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AN UNCONSTRAINED OPTIMIZATION TECHNIQUE FOR NONSMOOTH NONLINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. In this article, we consider an unconstrained minimization formulation of the nonlinear complementarity problem NCP(f) when the underlying functions are H-differentiable but not necessarily locally Lipschitzian or directionally differentiable. We show how, under appropriate regularity conditions on an H-differential of f, minimizing the merit function corresponding to f leads to a solution of the nonlinear complementarity problem. Our results give a unified treatment of such results for C^1-functions, semismooth-functions, and for locally Lipschitzian functions. We also show a result on the global convergence of a derivative-free descent algorithm for solving nonsmooth nonlinear complementarity problem.

Key words and phrases: Nonlinear complementarity problem, unconstrained minimization, NCP function, merit function, regularity conditions, nonsmooth function, descent algorithm.

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1. INTRODUCTION

We consider the nonlinear complementarity problem, denoted by NCP(f), which is to find a vector $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} \geq 0, \quad f(\bar{x}) \geq 0 \text{ and } \langle f(\bar{x}), \bar{x} \rangle = 0, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. This problem has a number of important applications in many fields, e.g., in operations research, economic equilibrium models and engineering sciences (in the form of contact problems, obstacle problems, equilibrium models,...), see [5], [17] for a more detailed description. Also, NCP(f) serves as a general framework for linear, quadratic, and nonlinear programming. Many methods have been developed for the solution of the nonlinear complementarity problem, see, e.g., [8], [11], [16], [17], and the references therein. Among these, one of the most popular approaches that has been studied extensively is to reformulate the NCP(f) as an unconstrained minimization

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problem through some merit function, see e.g., the survey paper by Fischer [8]. A function $\Psi : \mathbb{R}^n \to [0, \infty)$ is said to be a merit function for NCP(f) provided that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \ \bar{x} \text{ solves NCP}(f). \tag{1.2}$$

This leads to the following minimization problem:

$$\min_{x \in \mathbb{R}^n} \Psi(x). \tag{1.3}$$

NCP(f) is solvable if and only if the minimization problem (1.3) has a minimum value of zero. One way of constructing such a function is to define the so-called NCP function as follows:

A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP function if

$$\phi(a, b) = 0 \Leftrightarrow ab = 0, \quad a \geq 0, \, b \geq 0.$$

We call ϕ a nonnegative NCP function if $\phi(a, b) \geq 0$ on $\mathbb{R}^2$. Given ϕ for the problem NCP(f), we define

$$\Phi(x) = \begin{bmatrix} \phi(x_1, f_1(x)) \\ \vdots \\ \phi(x_i, f_i(x)) \\ \vdots \\ \phi(x_n, f_n(x)) \end{bmatrix} \tag{1.4}$$

and call $\Phi(x)$ an NCP function for NCP(f). We call Φ a nonnegative NCP function for NCP(f) if ϕ is nonnegative. If the NCP function is nonnegative, then we define the merit function Ψ at x by

$$\Psi(x) := \sum_{i=1}^{n} \Phi_i(x) = \sum_{i=1}^{n} \phi(x_i, f_i(x)), \tag{1.5}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$.

In this paper, we consider the following nonnegative NCP functions:

(1)

$$\begin{aligned} \Phi_i(x) &= \phi(x_i, f_i(x)) \\ &= \frac{\alpha}{2} \max{}^2\{0, x_i\, f_i(x)\} + \frac{1}{2} \left[\phi_{FB}(x_i, f_i(x))\right]^2 \\ &:= \frac{\alpha}{2} \max{}^2\{0, x_i\, f_i(x)\} + \frac{1}{2} \left[x_i + f_i(x) - \sqrt{x_i^2 + f_i(x)^2}\right]^2 \end{aligned} \tag{1.6}$$

where $\phi_{FB} : \mathbb{R}^2 \to \mathbb{R}$ is called the Fischer-Burmeister function and $\alpha \geq 0$ is a real parameter.

(2)

$$\begin{aligned} \Phi_i(x) &= \phi(x_i, f_i(x)) \\ &:= x_i f_i(x) + \frac{1}{2\alpha} \left[\max{}^2\{0, x_i - \alpha f_i(x)\} \right. \\ &\qquad \left. + \max{}^2\{0, f_i(x) - \alpha x_i\} - x_i^2 - f_i(x)^2\right], \end{aligned} \tag{1.7}$$

and where $\alpha > 1$ is any fixed parameter.

