begin{vmatrix} f(y) & f(y_1) & \cdots & f(y_n) \\ \boldsymbol{\omega}(y) & \boldsymbol{\omega}(y_1) & \cdots & \boldsymbol{\omega}(y_n) \end{vmatrix} = D_0(f(y) - \omega(y)). \tag{4.1}$$

Moreover, if $y_k \le y \le y_{k+1}$ and $(x_0, x_1, \ldots, x_n)$ denotes the increasing rearrangement of $(y; y_1, \ldots, y_n)$, the previous identity can be written into the form

$$\begin{vmatrix} f(x_0) & f(x_1) & \cdots & f(x_n) \\ \boldsymbol{\omega}(x_0) & \boldsymbol{\omega}(x_1) & \cdots & \boldsymbol{\omega}(x_n) \end{vmatrix} = (-1)^k D_0(f(y) - \omega(y)). \tag{4.2}$$

For the implication $(i) \implies (ii)$, observe that (4.1) guarantees the required interpolation property of ω in the points $y_1, \ldots, y_n$. Clearly, ω is uniquely determined. Suppose that $f : I \to \mathbb{R}$ is generalized n-convex with respect to $\boldsymbol{\omega}$. Then, the positivity of D_0 and formula (4.2) yield the inequalities to be proved. The implication $(ii) \implies (iii)$ is trivial. The proof of $(iii) \implies (i)$ is completely the same as the proof of the first assertion. □

In the standard setting and fixing $k = 1$, assertion (iii) gives the classical definition of standard convexity: a function is convex (in the standard sense) if and only if it is "under" the chord of the graph. Moreover, substituting $n = 2$, we also get a new characterization of generalized 2-convexity that completes Theorem 3.1. However, the most important application of Theorem 4.1 guarantees strong regularity properties for generalized convex functions.

Theorem 4.2. *Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be a Chebyshev system over an interval I. If $f : I \to \mathbb{R}$ is a generalized n-convex function with respect to this system and $n \ge 2$, then f is continuous on the interior of I. Furthermore, f is bounded on each compact subinterval of I.*

Proof. Choose $y_0 \in I^\circ$ and fix $x_0 < x_1 < \cdots < x_n$ in I so that $x_1 = y_0$ hold. Denote the generalized polynomials of $\omega_1, \ldots, \omega_n$ that interpolate ω_0 in the points $x_0 \ldots, x_{n-1}$ and $x_1, \ldots, x_n$ by $\omega^{(1)}$ and $\omega^{(2)}$, respectively. We assume that n is even (the argument in the odd case is analogous). Then, according to (ii) of Theorem 4.1, we have the inequalities

$$\omega^{(1)}(y) \ge \omega_0(y) \ge \omega^{(2)}(y) \qquad y \in [x_0, x_1],$$
$$\omega^{(1)}(y) \le \omega_0(y) \le \omega^{(2)}(y) \qquad y \in [x_1, x_2].$$

On the other hand, $\omega^{(1)}(y_0) = \omega_0(y_0)$ and $\omega^{(2)}(y_0) = \omega_0(y_0)$. Therefore, due to the continuity of the generalized polynomials $\omega^{(1)}$ and $\omega^{(2)}$, we get that both the left and right hand side limits of ω_0 exist at the point y_0 and

$$\lim_{y \to y_o - 0} \omega_0(y) = \omega_0(y_0),$$
$$\lim_{y \to y_o + 0} \omega_0(y) = \omega_0(y_0),$$

which yields the continuity of ω_0 at the interior point y_0 of I.

To prove the second assertion, we may assume that $I = [a, b]$. It is sufficient to show that ω_0 is locally bounded at the endpoints of I. Fix $x_0 < x_1 < \cdots < x_n$ in I so that $x_0 = a$ hold, and denote the generalized polynomials of $\omega_1, \dots, \omega_n$ that interpolate ω_0 in the points $x_0 \dots, x_{n-1}$ and $x_1, \dots, x_n$ by $\omega^{(1)}$ and $\omega^{(2)}$, respectively. We assume that n is even (the odd case is very similar). Then, by the previous theorem again, we have the inequalities

$$\omega^{(1)}(y) \geq \omega_0(y) \geq \omega^{(2)}(y) \qquad y \in [x_0, x_1].$$

On the other hand, the functions $\omega^{(1)}$ and $\omega^{(2)}$ are continuous, therefore bounded on $[a, b]$. Hence ω_0 is bounded in a right neighborhood of the endpoint a. It can be similarly proved that ω_0 is locally bounded at the left endpoint b. □

In particular, generalized convex functions are integrable on any compact subset of the domain. Let us also mention that the special case $n = 2$ gives the statement of Theorem 3.3 via another approach in the proof.

4.2. **Moment spaces induced by Chebyshev systems.** The geometric study of moment spaces induced by Chebyshev systems was systematically developed by M. G. Krein. Independently and simultaneously, S. Karlin and L. S. Shapley elaborated the geometry of moment spaces induced by the polynomial system. Some of the results of their research play a key role in further investigations.

Definition 4.1. *Let* $\boldsymbol{\omega} := (\omega_1, \dots, \omega_n)$ *be a Chebyshev system on* $[a, b]$ *and denote the set of all nondecreasing right continuous functions defined on* $[a, b]$ *by* $\mathscr{B}([a, b])$. *The set*

$$\mathscr{M}_n := \left\{ c \in \mathbb{R}^n \;\middle|\; c = \int_a^b \boldsymbol{\omega} d\sigma,\ \sigma \in \mathscr{B}([a, b]) \right\}$$

is called the moment space of $\boldsymbol{\omega}$.

It can be shown that $\mathscr{M}_n$ is a closed convex cone. More precisely, it is the smallest closed convex cone that contains the parameterized curve $\boldsymbol{\omega}(t)$ where t traverses the interval $[a, b]$. For details, see [16, pp. 38-41]. The following notion makes the formulation of many theorems quite convenient.

Definition 4.2. *The* index $I(c)$ *of a point* $c \in \mathscr{M}_n$ *is the minimal number of points* $\xi_1, \dots, \xi_{n_0}$ *in a representation*

$$c = \sum_{k=1}^{n_0} \alpha_k \boldsymbol{\omega}(\xi_k)$$

under the convention that $\boldsymbol{\omega}(a)$ *and* $\boldsymbol{\omega}(b)$ *are counted with half multiplicity, while* $\boldsymbol{\omega}(\xi)$ *for* $\xi \in]a, b[$ *receives a full count. The points* $\xi_1, \dots, \xi_{n_0}$ *are called the* roots *of the representation.*

By the celebrated theorem of Carathéodory (see [37]), each point belonging to the conical hull of a given subset of $\mathbb{R}^n$ can be represented as a cone combination involving at most n points of the subset. Due to the Chebyshev property of $\boldsymbol{\omega}$, a surprisingly better upper bound can be established: it turns out that the elements of $\mathscr{M}_n$ are cone combinations of approximately $n/2$ points of the range of $\boldsymbol{\omega}$. More precisely, the boundary and the interior of $\mathscr{M}_n$, denoted by $\operatorname{Bd}\mathscr{M}_n$ and $\operatorname{Int}\mathscr{M}_n$, can be characterized via the subsequent two theorems due to Krein and Markov.

Theorem C. ([16, Theorem 2.1. p. 42]) *A vector* $c \in \mathscr{M}_n$ *is a boundary point of* $\mathscr{M}_n$ *if and only if* $I(c) < n/2$. *Moreover, every* $c \in \operatorname{Bd}\mathscr{M}_n$ *admits a unique representation*

$$c = \sum_{k=1}^{n_0} \alpha_k \boldsymbol{\omega}(\xi_k) \qquad (\xi_k \in [a, b],\ \alpha_k > 0,\ k = 1, \dots, n_0)$$

where $n_0 \leq \frac{n+1}{2}$.

