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SOME INEQUALITIES OF PERTURBED TRAPEZOID TYPE

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ABSTRACT. A new generalized perturbed trapezoid type inequality is established by Peano kernel approach. Some related results are also given.

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1. INTRODUCTION

In recent years, some authors have considered the perturbed trapezoid inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le C(\Gamma_2-\gamma_2)(b-a)^3,$$

where $f:[a,b]\to\mathbb{R}$ is a twice differentiable mapping on (a,b) with $\gamma_2 = \inf_{x\in[a,b]} f''(x) > -\infty$ and $\Gamma_2 = \sup_{x\in[a,b]} f''(x) < +\infty$ while C is a constant. (e.g. see [1] – [8]) It seems that the best result $C = \frac{\sqrt{3}}{108}$ was separately and independently discovered by the authors of [5] and [8]. The perturbed trapezoid inequality has been established as

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{\sqrt{3}}{108}(\Gamma_2-\gamma_2)(b-a)^3. \tag{1.1}$$

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Moreover, we can also find in [5] the following two perturbed trapezoid inequalities as

$$(1.2)\qquad \left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{1}{384}(\Gamma_3-\gamma_3)(b-a)^4,$$

where $f:[a,b]\to\mathbb{R}$ is a third-order differentiable mapping on (a,b) with $\gamma_3=\inf_{x\in[a,b]} f'''(x) > -\infty$ and $\Gamma_3=\sup_{x\in[a,b]} f'''(x) < +\infty$, and

$$(1.3)\qquad \left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{1}{720}M_4(b-a)^5,$$

where $f:[a,b]\to\mathbb{R}$ is a fourth-order differentiable mapping on (a,b) with $M_4=\sup\limits_{x\in[a,b]}|f^{(4)}(x)| < +\infty$.

The purpose of this paper is to extend these above results to a more general version by choosing appropriate harmonic polynomials such as the Peano kernel. A new generalized perturbed trapezoid type inequality is established and some related results are also given.

2. For Differentiable Mappings With Bounded Derivatives

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be an n-times continuously differentiable mapping, $n\ge 2$ and such that $M_n := \sup_{x\in[a,b]}|f^{(n)}(x)| < \infty$. Then*

$$(2.1)\qquad \left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]}\frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}}f^{(2k)}\left(\frac{a+b}{2}\right)\right| \le M_n\times\begin{cases}\frac{\sqrt{3}(b-a)^3}{54} & \text{if } n=2;\\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n} & \text{if } n\ge 3,\end{cases}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.

Proof. It is not difficult to find the identity

$$(2.2)\qquad (-1)^n\int_a^b T_n(x)f^{(n)}(x)\,dx = \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]}\frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}}f^{(2k)}\left(\frac{a+b}{2}\right),$$

where $T_n(x)$ is the kernel given by

$$(2.3)\qquad T_n(x)=\begin{cases}\frac{(x-a)^n}{n!}-\frac{(b-a)(x-a)^{n-1}}{2(n-1)!}+\frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!} & \text{if } x\in\left[a,\frac{a+b}{2}\right],\\ \frac{(x-b)^n}{n!}+\frac{(b-a)(x-b)^{n-1}}{2(n-1)!}+\frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!} & \text{if } x\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Using the identity (2.2), we get

$$(2.4)\quad \left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| = \left|\int_a^b T_n(x) f^{(n)}(x)\,dx\right| \le M_n \int_a^b |T_n(x)|\,dx.$$

For brevity, we put

$$\begin{aligned} P_n(x) &:= \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!} \\ &= \frac{(x-a)^{n-2}}{n!}\left[(x-a)^2 - \frac{n(b-a)(x-a)}{2} + \frac{n(n-1)(b-a)^2}{12}\right], \\ x &\in \left[a, \frac{a+b}{2}\right] \end{aligned}$$

and

$$\begin{aligned} Q_n(x) &:= \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!} \\ &= \frac{(x-b)^{n-2}}{n!}\left[(x-b)^2 + \frac{n(b-a)(x-b)}{2} + \frac{n(n-1)(b-a)^2}{12}\right], \\ x &\in \left[\frac{a+b}{2}, b\right]. \end{aligned}$$