Yamada, Yamashita, and Fukushima [35] proposed the NCP function in (1.6) to solve the NCP in (1.1). In (1.6), when $\alpha = 0$, the NCP function reduced to the squared Fischer-Burmeister function. The NCP function in (1.7) was proposed by Mangasarian and Solodov

[23]. For f a C^1 function, Yamada, Yamashita, and Fukushima [35] proved that the $\Psi(x)$ corresponding to the NCP function in (1.6) is a C^1 and nonnegative merit function. For f a C^1 function, Mangasarian and Solodov [23] proved that the $\Psi(x)$ based on the NCP function in (1.7) is a C^1 and nonnegative merit function. Jiang [19] generalized some results in [23] to the case where the considered function is directionally differentiable.

In this paper, we extend/generalize these results to nonsmooth functions which admit H-differentiability, but are not necessarily locally Lipschitzian or directionally differentiable. Our results are applicable to any nonnegative NCP functions satisfying Lemma 3.4, but for simplicity, we consider the Yamada, Yamashita, and Fukushima function (1.6) and the implicit Lagrangian function (1.7).

The basic motivations of using the concepts of H-differentiability and H-differential are: H-differentiability implies continuity, any superset of an H-differential is an H-differential, and H-differentials enjoy simple sum, product, chain rules, a mean value theorem and a second order Taylor-like expansion, and inverse and implicit function theorems, see [13], [14], [15]. An H-differentiable function is not necessarily locally Lipschitzian or directionally differentiable. The Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [1], the Bouligand differential of a semismooth function [27], and the C-differential of Qi [28] are particular instances of H-differentials; moreover, the closure of the H-differential is an approximate Jacobian [18].

For some applications of H-differentiability to optimization problems, nonlinear complementarity problems and variational inequalities, see e.g. [31], [34] and [33].

The paper is organized as follows. In Section 2, we recall some definitions and basic facts which are needed in the subsequent analysis. In Section 3, we describe the H-differential of the Yamada, Yamashita, and Fukushima function, implicit Lagrangian function and their merit functions. Also, we show how, under appropriate regularity -conditions on an H-differential of f, finding local/global minimum of Ψ (or a 'stationary point' of Ψ) leads to a solution of the given nonlinear complementarity problem. Our results unify/extend various similar results proved in the literature for C^1, locally Lipschitzian, and semismooth functions [3], [4], [7], [9], [12], [17], [19], [20], [22], [23], [35], [36]. Moreover, we present a result on the global convergence of a derivative-free descent algorithm for solving a nonsmooth nonlinear complementarity problem.

2. PRELIMINARIES

Throughout this paper, we consider vectors in $\mathbb{R}^n$ as column vectors. Vector inequalities are interpreted componentwise. We denote the inner-product between two vectors x and y in $\mathbb{R}^n$ by either x^Ty or $\langle x, y\rangle$. For a matrix A, A_i denotes the ith row of A. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla f(\bar{x})$ denotes the Jacobian matrix of f at $\bar{x}$.

We need the following definitions from [2], [26].

Definition 2.1. A matrix $A \in \mathbb{R}^{n\times n}$ is called $\mathbf{P_0}$ (**P**) if $\forall x \in \mathbb{R}^n, x \neq 0$, there exists i such that $x_i \neq 0$ and $x_i\,(Ax)_i \geq 0\ (> 0)$, or equivalently, every principle minor of A is nonnegative (respectively, positive).

2.1. **H-differentiability and H-differentials.** We now recall the following definition from Gowda and Ravindran [15].

Definition 2.2. Given a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where Ω is an open set in $\mathbb{R}^n$ and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$ (also denoted by $T_F(x^*)$) of $\mathbb{R}^{m\times n}$ is an *H-differential of* F at x^* if for every sequence $\{x^k\} \subseteq \Omega$ converging to x^*, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that

$$F(x^{k_j}) - F(x^*) - A(x^{k_j} - x^*) = o(||x_j^k - x^*||). \tag{2.1}$$

We say that F is H-differentiable at x^* if F has an H-differential at x^*.

Remark 1. As observed in [34], if a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is H-differentiable at a point $\bar{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\bar{x}, \delta)$ of $\bar{x}$ with

$$||F(x) - F(\bar{x})|| \leq L||x - \bar{x}||, \ \forall x \in B(\bar{x}, \delta). \tag{2.2}$$

Conversely, if condition (2.2) holds, then $T(\bar{x}) := \mathbb{R}^{m \times n}$ can be taken as an H-differential of F at $\bar{x}$. We thus have, in (2.2), an alternate description of H-differentiability. However, as we see in the sequel, it is the identification of an appropriate H-differential that becomes important and relevant.