Theorem D. ([16, Theorem 3.1. p. 44; Remark 3.1. pp. 45-46; Corollary 3.1. p. 47.]) *For each* $c \in \operatorname{Int} \mathscr{M}_n$ *there exist precisely two representations of index* $I(c) = n/2$. *Distinguishing the even and odd cases, the representations in question are the following.*
Case $n = 2m$:

$$c = \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) \qquad (\,\xi_k \in]a, b[\,),$$

$$c = \beta_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m-1} \beta_k \boldsymbol{\omega}(\eta_k) + \beta_m \boldsymbol{\omega}(b) \qquad (\,\eta_k \in]a, b[\,);$$

Case $n = 2m + 1$:

$$c = \alpha_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) \qquad (\,\xi_k \in]a, b[\,),$$

$$c = \sum_{k=1}^{m} \beta_k \boldsymbol{\omega}(\eta_k) + \beta_{m+1} \boldsymbol{\omega}(b) \qquad (\,\eta_k \in]a, b[\,).$$

The roots of the representations in both cases strictly interlace.

Let $I \subset \mathbb{R}$ be a real interval and $\boldsymbol{\omega} := (\omega_1, \dots, \omega_n)$ be a Chebyshev system over I. Then, for pairwise distinct elements $t_1, \dots, t_n$ of I, the vectors $\boldsymbol{\omega}(t_1), \dots, \boldsymbol{\omega}(t_n)$ are linearly independent. This simple observation immediately implies

Theorem 4.3. *The coefficients and the roots of the representations above are uniquely determined.*

Now we present a sufficient condition for a point c to belong to the interior of the set $\mathscr{M}_n$. This condition guarantees that the inequalities of the main results have exactly the required form.

Theorem 4.4. *Let* $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ *be a Chebyshev system on* $[a, b]$ *and let* $\rho : [a, b] \to \mathbb{R}$ *be a positive integrable function. Then,*

$$c := \int_a^b \boldsymbol{\omega} \rho \in \operatorname{Int} \mathscr{M}_n.$$

Proof. Let us recall that $\mathscr{M}_n$ is a closed subset of $\mathbb{R}^n$. On the other hand, the positivity of ρ yields $c \in \mathscr{M}_n$, therefore it suffices to prove that $c \notin \operatorname{Bd} \mathscr{M}_n$. Assume indirectly that $c \in \operatorname{Bd} \mathscr{M}_n$. We shall distinguish two cases according to the parity of n.

Case $n = 2m + 1$. The indirect assumption and Theorem C implies $I(c) \leq m$ since $I(c)$ increases at most $1/2$. For simplicity, assume that $I(c) = m$. Then there are two further possibilities: the representation of c involves either m pairwise distinct interior base points $\xi_1 < \dots < \xi_m$ or $m - 1$ pairwise distinct interior base points $\xi_1 < \dots < \xi_{m-1}$ plus both the endpoints a and b, respectively. In the first case we have the representation

$$c = \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k).$$

Due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ, we arrive at

$$0 < \begin{vmatrix} \boldsymbol{\omega}(t_1)\rho(t_1) & \boldsymbol{\omega}(\xi_1) & \cdots & \boldsymbol{\omega}(t_m)\rho(t_m) & \boldsymbol{\omega}(\xi_m) & \boldsymbol{\omega}(t_{m+1})\rho(t_{m+1}) \end{vmatrix}$$

for $t_k \in]\xi_{k-1}, \xi_k[\quad (k = 1, \dots, m)$ where $\xi_0 := a$ and $\xi_{m+1} := b$. After integration with respect to $(t_1, \dots, t_{m+1})$ and using the above representation of c, we have

$$\begin{aligned} 0 &< \left| \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_1) \quad \cdots \quad \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_m) \quad \int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho \right| \\ &= \left| \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_1) \quad \cdots \quad \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_m) \quad \sum_{k=1}^{m+1} \int_{\xi_{k-1}}^{\xi_k} \boldsymbol{\omega}\rho \right| \\ &= \left| \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_1) \quad \cdots \quad \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_m) \quad \int_a^b \boldsymbol{\omega}\rho \right| \\ &= \left| \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_1) \quad \cdots \quad \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho \quad \boldsymbol{\omega}(\xi_m) \quad \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) \right| = 0 \end{aligned}$$

since the last column is the linear combination of the even indexed columns. Thus we get the desired contradiction.

Now consider the other case when c has the representation

$$c = \alpha_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m-1} \alpha_k \boldsymbol{\omega}(\xi_k) + \alpha_m \boldsymbol{\omega}(b).$$

Due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ again, we arrive at

$$0 < \left| \boldsymbol{\omega}(a) \quad \boldsymbol{\omega}(t_1)\rho(t_1) \quad \boldsymbol{\omega}(\xi_1) \quad \cdots \quad \boldsymbol{\omega}(\xi_{m-1}) \quad \boldsymbol{\omega}(t_m)\rho(t_m) \quad \boldsymbol{\omega}(b) \right|$$

for $t_k \in]\xi_{k-1}, \xi_k[\quad (k = 1, \dots, m)$ where $\xi_0 := a$ and $\xi_m := b$. An analogous argument to the previous one leads to contradiction.

Case $n = 2m$. Similarly to the odd case, now we may assume that $I(c) = m - 1/2$. Then there are two possibilities: the representation of c involves either the endpoint a and $m - 1$ pairwise distinct interior base points $\xi_1 < \cdots < \xi_{m-1}$ or the endpoint b and $m - 1$ pairwise distinct interior base points $\xi_1 < \cdots < \xi_{m-1}$. Applying the same method as above, both cases lead to contradiction again. □

4.3. **Hermite–Hadamard-type inequalities.** The main results concern the cases of even and odd order generalized convexity separately. First we establish Hermite–Hadamard-type inequalities for the odd order one.

Theorem 4.5. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{2m+1})$ be a Chebyshev system on $[a, b]$ and $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. There exist uniquely determined base points $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_m$ of $]a, b[$ such that*

$$\alpha_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) = \int_a^b \boldsymbol{\omega}\rho = \sum_{k=1}^{m} \beta_k \boldsymbol{\omega}(\eta_k) + \beta_{m+1} \boldsymbol{\omega}(b).$$

The coefficients $\alpha_0, \dots, \alpha_m$ and $\beta_1, \dots, \beta_{m+1}$ are positive and uniquely determined, too. Furthermore, for any generalized $\boldsymbol{\omega}$-convex function $f : [a, b] \to \mathbb{R}$, the following Hermite–Hadamard-type inequality holds

$$\alpha_0 f(a) + \sum_{k=1}^{m} \alpha_k f(\xi_k) \le \int_a^b f\rho \le \sum_{k=1}^{m} \beta_k f(\eta_k) + \beta_{m+1} f(b).$$