It is clear that $P_n(x)$ and $Q_n(x)$ are symmetric with respect to the line $x = \frac{a+b}{2}$ for n even, and symmetric with respect to the point $(\frac{a+b}{2}, 0)$ for n odd. Therefore,

$$\begin{aligned} \int_a^b |T_n(x)|\,dx &= 2\int_a^{\frac{a+b}{2}} |P_n(x)|\,dx \\ &= \frac{(b-a)^{n+1}}{n!2^n} \int_0^1 \left| t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right]\right| dt \end{aligned}$$

by substitution $x = a + \frac{b-a}{2}t$, and it is easy to find that $r_n(t) := t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right]$ is always nonnegative on $[0,1]$ for $n \ge 3$. Thus we have

$$\int_0^1 |r_n(t)|\,dt = \int_0^1 t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right] dt = \frac{n(n-2)}{3(n+1)}$$

for $n \ge 3$, and

$$\begin{aligned} \int_0^1 |r_2(t)|\,dt &= \int_0^1 \left|t^2 - 2t + \frac{2}{3}\right| dt \\ &= \int_0^{t_0} \left(t^2 - 2t + \frac{2}{3}\right) dt - \int_{t_0}^1 \left(t^2 - 2t + \frac{2}{3}\right) dt, \end{aligned}$$

where $t_0 = 1 - \frac{\sqrt{3}}{3}$ is the unique zero of $r_2(t)$ in $(0,1)$. Hence,

$$\int_a^b |T_n(x)|\,dx = \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3. \end{cases} \tag{2.5}$$

Consequently, the inequality (2.1) follows from (2.4) and (2.5). □

Remark 2.2. If in the inequality (2.1) we choose $n = 2, 3, 4$, then we get

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \leq \frac{\sqrt{3}}{54} M_2 (b-a)^3,$$

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \leq \frac{1}{192} M_3 (b-a)^4$$

and the inequality (1.3), respectively.

For convenience in further discussions, we will now collect some technical results related to (2.3) which are not difficult to obtain by elementary calculus as:

$$\int_a^b T_n(x)\,dx = \begin{cases} 0, & n \text{ odd}, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \text{ even}. \end{cases} \tag{2.6}$$

$$\max_{x\in[a,b]} |T_n(x)| = \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4. \end{cases} \tag{2.7}$$

$$\max_{x\in[a,b]} \left| T_{2m}(x) - \frac{1}{b-a}\int_a^b T_{2m}(x)\,dx \right| = \begin{cases} \frac{(b-a)^4}{720}, & m=2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases} \tag{2.8}$$

3. Bounds In Terms of Some Lebesgue Norms

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_\infty[a,b]$, then we have*

$$\begin{aligned} &\Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\ &\qquad - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg| \\ &\qquad\qquad \leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3, \end{cases} \end{aligned} \tag{3.1}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$ and $\|f^{(n)}\|_\infty := ess\sup_{x\in[a,b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a,b]$.

The proof of inequality (3.1) is similar to the proof of inequality (2.1) and so is omitted.

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)} (n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_1[a,b]$, then we have*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \|f^{(n)}\|_1 \times \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4, \end{cases} \tag{3.2}$$

where $\|f^{(n)}\|_1 := \int_a^b |f(x)|\,dx$ is the usual Lebesgue norm on $L_1[a,b]$.

Proof. By using the identity (2.2), we get

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| = \left| \int_a^b T_n(x) f^{(n)}(x)\,dx \right| \leq \max_{x\in[a,b]} |T_n(x)| \int_a^b |f^{(n)}(x)|\,dx.$$