Clearly any function locally Lipschitzian at $\bar{x}$ will satisfy (2.2). For real valued functions, condition (2.2) is known as the 'calmness' of F at $\bar{x}$. This concept has been well studied in the literature of nonsmooth analysis (see [30, Chapter 8]).

In the rest of this section we show that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function, the Bouligand subdifferential of a semismooth function, and the C-differential of a C-differentiable function are particular instances of H-differentials [15].

2.2. **Fréchet differentiable functions.** Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be Fréchet differentiable at $x^* \in \mathbb{R}^n$ with a Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla F(x^*)\}$ so that

$$F(x) - F(x^*) - \nabla F(x^*)(x - x^*) = o(||x - x^*||).$$

Then F is H-differentiable with $\{\nabla F(x^*)\}$ as an H-differential.

2.3. **Locally Lipschitzian functions.** Let $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitzian at each point of an open set Ω. For $x^* \in \Omega$, define the Bouligand subdifferential of F at x^* by

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\},$$

where Ω_F is the set of all points in Ω where F is Fréchet differentiable. Then, the (Clarke) generalized Jacobian [1]

$$\partial F(x^*) = co\partial_B F(x^*)$$

is an H-differential of F at x^*.

2.4. **Semismooth functions.** Consider a locally Lipschitzian function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that is semismooth at $x^* \in \Omega$ [24], [27], [29]. This means that for any sequence $x^k \to x^*$, and for any $V_k \in \partial F(x^k)$,

$$F(x^k) - F(x^*) - V_k(x^k - x^*) = o(||x^k - x^*||).$$

Then the Bouligand subdifferential

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}$$

is an H-differential of F at x^*. In particular, this holds if F is piecewise smooth, i.e., there exist continuously differentiable functions $F_j : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F(x) \in \{F_1(x), F_2(x), \ldots, F_J(x)\} \quad \forall\, x \in \mathbb{R}^n.$$

2.5. C**-differentiable functions.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C-differentiable [28] in a neighborhood D of x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset \mathbb{R}^{n\times n}$ satisfying the following condition at any $a \in D$: For any $V \in T(x)$,

$$F(x) - F(a) - V(x-a) = o(||x-a||).$$

Then, F is H-differentiable at x^* with $T(x^*)$ as an H-differential.

Remark 2. While the Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [1], the Bouligand differential of a semismooth function [27], and the C-differential of a C-differentiable function [28] are particular instances of H-differentials, the following simple example, taken from [13], shows that an H-differentiable function need not be locally Lipschitzian and/or directionally differentiable. Consider on $\mathbb{R}$,

$$F(x) = x \sin\left(\frac{1}{x}\right) \text{ for } x \neq 0 \text{ and } F(0) = 0.$$

Then F is H-differentiable on $\mathbb{R}$ with

$$T(0) = [-1,1] \text{ and } T(c) = \left\{\sin\left(\frac{1}{c}\right) - \frac{1}{c}\cos\left(\frac{1}{c}\right)\right\} \text{ for } c \neq 0.$$

We note that F is not locally Lipschitzian around zero. We also see that F is neither Fréchet differentiable nor directionally differentiable.

3. THE MAIN RESULTS

For a given H-differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, consider the associated Yamada, Yamashita, and Fukushima function/implicit Lagrangian function (or any nonnegative NCP function satisfying Lemma 3.4) Φ and the corresponding merit function $\Psi := \sum_{i=1}^n \Phi_i$. It should be recalled that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves NCP}(f).$$

3.1. H**-differentials of some NCP/merit functions.** First, we compute the H-differential of the merit function Ψ as given in (1.5). In what follows, e denotes the vector of ones.

Theorem 3.1. *Suppose Φ is H-differentiable at $\bar{x}$ with $T_\Phi(\bar{x})$ as an H-differential. Then $\Psi := \sum_{i=1}^n \Phi_i$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{e^T B : B \in T_\Phi(\bar{x})\}.$$

Proof. To describe an H-differential of Ψ as given in (1.5), let $\theta(x) = x_1 + \cdots + x_n$. Then $\Psi = \theta \circ \Phi$ so that by the chain rule for H-differentiability, we have $T_\Psi(\bar{x}) = (T_\theta \circ T_\Phi)(\bar{x})$ as an H-differential of Ψ at $\bar{x}$. Since $T_\theta(\bar{x}) = \{e^T\}$, we have

$$T_\Psi(\bar{x}) = \{e^T B : B \in T_\Phi(\bar{x})\}.$$

This completes the proof. □

Now, we describe the H-differential of the implicit Lagrangian function.