Proof. Let us note that $f\rho$ is integrable on $[a, b]$ by Theorem 4.2. The proofs of the left and right hand side inequalities need similar methods, therefore, we shall verify only the left hand side one. Theorem 4.4 guarantees that $\int_a^b \boldsymbol{\omega}\rho$ is an interior point of the moment space $\mathscr{M}_n$ hence (see Theorem D and Theorem 4.3) it has the representation

$$\int_a^b \boldsymbol{\omega}\rho = \alpha_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) \tag{4.3}$$

where the coefficients $\alpha_0,\dots,\alpha_m$ and interior base points $\xi_1,\dots,\xi_m$ are determined uniquely. Defining $\xi_0 := a$ and $\xi_{m+1} := b$, consider the following system of linear equations

$$\int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho = c_0\boldsymbol{\omega}(\xi_0) + \sum_{k=1}^{m}\Big(c_k^*\int_{\xi_{k-1}}^{\xi_k}\boldsymbol{\omega}\rho + c_k\boldsymbol{\omega}(\xi_k)\Big)$$

where the unknowns are $c_0, c_1^*, c_1, \dots, c_m^*, c_m$. Due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ, its base determinant

$$D := \begin{vmatrix} \boldsymbol{\omega}(\xi_0) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) \end{vmatrix}$$

is positive. Therefore, the system has a unique solution $(c_0, c_1^*, c_1, \dots, c_m^*, c_m)$. On the other hand, representation (4.3) shows that $(\alpha_0, -1, \alpha_1, \dots, -1, \alpha_m)$ is also a solution. Thus, $\alpha_0, \alpha_1, \dots, \alpha_n$ can be obtained by Cramer's Rule:

$$\begin{aligned}
\alpha_0 &= \frac{1}{D}\begin{vmatrix} \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) & \int_{\xi_m}^{\xi_{m+1}}\boldsymbol{\omega}\rho \end{vmatrix},\\
\alpha_k &= \frac{1}{D}\begin{vmatrix} \boldsymbol{\omega}(\xi_0) & \cdots & \int_{\xi_{k-1}}^{\xi_k}\boldsymbol{\omega}\rho & \int_{\xi_k}^{\xi_{k+1}}\boldsymbol{\omega}\rho & \cdots & \int_{\xi_m}^{\xi_{m+1}}\boldsymbol{\omega}\rho \end{vmatrix}.
\end{aligned}$$

Suppose now that $\omega_0 : [a,b] \to \mathbb{R}$ is a generalized $(2m+1)$-convex function with respect to $\boldsymbol{\omega}$. Then, for all elements t_k of $]\xi_k, \xi_{k+1}[$, the following inequality holds:

$$0 \ge \begin{vmatrix} f(\xi_0) & f(t_0) & \cdots & f(\xi_m) & f(t_m) \\ \boldsymbol{\omega}(\xi_0) & \boldsymbol{\omega}(t_0) & \cdots & \boldsymbol{\omega}(\xi_m) & \boldsymbol{\omega}(t_m) \end{vmatrix}.$$

Multiplying both sides by the positive $\rho(t_1)\cdots\rho(t_m)$ and integrating on the product $[\xi_0,\xi_1]\times\cdots\times[\xi_m,\xi_{m+1}]$ with respect to $(t_0,\dots,t_m)$, we arrive at the inequality

$$\begin{aligned}
0 &\ge \begin{vmatrix} f(\xi_0) & \int_{\xi_0}^{\xi_1} f\rho & \cdots & f(\xi_m) & \int_{\xi_m}^{\xi_{m+1}} f\rho \\ \boldsymbol{\omega}(\xi_0) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \cdots & \boldsymbol{\omega}(\xi_m) & \int_{\xi_m}^{\xi_{m+1}}\boldsymbol{\omega}\rho \end{vmatrix}\\
&= \begin{vmatrix} f(\xi_0) & \int_{\xi_0}^{\xi_1} f\rho & \cdots & f(\xi_m) & \int_{\xi_0}^{\xi_1} f\rho + \cdots + \int_{\xi_m}^{\xi_{m+1}} f\rho \\ \boldsymbol{\omega}(\xi_0) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \cdots & \boldsymbol{\omega}(\xi_m) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho + \cdots + \int_{\xi_m}^{\xi_{m+1}}\boldsymbol{\omega}\rho \end{vmatrix}\\
&= \begin{vmatrix} f(\xi_0) & \int_{\xi_0}^{\xi_1} f\rho & \cdots & f(\xi_m) & \int_a^b f\rho \\ \boldsymbol{\omega}(\xi_0) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \cdots & \boldsymbol{\omega}(\xi_m) & \int_a^b\boldsymbol{\omega}\rho \end{vmatrix}.
\end{aligned}$$

Observe that the adjoint determinants of each element $\int_{\xi_k}^{\xi_{k+1}} f\rho$ in the last expression are equal to zero since their columns are linearly dependent due to (4.3). Therefore, applying the expansion theorem to the first row, it follows that

$$\begin{aligned}
0 \le{}& \begin{vmatrix} \boldsymbol{\omega}(\xi_0) & \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) \end{vmatrix}\cdot\int_a^b f\rho\\
&- \begin{vmatrix} \int_{\xi_0}^{\xi_1}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m}\boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) & \int_a^b\boldsymbol{\omega}\rho \end{vmatrix} f(\xi_0)\\
&- \sum_{k=1}^{m}\begin{vmatrix} \boldsymbol{\omega}(\xi_0) & \cdots & \int_{\xi_{k-1}}^{\xi_k}\boldsymbol{\omega}\rho & \int_{\xi_k}^{\xi_{k+1}}\boldsymbol{\omega}\rho & \cdots & \int_a^b\boldsymbol{\omega}\rho \end{vmatrix} f(\xi_k).
\end{aligned}$$

Here the coefficient of $\int_a^b f\rho$ is the positive determinant D, while the the coefficients of $f(\xi_0)$, $\dots, f(\xi_m)$ are exactly the numerators of $\alpha_0,\dots,\alpha_m$ (see above), since the last column $\int_a^b \boldsymbol{\omega}$ can be replaced by $\int_{\xi_m}^{\xi_{m+1}}\boldsymbol{\omega}\rho$. After rearranging, we get the left hand side of the Hermite–Hadamard-type inequality. □

Theorem 4.6. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{2m})$ be a Chebyshev system on $[a, b]$ and $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Then, there exist uniquely determined base points $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_{m-1}$ of $]a, b[$ such that*

$$\sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) = \int_a^b \boldsymbol{\omega}\rho = \beta_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m-1} \beta_k \boldsymbol{\omega}(\eta_k) + \beta_m \boldsymbol{\omega}(b).$$

The coefficients $\alpha_1, \dots, \alpha_m$ and $\beta_0, \dots, \beta_m$ are positive and uniquely determined, too. Furthermore, for any generalized $\boldsymbol{\omega}$-convex function $f : [a, b] \to \mathbb{R}$, the following Hermite–Hadamard-type inequality holds

$$\sum_{k=1}^{m} \alpha_k f(\xi_k) \le \int_a^b f\rho \le \beta_0 f(a) + \sum_{k=1}^{m-1} \beta_k f(\eta_k) + \beta_m f(b).$$

Proof. To prove the left hand side inequality, take the unique interior base points $\xi_1, \dots, \xi_m$ and coefficients $\alpha_1, \dots, \alpha_m$ fulfilling the representation

$$\int_a^b \boldsymbol{\omega}\rho = \sum_{k=1}^{m} \alpha_k \boldsymbol{\omega}(\xi_k) \tag{4.4}$$

guaranteed by Theorem 4.4. Defining $\xi_0 := a$ and $\xi_{m+1} := b$, consider the following system of linear equations

$$\int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho = \sum_{k=1}^{m} \Big(c_k^* \int_{\xi_{k-1}}^{\xi_k} \boldsymbol{\omega}\rho + c_k \boldsymbol{\omega}(\xi_k) \Big)$$

where the unknowns are $c_1^*, c_1, \dots, c_m^*, c_m$. Due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ, its base determinant

$$D_1 := \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) \end{vmatrix}$$

is positive, hence the system has a unique solution $(c_1^*, c_1, \dots, c_m^*, c_m)$. On the other hand, the representation (4.4) shows that $(-1, \alpha_1, \dots, -1, \alpha_m)$ is also a solution. Thus, the coefficients can be obtained by Cramer's Rule:

$$\alpha_1 = \frac{1}{D_1} \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \int_{\xi_1}^{\xi_2} \boldsymbol{\omega}\rho & \cdots & \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) & \int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho \end{vmatrix},$$
$$\alpha_k = \frac{1}{D_1} \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \cdots & \int_{\xi_{k-1}}^{\xi_k} \boldsymbol{\omega}\rho & \int_{\xi_k}^{\xi_{k+1}} \boldsymbol{\omega}\rho & \cdots & \int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho \end{vmatrix}.$$