Then the inequality (3.2) follows from (2.7). □

4. NON-SYMMETRIC BOUNDS

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n)} (n \geq 2)$ is integrable with $\gamma_n = \inf_{x\in[a,b]} f^{(n)}(x) > -\infty$ and $\Gamma_n = \sup_{x\in[a,b]} f^{(n)}(x) < +\infty$. Then we have*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{\Gamma_n - \gamma_n}{2} \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3 \text{ and odd,} \end{cases} \tag{4.1}$$

$$
\begin{aligned}
(4.2)\quad &\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\
&\qquad - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right)\Bigg| \\
&\qquad\qquad \le [f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)] \\
&\qquad\qquad\qquad \times \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \ge 5 \textit{ and odd,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(4.3)\quad &\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\
&\qquad - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right)\Bigg| \\
&\qquad\qquad \le [\Gamma_n(b-a) - f^{(n-1)}(b) + f^{(n-1)}(a)] \\
&\qquad\qquad\qquad \times \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \ge 5 \textit{ and odd,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\quad &\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\
&\qquad - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\
&\qquad\quad - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)]\Bigg| \\
&\qquad\qquad \le [f^{(2m-1)}(b) - f^{(2m-1)}(a) - \gamma_{2m}(b-a)] \\
&\qquad\qquad\qquad \times \begin{cases} \frac{(b-a)^4}{720}, & m=2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \ge 3, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(4.5)\quad &\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\
&\qquad - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\
&\qquad\quad - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)]\Bigg|
\end{aligned}
$$

$$\leq [\Gamma_{2m}(b-a) - f^{(2m-1)}(b) + f^{(2m-1)}(a)] \begin{cases} \frac{(b-a)^4}{720}, & m = 2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases}$$

Proof. For n odd and $n = 2$, by (2.2) and (2.6) we get

$$(-1)^n \int_a^b T_n(x)[f^{(n)}(x) - C]\,dx$$
$$= \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]$$
$$- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right).$$

where $C \in \mathbb{R}$ is a constant.

If we choose $C = \frac{\gamma_n + \Gamma_n}{2}$, then we have

$$\Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]$$
$$- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg| \leq \frac{\Gamma_n - \gamma_n}{2} \int_a^b |T_n(x)|\,dx.$$

and hence the inequality (4.1) follows from (2.5).

If we choose $C = \gamma_n$, then we have

$$\Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]$$
$$- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg| \leq \max_{x\in[a,b]} |T_n(x)| \int_a^b |f^{(n)}(x) - \gamma_n|\,dx,$$

and hence the inequality (4.2) follows from (2.7).

Similarly we can prove that the inequality (4.3) holds.

By (2.2) and (2.6) we can also get

$$\Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]$$
$$- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right)$$
$$- \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)] \Bigg|$$
$$= \left| \int_a^b \left[T_{2m}(x) - \frac{1}{b-a}\int_a^b T_{2m}(x)\,dx \right] [f^{2m}(x) - C]\,dx \right|,$$

where $C \in \mathbb{R}$ is a constant.

If we choose $C = \gamma_{2m}$, then we have

$$\begin{aligned}\Bigg|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \\ - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\ - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b)-f^{(2m-1)}(a)]\Bigg| \\ \leq \max_{x\in[a,b]}\left|T_{2m}(x) - \frac{1}{b-a}\int_a^b T_{2m}(x)\,dx\right| \int_a^b |f^{(2m)}(x)-\gamma_{2m}|\,dx\end{aligned}$$

and hence the inequality (4.4) follows from (2.8). □

Similarly we can prove that the inequality (4.5) holds.

Remark 4.2. It is not difficult to find that the inequality (4.1) is sharp in the sense that we can choose f to attain the equality in (4.1). Indeed, for $n = 2$, we construct the function $f(x) = \int_a^x \left(\int_a^y j(z)\,dz\right) dy$, where

$$j(x) = \begin{cases} \Gamma_2, & a \leq x < \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6}, \\ \gamma_2, & \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6} \leq x < \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6}, \\ \Gamma_2, & \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6} \leq x \leq b, \end{cases}$$

and for $n \geq 3$ and odd, we construct the function

$$f(x) = \int_a^x \left(\int_a^{y_n} \left(\cdots \int_a^{y_2} j(y_1)\,dy_1 \cdots\right) dy_{n-1}\right) dy_n,$$

where

$$j(x) = \begin{cases} \Gamma_n, & a \leq x < \frac{a+b}{2}, \\ \gamma_n, & \frac{a+b}{2} \leq x \leq b. \end{cases}$$

Remark 4.3. If in the inequality (4.1) we choose $n = 2, 3$, then we recapture the inequalities (1.1) and (1.2), respectively.

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