Theorem 3.2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with $T(\bar{x})$ as an H-differential. Consider Φ as in (1.7). Then the implicit Lagrangian function $\Psi := \sum_{i=1}^n \Phi_i$*

is H-differentiable with an H-differential $T_\Psi(\bar{x})$ consisting of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are column vectors with entries defined by

$$v_i = \bar{x}_i + \frac{1}{\alpha}\left[-\alpha \max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} + \max\{0, f_i(\bar{x}) - \alpha \bar{x}_i\} - f_i(\bar{x})\right], \tag{3.1}$$
$$w_i = f_i(\bar{x}) + \frac{1}{\alpha}\left[\max\{0, \bar{x}_i - \alpha f_i(\bar{x})\} - \bar{x}_i - \alpha \max\{0, f_i(\bar{x}) - \alpha \bar{x}_i\}\right].$$

Proof. First we show that an H-differential of

$$\Phi(x) := x * f(x) + \frac{1}{2\alpha}\left[\max{}^2\{0, x - \alpha f(x)\} + \max{}^2\{0, f(x) - \alpha x\} - x^2 - f(x)^2\right] \tag{3.2}$$

is given by

$$T_\Phi(\bar{x}) = \{B = VA + W : \ A \in T(\bar{x}),\ V = \operatorname{diag}(v_i) \text{ and } W = \operatorname{diag}(w_i)$$
$$\text{where } v_i, w_i \text{ satisfy } (3.1)\}.$$

Let $g(x) = \max\{0, x - \alpha f(x)\}$, $h(x) = \max\{0, f(x) - \alpha x\}$. For each $A \in T(\bar{x})$, let A' and A'' be matrices such that for $i = 1, \dots, n$,

$$A'_i \in \begin{cases} \{e_i - \alpha A_i\} & \text{if } \bar{x}_i - \alpha f_i(\bar{x}) > 0 \\ \{0, e_i - \alpha A_i\} & \text{if } \bar{x}_i - \alpha f_i(\bar{x}) = 0 \\ \{0\} & \text{if } \bar{x}_i - \alpha f_i(\bar{x}) < 0, \end{cases} \tag{3.3}$$

and

$$A''_i \in \begin{cases} \{A_i - \alpha e_i\} & \text{if } f_i(\bar{x}) - \alpha \bar{x}_i > 0 \\ \{0, A_i - \alpha e_i\} & \text{if } f_i(\bar{x}) - \alpha \bar{x}_i = 0 \\ \{0\} & \text{if } f_i(\bar{x}) - \alpha \bar{x}_i < 0. \end{cases} \tag{3.4}$$

Then it can be easily verified that $T_g(\bar{x}) = \{A' | A \in T(\bar{x})\}$ and $T_h(\bar{x}) = \{A'' | A \in T(\bar{x})\}$ are H-differentials of g and h, respectively. Now simple calculations show that $T_\Phi(\bar{x})$ consists of matrices of the form

$$B = [\operatorname{diag}(\bar{x})\, A + \operatorname{diag}(f(\bar{x}))] \tag{3.5}$$
$$+ \frac{1}{2\alpha}\left[2 \operatorname{diag}(g(\bar{x}))\, A' + 2 \operatorname{diag}(h(\bar{x}))\, A'' - 2 \operatorname{diag}(\bar{x}) - 2 \operatorname{diag}(f(\bar{x}))\right],$$

where A' and A'' for $A \in T(\bar{x})$ are defined by (3.3) and (3.4), respectively.

Since $g_i(x) = 0$ when $x_i - \alpha f_i(x) \leq 0$, we have

$$\operatorname{diag}(g(\bar{x}))\, A' = \operatorname{diag}(g(\bar{x}))(I - \alpha A).$$

Similarly, $\operatorname{diag}(h(\bar{x}))\, A'' = \operatorname{diag}(h(\bar{x}))(A - \alpha I)$.