Suppose now that $f : [a, b] \to \mathbb{R}$ is a generalized $(2m)$-convex function with respect to $\boldsymbol{\omega}$. Then, for all elements t_k of $]\xi_k, \xi_{k+1}[$, the following inequality holds:

$$0 \le \begin{vmatrix} f(t_0) & f(\xi_1) & \cdots & f(\xi_m) & f(t_m) \\ \boldsymbol{\omega}(t_0) & \boldsymbol{\omega}(\xi_1) & \cdots & \boldsymbol{\omega}(\xi_m) & \boldsymbol{\omega}(t_m) \end{vmatrix}.$$

Therefore,

$$\begin{aligned} 0 &\le \begin{vmatrix} \int_{\xi_0}^{\xi_1} f\rho & f(\xi_1) & \cdots & f(\xi_m) & \int_{\xi_m}^{\xi_{m+1}} f\rho \\ \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \boldsymbol{\omega}(\xi_m) & \int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho \end{vmatrix} \\ &= \begin{vmatrix} \int_{\xi_0}^{\xi_1} f\rho & f(\xi_1) & \cdots & f(\xi_m) & \int_{\xi_0}^{\xi_1} f\rho + \cdots + \int_{\xi_m}^{\xi_{m+1}} f\rho \\ \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \boldsymbol{\omega}(\xi_m) & \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho + \cdots + \int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho \end{vmatrix} \\ &= \begin{vmatrix} \int_{\xi_0}^{\xi_1} f\rho & f(\xi_1) & \cdots & f(\xi_m) & \int_a^b f\rho \\ \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \boldsymbol{\omega}(\xi_m) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix}. \end{aligned}$$

In the last expression, the adjoint determinant of each element $\int_{\xi_k}^{\xi_{k+1}} f\rho$ are equal to zero since their columns are linearly dependent due to (4.4). Applying the expansion theorem to the first row, we arrive at the inequality

$$0 \le \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_1) & \cdots & \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) \end{vmatrix} \cdot \int_a^b f\rho$$
$$- \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \int_{\xi_1}^{\xi_2} \boldsymbol{\omega}\rho & \cdots & \int_{\xi_{m-1}}^{\xi_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\xi_m) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} f(\xi_1)$$
$$- \sum_{k=2}^{m} \begin{vmatrix} \int_{\xi_0}^{\xi_1} \boldsymbol{\omega}\rho & \cdots & \int_{\xi_{k-1}}^{\xi_k} \boldsymbol{\omega}\rho & \int_{\xi_k}^{\xi_{k+1}} \boldsymbol{\omega}\rho & \cdots & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} f(\xi_k).$$

Here the coefficient of $\int_a^b f\rho$ is the positive D_1; moreover, the coefficients of $f(\xi_1), \dots, f(\xi_m)$ are exactly the numerators of $\alpha_1, \dots, \alpha_m$ since the last column $\int_a^b \boldsymbol{\omega}\rho$ can be replaced by $\int_{\xi_m}^{\xi_{m+1}} \boldsymbol{\omega}\rho$. After rearranging, we get the left hand side of the Hermite–Hadamard-type inequality.

For the right hand side inequality, take the uniquely determined interior base points $\eta_1, \dots, \eta_{m-1}$ and coefficients $\beta_0, \dots, \beta_m$ so that the representation

$$\int_a^b \boldsymbol{\omega}\rho = \beta_0 \boldsymbol{\omega}(a) + \sum_{k=1}^{m-1} \beta_k \boldsymbol{\omega}(\eta_k) + \beta_m \boldsymbol{\omega}(b) \tag{4.5}$$

holds. Defining $\eta_0 := a$ and $\eta_m := b$, consider the following system of linear equations

$$\int_{\eta_{m-1}}^{\eta_m} \boldsymbol{\omega}\rho = c_0 \boldsymbol{\omega}(\eta_0) + \sum_{k=1}^{m-1} \left(c_k^* \int_{\eta_{k-1}}^{\eta_k} \boldsymbol{\omega}\rho + c_k \boldsymbol{\omega}(\eta_k) \right) + c_m \boldsymbol{\omega}(\eta_m),$$

where the unknowns are $c_0, c_1^*, c_1, \dots, c_{m-1}^*, c_{m-1}, c_m$. Due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ, its base determinant

$$D_2 := \begin{vmatrix} \boldsymbol{\omega}(\eta_0) & \int_{\eta_0}^{\eta_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_1) & \cdots & \int_{\eta_{m-2}}^{\eta_{m-1}} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_{m-1}) & \boldsymbol{\omega}(\eta_m) \end{vmatrix}$$

is positive, hence the system has a unique solution $c_0, c_1^*, c_1, \dots, c_{m-1}^*, c_{m-1}, c_m$. The representation (4.5) shows that $(\beta_0, -1, \beta_1, \dots, \beta_{m-1}, -1, \beta_m)$ is also a solution, therefore Cramer's Rule can be applied:

$$\beta_0 = \frac{1}{D_2} \begin{vmatrix} \int_{\eta_0}^{\eta_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_1) & \cdots & \boldsymbol{\omega}(\eta_{m-1}) & \int_{\eta_{m-1}}^{\eta_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_m) \end{vmatrix},$$
$$\beta_k = \frac{1}{D_2} \begin{vmatrix} \boldsymbol{\omega}(\eta_0) & \cdots & \int_{\eta_{k-1}}^{\eta_k} \boldsymbol{\omega}\rho & \int_{\eta_k}^{\eta_{k+1}} \boldsymbol{\omega}\rho & \cdots & \int_{\eta_{m-1}}^{\eta_m} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_m) \end{vmatrix},$$
$$\beta_m = \frac{1}{D_2} \begin{vmatrix} \boldsymbol{\omega}(\eta_0) & \int_{\eta_0}^{\eta_1} \boldsymbol{\omega}\rho & \cdots & \int_{\eta_{m-2}}^{\eta_{m-1}} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_{m-1}) & \int_{\eta_{m-1}}^{\eta_m} \boldsymbol{\omega}\rho \end{vmatrix}.$$

These coefficients are positive since even changes are needed to transfer the column $\int_{\eta_{m-1}}^{\eta_m} \boldsymbol{\omega}\rho$ to the adequate place.

If a function $f : [a, b] \to \mathbb{R}$ is a generalized $(2m)$-convex with respect to $\boldsymbol{\omega}$, then we arrive at the inequality

$$0 \le \begin{vmatrix} f(\eta_0) & \int_{\eta_0}^{\eta_1} f\rho & f(\eta_1) & \cdots & \int_{\eta_{m-2}}^{\eta_{m-1}} f\rho & f(\eta_m) & \int_a^b f\rho \\ \boldsymbol{\omega}(\eta_0) & \int_{\eta_0}^{\eta_1} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_1) & \cdots & \int_{\eta_{m-2}}^{\eta_{m-1}} \boldsymbol{\omega}\rho & \boldsymbol{\omega}(\eta_m) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix},$$

whence an analogous argument to the previous one completes the proof. $\square$

4.4. **An alternative approach in a particular case.** To prove the main results, the main point is the existence of the representations of Theorem D. These representations can also be considered as systems of nonlinear equations where the unknowns are the coefficients and the base points. The number of the equations and the unknowns coincide in each case. In those cases when only one interior base point is involved, the solubility of the system of equations can directly be verified without applying the Krein–Markov theory of moment spaces.