Therefore, (3.5) becomes

$$B = \Bigg[\operatorname{diag}(\bar{x}) + \frac{1}{\alpha}\left[-\alpha \operatorname{diag}(\max\{0, \bar{x} - \alpha f(\bar{x})\}) + \operatorname{diag}(\max\{0, f(\bar{x}) - \alpha \bar{x}\}) \right. \tag{3.6}$$
$$\left. - \operatorname{diag}(f(\bar{x}))\right]\Bigg] A + \Bigg[\operatorname{diag}(f(\bar{x})) + \frac{1}{\alpha}\left[\operatorname{diag}(\max\{0, \bar{x} - \alpha f(\bar{x})\}) \right.$$
$$\left. - \alpha \operatorname{diag}(\max\{0, f(\bar{x}) - \alpha \bar{x}\})\right]\Bigg] = VA + W,$$

where V and W are diagonal matrices with diagonal entries given by (3.1). By Theorem 3.1, we have

$$T_\Psi(\bar{x}) = \{e^T(VA+W) = v^TA + w^T : A \in T(\bar{x}), v \text{ and } w \text{ are vectors in } \mathbb{R}^n \text{ with components defined by (3.1)}\}. \tag{3.7}$$

This completes the proof. □

We describe the H-differential of the Yamada, Yamashita, and Fukushima function.

Theorem 3.3. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with $T(\bar{x})$ as an H-differential. Consider the associated Yamada, Yamashita, and Fukushima function*

$$\Phi(x) = \frac{\alpha}{2}\,[x * f(x)]_+^2 + \frac{1}{2}\left[x + f(x) - \sqrt{x^2 + f(x)^2}\right]^2, \tag{3.8}$$

where all the operations are performed componentwise, $x_+ = \max\{0, x\}$ and $\alpha \geq 0$ is a real parameter. Let

$$J(\bar{x}) = \{i : f_i(\bar{x}) = 0 = \bar{x}_i\} \text{ and } K(\bar{x}) = \{i : \bar{x}_i > 0,\ f_i(\bar{x}) > 0\}.$$

Then the H-differential of Φ is given by

$$T_\Phi(\bar{x}) = \{VA + W :\ (A, V, W, d) \in \Gamma\},$$

where Γ is the set of all quadruples (A, V, W, d) with $A \in T(\bar{x})$, $||d|| = 1$, $V = \operatorname{diag}(v_i)$ and $W = \operatorname{diag}(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \phi_{FB}(\bar{x}_i, f_i(\bar{x}))\left(1 - \frac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + \alpha\,\bar{x}_i^2\, f_i(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \phi_{FB}(d_i, A_i\, d)\left(1 - \frac{A_i d}{\sqrt{d_i^2 + (A_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 > 0 \\ \phi_{FB}(\bar{x}_i, f_i(\bar{x}))\left(1 - \frac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + \alpha\,\bar{x}_i^2\, f_i(\bar{x}) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 = 0, \end{cases} \tag{3.9}$$

$$w_i = \begin{cases} \phi_{FB}(\bar{x}_i, f_i(\bar{x}))\left(1 - \frac{\bar{x}_i}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + \alpha\,\bar{x}_i\, f_i^2(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \phi_{FB}(d_i, A_i\, d)\left(1 - \frac{d_i}{\sqrt{d_i^2 + (A_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 > 0 \\ \phi_{FB}(\bar{x}_i, f_i(\bar{x}))\left(1 - \frac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + \alpha\,\bar{x}_i\, f_i^2(\bar{x}) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 = 0. \end{cases}$$

Proof. Similar to the calculation and analysis of Examples 5-7 in [34]. □

Remark 3.

- The calculation in Theorem 3.3 relies on the observation that the following is an H-differential of the one variable function $t \mapsto t_+$ at any $\bar{t}$:

$$\Delta(\bar{t}) = \begin{cases} \{1\} & \text{if } \bar{t} > 0 \\ \{0,1\} & \text{if } \bar{t} = 0 \\ \{0\} & \text{if } \bar{t} < 0. \end{cases}$$

- By Theorem 3.1, the H-differential $T_\Psi(\bar{x})$ of $\Psi(\bar{x}) = \sum_{i=1}^n \Phi_i(x)$ on the basis of the square Fischer-Burmeister function consists of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are column vectors with entries defined by (3.9).

We conclude this subsection with the following lemma that will be needed in the sequel. The proof is similar to Lemma 3.1 of [12].

Lemma 3.4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with $T(\bar{x})$ as an H-differential. Suppose that Φ is defined as in Theorems 3.2 – 3.3, H-differentiable with an H-differential $T_\Phi(\bar{x})$ as given by*

$$T_\Phi(\bar{x}) = \{VA + W : \ A \in T(\bar{x}), \ V = \operatorname{diag}(v_i) \text{ and } W = \operatorname{diag}(w_i)\}, \tag{3.10}$$

and Ψ is H-differentiable with an H-differential $T_\Psi(\bar{x})$. Then Φ is nonnegative and the following properties hold:

$$\left.\begin{array}{rl} (\mathbf{i}) & \bar{x} \text{ solves NCP(f) } \Leftrightarrow \Phi(\bar{x}) = 0. \\ (\mathbf{ii}) & \text{For } i \in \{1,\dots,n\}, \ v_i w_i \geq 0. \\ (\mathbf{iii}) & \text{For } i \in \{1,\dots,n\}, \ \Phi_i(\bar{x}) = 0 \Leftrightarrow (v_i, w_i) = (0,0). \\ (\mathbf{iv}) & \text{For } i \in \{1,\dots,n\} \text{ with } \bar{x}_i \geq 0 \text{ and } f(\bar{x}_i) \geq 0, \text{ we have } v_i \geq 0. \\ (\mathbf{v}) & \text{If } 0 \in T_\Psi(\bar{x}), \text{then } \Phi(\bar{x}) = 0 \Leftrightarrow v = 0. \end{array}\right\} \tag{3.11}$$

In the following subsection, we show that under appropriate regularity conditions, a vector $\bar{x}$ is a solution of the NCP(f) if and only if zero belongs to $T_\Psi(\bar{x})$ (when the underlying functions are H-differentiable.)

3.2. **Minimizing the merit function under regularity (strict regularity) conditions.** We generalize the concept of a regular (strictly regular) point from [3], [6], [22], [25].

For a given H-differentiable function f and $\bar{x} \in \mathbb{R}^n$, we define the following index sets:

$$\mathcal{P}(\bar{x}) := \{i : v_i > 0\}, \qquad \mathcal{N}(\bar{x}) := \{i : v_i < 0\},$$

$$\mathcal{C}(\bar{x}) := \{i : v_i = 0\}, \qquad \mathcal{R}(\bar{x}) := \mathcal{P}(\bar{x}) \cup \mathcal{N}(\bar{x}),$$

where v_i are the entries of V in (3.10) (e.g., v_i is defined in Theorems 3.2 – 3.3).

Definition 3.1. Consider f, Φ as in (1.6) or (1.7), and Ψ as (1.5). A vector $x^* \in \mathbb{R}^n$ is called *strictly regular* if, for every nonzero vector $z \in \mathbb{R}^n$ such that

$$z_\mathcal{C} = 0, \ z_\mathcal{P} > 0, \ z_\mathcal{N} < 0, \tag{3.12}$$

there exists a vector $s \in \mathbb{R}^n$ such that

$$s_\mathcal{P} \geq 0, \ s_\mathcal{N} \leq 0, \ s_\mathcal{C} = 0, \text{ and} \tag{3.13}$$

$$s^T A^T z > 0 \quad \text{for all } A \in T(x^*). \tag{3.14}$$

Remark 4. It is possible for Φ in Definition 3.1 to be any nonnegative NCP function satisfying Lemma 3.4.

Theorem 3.5. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Assume Φ is defined as in Theorems 3.2 – 3.3. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\},$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w are vectors in $\mathbb{R}^n$ satisfying properties (ii), (iii), and (v) in (3.11).

Then $\bar{x}$ solves NCP(f) if and only if $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point.

Proof. The 'if' part of the theorem follows easily from the definitions. Now suppose that $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point. Then for some $v^T A + w^T \in T_\Psi(\bar{x})$,

$$0 = v^T A + w^T \Rightarrow A^T v + w = 0. \tag{3.15}$$

We claim that $\Phi(\bar{x}) = 0$. Assume the contrary that $\bar{x}$ is not a solution of NCP(f). Then by property (v) in (3.11), we have v as a nonzero vector satisfying $v_{\mathcal{C}} = 0,\ v_{\mathcal{P}} > 0,\ v_{\mathcal{N}} < 0$. Since $\bar{x}$ is a strictly regular point, and $v_i w_i \geq 0$ by property (ii) in (3.11), by taking a vector $s \in \mathbb{R}^n$ satisfying (3.13) and (3.14), we have

$$s^T A^T v > 0 \tag{3.16}$$

and

$$s^T w = s_{\mathcal{C}}^T w_{\mathcal{C}} + s_{\mathcal{P}}^T w_{\mathcal{P}} + s_{\mathcal{N}}^T w_{\mathcal{N}} \geq 0. \tag{3.17}$$

Thus we have $s^T(A^T v + w) = s^T A^T v + s^T w > 0$. We reach a contradiction to (3.15). Hence, $\bar{x}$ is a solution of NCP(f). □

Now we state a consequence of the above theorem.

Theorem 3.6. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Assume Φ is defined as in Theorems 3.2 – 3.3.*

Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\},$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w are vectors in $\mathbb{R}^n$ satisfying properties (ii), (iii), and (v) in (3.11).