Theorem 4.7. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be a Chebyshev system on $[a, b]$ and $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Then, there exist unique elements ξ, η of $]a, b[$ and uniquely determined positive coefficients c_1, c_2 and d_1, d_2 such that*

$$c_1\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi) = \int_a^b \boldsymbol{\omega}\rho = d_1\boldsymbol{\omega}(\eta) + d_2\boldsymbol{\omega}(b).$$

Furthermore, if a function $f : [a, b] \to \mathbb{R}$ is generalized 3-convex with respect to $\boldsymbol{\omega}$, then the following Hermite–Hadamard-type inequality holds

$$c_1f(a) + c_2f(\xi) \le \int_a^b f\rho \le d_1f(\eta) + d_2f(b).$$

Proof. We shall restrict the process of the proof only on the existence of the interior point ξ. To do this, define the function $F : [a, b] \to \mathbb{R}$ by the formula

$$F(x) := \begin{vmatrix} \boldsymbol{\omega}(a) & \int_a^x \boldsymbol{\omega}\rho & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} := \begin{vmatrix} \omega_1(a) & \int_a^x \omega_1\rho & \int_a^b \omega_1\rho \\ \omega_2(a) & \int_a^x \omega_2\rho & \int_a^b \omega_2\rho \\ \omega_3(a) & \int_a^x \omega_3\rho & \int_a^b \omega_3\rho \end{vmatrix}.$$

Then, F is continuous on $[a, b]$ and $F(a) = F(b) = 0$. Further on, $F(x) \ne 0$ if $x \in]a, b[$ due to the Chebyshev property of $\boldsymbol{\omega}$ and the positivity of ρ. For simplicity, we may assume that F is positive on $]a, b[$. Therefore, by Weierstrass' theorem, there exists $\xi \in]a, b[$ such that

$$F(\xi) = \max_{[a,b]} F.$$

Assume that $x \in]\xi, b]$. Then, the maximal property of ξ yields the inequality

$$0 \ge \frac{F(x) - F(\xi)}{\int_\xi^x \rho} = \begin{vmatrix} \boldsymbol{\omega}(a) & \frac{\int_\xi^x \boldsymbol{\omega}\rho}{\int_\xi^x \rho} & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix}.$$

The central column of the determinant tends to $\boldsymbol{\omega}(\xi)$ as x tends to ξ since the following estimations are valid for $k = 1, 2, 3$:

$$\min_{[\xi,x]} \omega_k = \frac{\min_{[\xi,x]} \omega_k \int_\xi^x \rho}{\int_\xi^x \rho} \le \frac{\int_\xi^x \omega_k\rho}{\int_\xi^x \rho} \le \frac{\max_{[\xi,x]} \omega_k \int_\xi^x \rho}{\int_\xi^x \rho} = \max_{[\xi,x]} \omega_k.$$

Therefore

$$\begin{vmatrix} \boldsymbol{\omega}(a) & \boldsymbol{\omega}(\xi) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} \le 0.$$

Choosing $x \in [a, \xi[$ and using the maximal property of ξ again, we get the opposite inequality with the same argument and arrive at the identity

$$\begin{vmatrix} \boldsymbol{\omega}(a) & \boldsymbol{\omega}(\xi) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} = 0.$$

Thus, the linear independence of $\boldsymbol{\omega}(a)$ and $\boldsymbol{\omega}(\xi)$ yields that there exist coefficients c_1 and c_2 such that

$$c_1\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi) = \int_a^b \boldsymbol{\omega}\rho.$$

The right hand side inequality can be verified with an analogous argument, therefore the proof is omitted. □

Let us note, that if the weight function ρ is continuous, then the function F is differentiable and Rolle's mean-value theorem can directly be applied.

The representations of Theorem 4.7 are linear with respect to the coefficients. Therefore, in concrete cases, the main difficulty lies in determining the interior base points ξ and η. Without claiming completeness, we list some examples of when they can be determined explicitly.

Example 1. *If the Chebyshev system $(\omega_1, \omega_2, \omega_3)$ is defined on $[a, b]$ by $\omega_1(x) = 1$, $\omega_2(x) = \sinh x$, $\omega_3(x) = \cosh x$ and $\rho \equiv 1$, then*

$$\begin{aligned}\xi &= 2\,\mathrm{artanh}\left(\frac{\sinh b - \sinh a - (b-a)\cosh a}{\cosh b - \cosh a - (b-a)\sinh a}\right) - a,\\ \eta &= 2\,\mathrm{artanh}\left(\frac{\sinh b - \sinh a - (b-a)\cosh b}{\cosh b - \cosh a - (b-a)\sinh b}\right) - b.\end{aligned}$$

Proof. With the above setting, the left hand side representation of Theorem 4.5 reduces to the following system of nonlinear equations

$$\begin{aligned}c_1 + c_2 &= \int_a^b 1dx = b - a,\\ c_1 \sinh a + c_2 \sinh \xi &= \int_a^b \sinh x dx = \cosh b - \cosh a,\\ c_1 \cosh a + c_2 \cosh \xi &= \int_a^b \cosh x dx = \sinh b - \sinh a,\end{aligned}$$

where the three unknowns are c_1, c_2 and ξ, respectively. Multiplying the first equation by $\sinh a$ and subtracting it from the second one, then multiplying again the first equation by $\cosh a$ and subtracting it from the third one, the coefficient c_1 can be eliminated and it follows

$$\begin{aligned}c_2(\sinh \xi - \sinh a) &= \cosh b - \cosh a - (b-a)\sinh a\\ c_2(\cosh \xi - \cosh a) &= \sinh b - \sinh a - (b-a)\cosh a.\end{aligned}$$

Applying the well known additional properties of hyperbolic functions for the identities $\xi = (\xi + a)/2 + (\xi - a)/2$ and $a = (\xi + a)/2 - (\xi - a)/2$, the left hand side of both equations can be written in product form:

$$\begin{aligned}2c_2 \cosh\left(\frac{\xi + a}{2}\right)\sinh\left(\frac{\xi - a}{2}\right) &= \cosh b - \cosh a - (b-a)\sinh a,\\ 2c_2 \sinh\left(\frac{\xi + a}{2}\right)\sinh\left(\frac{\xi - a}{2}\right) &= \sinh b - \sinh a - (b-a)\cosh a.\end{aligned}$$

The left hand side of the first equation differs from zero since $\xi \neq a$. Therefore, dividing the second equation by the first one, we get the equation

$$\tanh\left(\frac{\xi + a}{2}\right) = \frac{\sinh b - \sinh a - (b-a)\cosh a}{\cosh b - \cosh a - (b-a)\sinh a},$$

whence the desired expression of ξ is obtained. For determining η, we shall consider the following system of nonlinear equations:

$$\begin{aligned} d_1 + d_2 &= b - a, \\ d_1 \sinh \eta + d_2 \sinh b &= \cosh b - \cosh a, \\ d_1 \cosh \eta + d_2 \cosh b &= \sinh b - \sinh a. \end{aligned}$$

In this case, the coefficient d_2 can be eliminated with a similar method to the previous one. The new system of equations, due to the additional formulae again, can be written in the form

$$\begin{aligned} 2d_1 \cosh\left(\frac{b+\eta}{2}\right) \sinh\left(\frac{b-\eta}{2}\right) &= \cosh b - \cosh a - (b-a)\sinh b, \\ 2d_1 \sinh\left(\frac{b+\eta}{2}\right) \sinh\left(\frac{b-\eta}{2}\right) &= \sinh b - \sinh a - (b-a)\cosh b. \end{aligned}$$

This system, analogously to the previous case, yields the equation

$$\tanh\left(\frac{b+\eta}{2}\right) = \frac{\sinh b - \sinh a - (b-a)\cosh b}{\cosh b - \cosh a - (b-a)\sinh b},$$

whence the base point η can be expressed easily. □

The proofs of the subsequent examples are similar to the previous one, therefore they are omitted.