Further suppose that $T(\bar{x})$ consists of positive-definite matrices. Then

$$\Phi(\bar{x}) = 0 \Leftrightarrow 0 \in T_\Psi(\bar{x}).$$

Proof. The proof follows by taking $s = z$ in Definition 3.1 for a strictly regular point and by using Theorem 3.5. □

Before we state the next theorem, we recall a definition from [32].

Definition 3.2. Consider a nonempty set $\mathcal{C}$ in $\mathbb{R}^{n\times n}$. We say that a matrix A is a *row representative of* $\mathcal{C}$ if for each index $i = 1, 2, \ldots, n$, the ith row of A is the ith row of some matrix $C \in \mathcal{C}$. We say that $\mathcal{C}$ has the *row-$\mathbf{P}_0$-property* (*row-$\mathbf{P}$-property*) if every row representative of $\mathcal{C}$ is a $\mathbf{P}_0$-matrix ($\mathbf{P}$-matrix). We say that $\mathcal{C}$ has the *column-$\mathbf{P}_0$-property* (*column-$\mathbf{P}$-property*) if $\mathcal{C}^T = \{A^T : A \in \mathcal{C}\}$ has the *row-$\mathbf{P}_0$-property* (*row-$\mathbf{P}$-property*).

Theorem 3.7. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Assume Φ is defined as in Theorems 3.2 – 3.3. Assume that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\},$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w are vectors in $\mathbb{R}^n$ satisfying properties (ii), (iii), and (v) in (3.11).

Further, suppose that $T(\bar{x})$ has the column-$\mathbf{P}$-property. Then

$$\bar{x} \text{ solves NCP}(f) \text{ if and only if } 0 \in T_\Psi(\bar{x}).$$

Proof. In view of Theorem 3.5, it is enough to show that $\bar{x}$ is a strictly regular point. To see this, let v be a nonzero vector satisfying (3.12). Since $T(\bar{x})$ has the column-$\mathbf{P}$-property, by Theorem 2 in [32], there exists an index j such that $v_j \left[A^T v\right]_j > 0 \ \forall A \in T(\bar{x})$. Choose $s \in \mathbb{R}^n$ so that $s_j = v_j$ and $s_i = 0$ for all $i \neq j$. Then $s^T A^T v = v_j \left[A^T v\right]_j > 0 \ \forall A \in T(\bar{x})$. Hence $\bar{x}$ is a strictly regular point. □

Theorem 3.8. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Assume that Φ is defined as in Theorems 3.2 – 3.3. Suppose $\Psi := \sum_{i=1}^n \Phi_i$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\},$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w are vectors in $\mathbb{R}^n$ satisfying properties (iii) and (v) in (3.11), and $v_i\, w_i > 0$ whenever $\Phi_i(\bar{x}) \neq 0$.

Further, suppose that $T(\bar{x})$ consists of $\mathbf{P}_0$-matrices. Then

$$\Phi(\bar{x}) = 0 \Leftrightarrow 0 \in T_\Psi(\bar{x}).$$

Proof. The proof is similar to that of Theorem 3.7. □

As a consequence of the above theorem, we have the following corollary.

Corollary 3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzian. Let Φ be the square Fischer-Burmeister function. Suppose that $\Psi := \sum_{i=1}^n \Phi_i(\bar{x})$. Further, assume that $\partial_B f(\bar{x})$ has the column-$\mathbf{P_0}$-property. Then*

$$\Psi(\bar{x}) = 0 \Leftrightarrow 0 \in \partial\Psi(\bar{x}).$$

Proof. We note that by Corollary 1 in [34], every matrix in $\partial f(\bar{x}) = co\, \partial_B f(\bar{x})$ is a $\mathbf{P}_0$-matrix. In fact, by applying Theorem 3.8 with $T_f(x) = \partial f(x)$ and using a result by Fischer in [9] that $\partial\Psi(x) \subseteq T_{\Psi(x)}$ for all x, we obtain the result. □

Remark 5.

- The usefulness of Corollary 3.9 is seen when the function f is piecewise smooth, in which case $\partial_B f(\bar{x})$ consists of a finite number of matrices.
- It is noted that in [31] $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathbf{P}_0(\mathbf{P})$-function if f is H-differentiable on $\mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, an H-differential $T_f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices. A simple consequence of this result is the following:

Corollary 3.10. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Assume that Φ is defined as in Theorems 3.2 – 3.3. Suppose that $\Psi := \sum_{i=1}^n \Phi_i$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{v^T A + w^T : (A, v, w) \in \Omega\},$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w are vectors in $\mathbb{R}^n$ satisfying properties (iii) and (v) in (3.11), and $v_i\, w_i > 0$ whenever $\Phi_i(\bar{x}) \neq 0$.