Example 2. *If the Chebyshev system* $(\omega_1, \omega_2, \omega_3)$ *is defined on* $[a,b] \subset]-\pi,\pi[$ *by* $\omega_1(x) = 1$, $\omega_2(x) = \sin x$, $\omega_3(x) = \cos x$ *and* $\rho \equiv 1$, *then*

$$\begin{aligned} \xi &= 2\arctan\left(\frac{\sin a - \sin b + (b-a)\cos a}{\cos a - \cos b - (b-a)\sin a}\right) - a, \\ \eta &= 2\arctan\left(\frac{\sin a - \sin b + (b-a)\cos b}{\cos a - \cos b - (b-a)\sin b}\right) - b. \end{aligned}$$

Example 3. *If the Chebyshev system* $(\omega_1, \omega_2, \omega_3)$ *is defined on* $[a,b]$ *by* $\omega_1(x) = 1$, $\omega_2(x) = \exp x$, $\omega_3(x) = \exp 2x$ *and* $\rho \equiv 1$, *then*

$$\begin{aligned} \xi &= \log\left(\frac{\exp 2b - \exp 2a - 2(b-a)\exp 2a}{2(\exp b - \exp a - (b-a)\exp a)} - \exp a\right), \\ \eta &= \log\left(\frac{\exp 2b - \exp 2a - 2(b-a)\exp 2b}{2(\exp b - \exp a - (b-a)\exp b)} - \exp b\right). \end{aligned}$$

Example 4. *If, for* $p > 0$, *the Chebyshev system* $(\omega_1, \omega_2, \omega_3)$ *is defined on* $[a,b] \subset [0,+\infty[$ *by* $\omega_1(x) = 1$, $\omega_2(x) = x^p$, $\omega_3(x) = x^{2p}$ *and* $\rho \equiv 1$, *then*

$$\begin{aligned} \xi &= \left(\frac{p+1}{2p+1} \cdot \frac{b^{2p+1} - a^{2p+1} - (2p+1)(b-a)a^{2p}}{b^{p+1} - a^{p+1} - (p+1)(b-a)a^p} - a^p\right)^{1/p}, \\ \eta &= \left(\frac{p+1}{2p+1} \cdot \frac{b^{2p+1} - a^{2p+1} - (2p+1)(b-a)b^{2p}}{b^{p+1} - a^{p+1} - (p+1)(b-a)b^p} - b^p\right)^{1/p}. \end{aligned}$$

The particular case $p = 1$ of the last example gives a corollary of Theorem 2.8 for polynomially 3-convex functions. For 3 dimensional Chebyshev systems generated by arbitrary power functions, the interior base points in general, cannot be expressed explicitly.

The proof of Theorem 4.7 is applicable for generalized 2-convexity, and gives a different approach to that followed in Theorem 3.4. We can also state the right hand side Hermite–Hadamard-type inequality for generalized 4-convex functions.

Theorem 4.8. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ be a Chebyshev system on $[a,b]$ and $\rho : [a,b] \to \mathbb{R}$ be a positive integrable function. Then, there exist a unique element ξ of $]a,b[$ and uniquely determined positive coefficients c_1, c_2, c_3 such that*

$$\int_a^b \boldsymbol{\omega}\rho = c_1\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi) + c_3\boldsymbol{\omega}(b).$$

Furthermore, if a function $f : [a,b] \to \mathbb{R}$ is generalized 4-convex with respect to $\boldsymbol{\omega}$, then the following Hermite–Hadamard-type inequality holds

$$\int_a^b f\rho \le c_1 f(a) + c_2 f(\xi) + c_3 f(b).$$

Hint. Apply the same argument as in the proof of Theorem 4.7 for the function $F : [a,b] \to \mathbb{R}$ defined by the formula

$$F(x) := \left| \boldsymbol{\omega}(a) \;\; \int_a^x \boldsymbol{\omega}\rho \;\; \boldsymbol{\omega}(b) \;\; \int_a^b \boldsymbol{\omega}\rho \right| := \begin{vmatrix} \omega_1(a) & \int_a^x \omega_1\rho & \omega_1(b) & \int_a^b \omega_1\rho \\ \omega_2(a) & \int_a^x \omega_2\rho & \omega_2(b) & \int_a^b \omega_2\rho \\ \omega_3(a) & \int_a^x \omega_3\rho & \omega_3(b) & \int_a^b \omega_3\rho \\ \omega_4(a) & \int_a^x \omega_4\rho & \omega_4(b) & \int_a^b \omega_4\rho \end{vmatrix}.$$

□

For example, if $\boldsymbol{\omega}(x) := (\cosh x, \sinh x, \cosh 2x, \sinh 2x)$, then one can check that the interior base point of the inequality is exactly the midpoint of the domain. Unfortunately, the method fails if someone tries to use it for proving the left hand side of the Hermite–Hadamard-type inequality for a generalized 4-convex function since, by the even case of Theorem D, the existence of two interior base points should be guaranteed. For similar reasons, the "existence" part in the proof of Theorem 4.7 cannot be applied for generalized n-convex functions if $n > 4$.

5. Characterizations via Hermite–Hadamard Inequalities

Under some weak regularity conditions, the Hermite–Hadamard-inequality *characterizes* (standard) convexity (see [17, Excersice 8. p. 205]). The aim of this section is to verify analogous results for (ω_1, ω_2)-convexity. To do this, the most important auxiliary tool turns out to be some characterization properties of continuous, *non* generalized 2-convex functions.

5.1. Further properties of generalized lines.

In what follows, two properties of generalized lines are crucial. The first one improves the statement of Lemma 3.2 and states that, on compact intervals, generalized lines are uniformly non bounded.

Lemma 5.1. *Let (ω_1, ω_2) be a Chebyshev system on an interval I. Then, for any compact subinterval of I and positive number K, there exists $\omega \in \mathscr{L}(\omega_1, \omega_2)$ such that $\omega > K$ on the compact subinterval.*

Proof. According to Lemma 3.2, there exist coefficients α, β such that the generalized line $\alpha\omega_1 + \beta\omega_2$ is positive on the interior of I. Therefore, if $[x,y]$ is a compact subinterval of I, $m := \min\{\alpha\omega_1(t) + \beta\omega_2(t) \mid t \in [x,y]\} > 0$. Defining the coefficients α^* and β^* by the formulae

$$\alpha^* := \frac{\alpha K}{m} \qquad \beta^* := \frac{\beta K}{m},$$

the generalized line $\omega := \alpha^*\omega_1 + \beta^*\omega_2$ is strictly greater than K on $[x,y]$. □

The second important property concerns the convergence of generalized lines. It turns out that pointwise convergence is not only a necessary but a sufficient condition for the uniform convergence of sequences of generalized lines. Let us note that an analogous result remains true for generalized polynomials in the higher-order case.

Lemma 5.2. *Let (ω_1, ω_2) be a Chebyshev system on an interval I, furthermore, let $\omega = \alpha\omega_1 + \beta\omega_2$ and $\omega_n = \alpha_n\omega_1 + \beta_n\omega_2$ $(n \in \mathbb{N})$ be generalized lines. Then, the following statements are equivalent:*

(i) *there exist elements $x < y$ of I such that $\omega_n(x) \to \omega(x)$ and $\omega_n(y) \to \omega(y)$;*
(ii) *the sequences α_n and β_n are convergent, with $\alpha_n \to \alpha$ and $\beta_n \to \beta$;*
(iii) *$\omega_n \to \omega$ uniformly on each compact subset of I.*

Proof. $(i) \Rightarrow (ii)$. Applying Cramer's Rule and the convergence properties of $\omega_n(x)$ and $\omega_n(y)$, one can easily get that

$$\alpha = \frac{\begin{vmatrix} \omega(x) & \omega_2(x) \\ \omega(y) & \omega_2(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_2(x) \\ \omega_1(y) & \omega_2(y) \end{vmatrix}} = \lim_{n\to\infty} \frac{\begin{vmatrix} \omega_n(x) & \omega_2(x) \\ \omega_n(y) & \omega_2(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_2(x) \\ \omega_1(y) & \omega_2(y) \end{vmatrix}} = \lim_{n\to\infty} \alpha_n.$$

The convergence of β_n can be obtained similarly.

$(ii) \Rightarrow (iii)$. Let $[x, y]$ be a compact subinterval of I, and $t \in [x, y]$ arbitrary. Due to the continuity of the functions ω_1 and ω_2, there exists $K > 0$ such that

$$\max\left\{\sup_{[x,y]} |\, \omega_1(t)\,|, \sup_{[x,y]} |\, \omega_2(t)\,|\right\} \le K.$$

Therefore,

$$\begin{aligned} |\, \omega_n(t) - \omega(t)\,| &= |\, \alpha_n\omega_1(t) - \alpha\omega_1(t) + \beta_n\omega_2(t) - \beta\omega_2(t)\,| \\ &\le |\, \alpha_n - \alpha\,||\, \omega_1(t)\,| + |\, \beta_n - \beta\,||\, \omega_2(t)\,| \\ &\le K\big(|\, \alpha_n - \alpha\,| + |\, \beta_n - \beta\,|\big) \to 0 \end{aligned}$$

as $n \to \infty$; hence $\omega_n \to \omega$ uniformly on $[x, y]$.

$(iii) \Rightarrow (i)$. Trivial. □

Under the assumption of continuity, if a function is not convex, then it must be locally strictly concave somewhere. The following theorem generalizes this result for non (ω_1, ω_2)-convexity.

Theorem 5.1. *Let (ω_1, ω_2) be a Chebyshev system on an interval I. Furthermore, let $f : I \to \mathbb{R}$ be a continuous function. Then, the following assertions are equivalent:*

(i) *f is not (ω_1, ω_2)-convex;*
(ii) *there exist elements $x < y$ of I such that $\omega < f$ on $]x, y[$ where ω is the generalized line determined by the properties*

$$\omega(x) = f(x), \qquad \omega(y) = f(y);$$

(iii) *there exist elements $x < p < y$ of I and a generalized line ω such that $\omega \ge f$ on $[x, y]$. Moreover*

$$f(x) < \omega(x), \quad f(p) = \omega(p), \quad f(y) < \omega(y);$$

(iv) *there exists* $p \in I^\circ$ *such that* f *is* locally strictly (ω_1, ω_2)-concave at p, *that is, there exist elements* $x < p < y$ *of* I *such that, for all* $x < u < p < v < y$, *the following inequality holds:*

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} < 0.$$

Proof. $(i) \Rightarrow (ii)$. If f is not (ω_1, ω_2)-convex, then there exist elements $x_0 < p < y_0$ of I such that $\omega(p) < f(p)$, where ω is the generalized line determined by the properties $\omega(x_0) = f(x_0)$ and $\omega(y_0) = f(y_0)$ (see assertion (vi) of Theorem 3.1). Define the function $F : [x_0, y_0] \to \mathbb{R}$ by $F := f - \omega$, and the elements x and y by the formulae

$$x := \sup\{\, t \,|\, F(t) = 0,\ x_0 \le t < p \,\},$$
$$y := \inf\{\, t \,|\, F(t) = 0,\ p < t \le y_0 \,\}.$$

Clearly, $x_0 \le x < p < y \le y_0$ hold; moreover, $F(x) = F(y) = 0$ and $F > 0$ on $]x, y[$ due to the continuity of F. That is, $\omega(x) = f(x)$, $\omega(y) = f(y)$ and $f(t) > \omega(t)$ for all $t \in]x, y[$.

$(ii) \Rightarrow (iii)$. Take the elements $x < y$ of I and the generalized line ω fulfilling the properties $\omega(x) = f(x)$, $\omega(y) = f(y)$ and $\omega|_{]x,y[} < f|_{]x,y[}$. Define, for all $t \in \mathbb{R}$, the family of "parallel" generalized lines ω_t by the conditions

$$\omega_t(x) = \omega(x) + t, \qquad \omega_t(y) = \omega(y) + t.$$

Observe first that $\omega_t|_{[x,y]} > f|_{[x,y]}$ for "sufficiently large" t. Indeed, take the generalized line ω^* satisfying the inequality $\omega^*|_{[x,y]} > \max f|_{[x,y]}$ and choose $t > 0$ such that $\omega_t(x) > \omega^*(x)$ and $\omega_t(y) > \omega^*(y)$ hold. (The existence of ω^* is guaranteed by Lemma 5.1.) Then, $\omega_t|_{[x,y]} > \omega^*|_{[x,y]}$ due to Lemma 3.1 hence $\omega_t|_{[x,y]} > f|_{[x,y]}$. On the other hand, a similar argument to the previous one yields the inequalities $\omega_t|_{[x,y]} < \omega|_{[x,y]} \le f|_{[x,y]}$ for all $t < 0$. Therefore,

$$t_0 := \inf\{t \in \mathbb{R} \,|\, \omega_t|_{[x,y]} > f|_{[x,y]}\} \in \mathbb{R}.$$

By definition, $\omega_{t_0} \ge f$ on $[x, y]$. Assume indirectly that this inequality is strict. Then, according to the continuity of ω_{t_0} and f, there exists $\varepsilon > 0$ such that

$$f + \varepsilon < \omega_{t_0}$$

on $[x, y]$. Consider the sequence of generalized lines ω_n determined by the conditions

$$\omega_n(x) := \omega(x) + t_0 - \frac{1}{n}, \qquad \omega_n(y) := \omega(y) + t_0 - \frac{1}{n}.$$

Lemma 3.1 implies that (ω_n) is strictly monotone increasing; further, according to Lemma 5.2, $\omega_n \to \omega_{t_0}$ uniformly on the compact interval $[x, y]$ since $\omega_n(x) \to \omega_{t_0}(x)$ and $\omega_n(y) \to \omega_{t_0}(y)$. Hence, there exists an $n_0 \in \mathbb{N}$ satisfying the inequalities

$$\omega_{n_0} < \omega_{t_0} < \omega_{n_0} + \frac{\varepsilon}{2}.$$

Comparing this to the previous one, it follows that

$$f + \frac{\varepsilon}{2} < \omega_{n_0} < \omega_{t_0},$$

which contradicts the definition of t_0 since ω_n can also be written in the form $\omega_{t_0 - 1/n}$. Therefore, the choice ω_{t_0} satisfies the requirements.

$(iii) \Rightarrow (iv)$. Due to the continuity of the functions f and ω, we may assume that p is the minimal element of $]x, y[$ fulfilling the properties of the assertion. Then, $f(u) < \omega(u)$ if

$x < u < p$ and $f(v) \leq \omega(v)$ if $p < v < y$. Therefore,

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} < \begin{vmatrix} \omega(u) & \omega(p) & \omega(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix}$$

since the adjoint determinants of $f(u)$ and $f(v)$ are positive, furthermore, f and ω coincide at p. However, ω is a linear combination of ω_1 and ω_2, hence the left hand side of the previous inequality equals zero.

$(iv) \Rightarrow (i)$. Trivial. □

The next result shows that (ω_1, ω_2)-convexity, similarly to the standard one, is a pointwise property.

Corollary 5.1. *Let (ω_1, ω_2) be a Chebyshev system over the open interval I, furthermore $f : I \to \mathbb{R}$ is a given function. Then, the following assertions are equivalent:*

(i) *f is (ω_1, ω_2)-convex;*

(ii) *f is* locally (ω_1, ω_2)-convex, *that is, each element of the domain has a neighborhood where it is (ω_1, ω_2)-convex;*

(iii) *f is continuous and, for all $p \in I$, there exist elements $x < p < y$ of I such that*

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} \geq 0$$

for all $x < u < p < v < y$ (i. e., f is locally convex *at each point).*

Hint. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial. For the implication $(iii) \Rightarrow (i)$, the last assertion of Corollary 5.1 can be applied, which, in the case of indirect assumption, immediately leads to contradiction. □

5.2. Hermite–Hadamard-type inequalities and (ω_1, ω_2)-convexity.

The main results are presented in three theorems. The first and the second ones concern the left and right hand side inequalities of Theorem 3.4 independently, while the third one is analogous to the classical Jensen inequality.

Theorem 5.2. *Let (ω_1, ω_2) be a Chebyshev system on an interval $[a, b]$ such that ω_1 is positive on $]a, b[$, furthermore $\rho : [a, b] \to \mathbb{R}$ is a positive integrable function. Define, for all elements $x < y$ of $[a, b]$, the functions $\xi(x, y)$ and $c(x, y)$ by the formulae*

$$\xi(x, y) := \left(\frac{\omega_2}{\omega_1}\right)^{-1} \left(\frac{\int_x^y \omega_2\rho}{\int_x^y \omega_1\rho}\right), \qquad c(x, y) = \frac{\int_x^y \omega_1\rho}{\omega_1(\xi(x, y))}.$$

Then, a continuous function $f : [a, b] \to \mathbb{R}$ is generalized convex with respect to (ω_1, ω_2) if and only if, for all elements $x < y$ of $[a, b]$, it satisfies the inequality

$$c(x, y) f\big(\xi(x, y)\big) \leq \int_x^y f\rho.$$

Proof. The necessity is due to Theorem 3.4. For the converse assertion, note first that the mapping $(x, y) \mapsto \xi(x, y)$ is continuous in each variable and takes its value between x and y since it is a Lagrange-type mean-value. Further, $c(x, y)$ and $\xi(x, y)$ are constructed so that all generalized lines (i.e., the linear combinations of ω_1 and ω_2) are solutions of the functional equation

$$c(x, y)\omega\big(\xi(x, y)\big) = \int_x^y \omega\rho \qquad (x < y). \tag{5.1}$$

(For details, see the proof of Theorem 3.4.) Assume that f satisfies the inequality of the theorem and, indirectly, is not (ω_1, ω_2)-convex. Then, according to assertion (iii) of Theorem 5.1, there exist elements $x < p < y$ of I and a generalized line ω such that $f \le \omega$ on $[x, y]$ and

$$f(x) < \omega(x), \quad f(p) = \omega(p), \quad f(y) < \omega(y).$$

If, for example, $p \le \xi(x, y)$, then there is $u \in]p, y]$ such that $p = \xi(x, u)$ since ξ is a Lagrange-type mean-value. The inequality $f(x) < \omega(x)$ and the continuity of f implies that $f < \omega$ on a right hand side neighborhood of x hence, applying (5.1), it follows that

$$c(x, u)f\big(\xi(x, u)\big) \le \int_x^u f\rho < \int_x^u \omega\rho = c(x, u)\omega\big(\xi(x, u)\big).$$

On the other hand, both sides have the common value $c(x, u)f(p)$, which is a contradiction. □

Theorem 5.3. *Let (ω_1, ω_2) be a Chebyshev system over an interval $[a, b]$ such that ω_1 is positive on $]a, b[$, furthermore $\rho : [a, b] \to \mathbb{R}$ is a positive integrable function. Define, for all elements $x < y$ of $[a, b]$, the functions $c_1(x, y)$ and $c_2(x, y)$ by the formulae*

$$c_1(x, y) = \frac{\begin{vmatrix} \int_x^y \omega_1\rho & \omega_1(y) \\ \int_x^y \omega_2\rho & \omega_2(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}}, \qquad c_2(x, y) = \frac{\begin{vmatrix} \omega_1(x) & \int_x^y \omega_1\rho \\ \omega_2(x) & \int_x^y \omega_2\rho \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}}.$$

Then, a continuous function $f : [a, b] \to \mathbb{R}$ is generalized convex with respect to (ω_1, ω_2) if and only if, for all elements $x < y$ of $[a, b]$, it satisfies the inequality

$$\int_x^y f\rho \le c_1(x, y)f(x) + c_2(x, y)f(y).$$

Proof. The necessity is due to Theorem 3.4 again. Conversely, note first that $c_1(x, y)$ and $c_2(x, y)$ are constructed such that all generalized lines (i.e., the linear combinations of ω_1 and ω_2) are the solutions of the functional equation

$$\int_x^y \omega\rho = c_1(x, y)\omega(x) + c_2(x, y)\omega(y). \tag{5.2}$$

(For details, see the proof of Theorem 3.4.) Assume indirectly that f is not (ω_1, ω_2)-convex. Then, according to assertion (ii) of Theorem 5.1, there exist elements $x < y$ of I and a generalized line ω such that $\omega(x) = f(x)$, $\omega(y) = f(y)$ and $\omega < f$ on $]x, y[$. Therefore,

$$\begin{aligned} \int_x^y \omega\rho < \int_x^y f\rho &\le c_1(x, y)f(x) + c_2(x, y)f(y) \\ &= c_1(x, y)\omega(x) + c_2(x, y)\omega(y), \end{aligned}$$

which contradicts (5.2). □

Theorem 5.4. *Let (ω_1, ω_2) be a Chebyshev system on I and $f : I \to \mathbb{R}$ be a continuous function. Keeping the notations of Theorem 5.3 and Theorem 5.2, f is (ω_1, ω_2)-convex if and only if, for all elements $x < y$ of I, it satisfies the inequality*

$$c(x, y)f\big(\xi(x, y)\big) \le c_1(x, y)f(x) + c_2(x, y)f(y).$$

Proof. The necessity part has already been proved in Theorem 3.4. For the sufficiency, observe first that the functions c, c_1, c_2 and ξ are constructed so that all the generalized lines are solutions of the functional equation

$$c(x, y)\omega\big(\xi(x, y)\big) = c_1(x, y)\omega(x) + c_2(x, y)\omega(y) \qquad (x < y)$$

since both sides have the common value $\int_x^y \omega\rho$. Assume indirectly that a function $f : I \to \mathbb{R}$ satisfies the inequality of the theorem and is not generalized convex with respect to (ω_1, ω_2). Then, there exist elements $x < y$ of I and a generalized line ω fulfilling the conditions

$$\omega(x) = f(x), \qquad \omega|_{]x,y[} < f|_{]x,y[}, \qquad \omega(y) = f(y)$$

due to Theorem 5.1. Therefore, taking the above observation into consideration, one can immediately get that

$$\begin{aligned} c(x,y)f\big(\xi(x,y)\big) &\le c_1(x,y)f(x) + c_2(x,y)f(y) \\ &= c_1(x,y)\omega(x) + c_2(x,y)\omega(y) \\ &= c(x,y)\omega\big(\xi(x,y)\big) < c(x,y)f\big(\xi(x,y)\big), \end{aligned}$$

which is a contradiction. □

To give a unified view, the previous results are combined in the subsequent corollary. This corollary, Theorem 3.1, Corollary 3.2, Theorem 3.2 and Corollary 5.1 together are a comprehensive characterization of generalized convexity induced by two dimensional Chebyshev systems.

Corollary 5.2. *Let (ω_1, ω_2) be a Chebyshev system on I such that ω_1 is positive on I°, further, $\rho : I \to \mathbb{R}$ is a positive integrable function. Keeping the notations of Theorem 5.2, Theorem 5.3 and Theorem 5.4, the following assertions are equivalent for any function $f : I \to \mathbb{R}$:*

(i) *f is generalized convex with respect to (ω_1, ω_2);*
(ii) *f is continuous and, for all elements $x < y$ of I, satisfies the inequality*

$$c(x,y)f\big(\xi(x,y)\big) \le \int_x^y f\rho;$$

(iii) *f is continuous and, for all elements $x < y$ of I, satisfies the inequality*

$$\int_x^y f\rho \le c_1(x,y)f(x) + c_2(x,y)f(y);$$

(iv) *f is continuous and, for all elements $x < y$ of I, satisfies the inequality*

$$c(x,y)f\big(\xi(x,y)\big) \le c_1(x,y)f(x) + c_2(x,y)f(y).$$

The question arises, quite evidently, *whether Hermite–Hadamard-type inequalities also characterize generalized convexity in the general case or not*. To give an affirmative answer even in the polynomial case remains an open problem and may be the subject of further studies.

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