Further, suppose that f is a $\mathbf{P}_0$-function. Then

$$\Phi(\bar{x}) = 0 \Leftrightarrow 0 \in T_{\Psi}(\bar{x}).$$

4. DERIVATIVE-FREE DESCENT METHOD

We present a derivative-free descent method which does not require the computation of the derivatives of the function f involved in the NCP and the merit function Ψ.

Descent direction methods were proposed for when f is smooth by the authors in [12], [36] for solving NCP functions in (1.6) and (1.7), respectively. The author in [9] obtained similar results for when f is locally Lipschitzian.

Now our goal is to formulate the derivative-free line search algorithm according to [10], [12]. We define the search direction $s(x) := -\nabla_2\Psi_1(x, f(x))$, for all $x \in \mathbb{R}^n$ where

$$\Psi(x) = \Psi_1(x, f(x)) := \sum_{i=1}^{n} \Phi_i(\bar{x}) = \sum_{i=1}^{n} \phi(x_i, f_i(x)) \text{ as in (1.6) and (1.7).}$$

Then we define the function $\theta : \mathbb{R}^n \to \mathbb{R}$ by

$$\theta(x) = \nabla_1\Psi_1(x, f(x))^T \nabla_2\Psi_1(x, f(x)),$$

where $\nabla_1\Psi_1$ and $\nabla_2\Psi_1$ denote the partial derivatives of Ψ_1 with respect to the first variable and the second variable.

Here is the algorithm.

Algorithm 4.1. Given $\alpha, \beta \in (0, 1)$, $x^0 \in \mathbb{R}^n$, for $k = 0, 1, 2, ...$, do the following steps:

(i) If $\Psi(x^k) = 0$, stop.
(ii) Set $s^k = s(x^k)$ and choose $t_k \in \{\alpha^j | j \in N\}$ as large as possible such that

$$\Psi(x^k + t_k s^k) \leq \Psi(x^k) - \beta t_k \theta(x^k).$$

(iii) Set $x^{k+1} = x^k + t_k s^k$. Return to (i).

The following definition is needed in Theorem 4.1.

Definition 4.1 ([10])**.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called comonotone at $x \in \mathbb{R}^n$ in the direction $u \in \mathbb{R}^n$ if there exists $\nu_{(x,u)} > 0$ so that the following inequality holds:

$$\langle f(x + tu) - f(x), u\rangle \geq \nu_{(x,u)} ||f(x + tu) - f(x)||,$$

for all $t \geq 0$ sufficiently small.

The following theorem shows the convergence of Algorithm 4.1.

Theorem 4.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable and a monotone map. If f is comonotone at each $x \in \mathbb{R}^n$ in each direction $u \in \mathbb{R}^n$ for which the relation*

$$\limsup_{t\downarrow 0} ||f(x + tu) - f(x)||/t = +\infty$$

is satisfied , then Algorithm 4.1 is well defined and any accumulation point of the sequence $\{x^k\}$ generated by Algorithm 4.1 solves the nonlinear complementarity problem.

Proof. Since H-differentiability implies continuity as noted in [15], the result now follows from Theorem 5.1 in [10]. $\square$

Remark 6. Note that an accumulation point of the sequence $\{x^k\}$ generated by Algorithm 4.1 exists if the level set

$$L(x^0) := \{x \in \mathbb{R}^n | \Psi(x) \leq \Psi(x^0)\}$$

is bounded. The boundedness of $L(x^0)$ can be established under the assumption that f is a uniform $\mathbf{P}$-function, see [9], [19].

Remark 7. Note that the following property does not hold for an implicit Lagrangian function,

$$v_i w_i = 0 \not\Rightarrow v_i = 0 = w_i \quad \text{for all } i. \tag{4.1}$$

For example, when

$$f_i(x) - \alpha x_i > 0 \quad \text{and} \quad x_i - \alpha f_i(x) \leq 0.$$

For the Yamada, Yamashita and Fukushima function, we have

$$v_i w_i = 0 \Longleftrightarrow v_i = 0 = w_i \quad \text{for all } i. \tag{4.2}$$

The property (4.2) is important in proving the convergence of our algorithm. Thus, the interested reader can show that the proof of Theorem 5.1 in [10] is not applicable to an NCP function based on an implicit Lagrangian function due to the property (4.1). Therefore, we cannot prove the convergence of the algorithm in [10] for an NCP function based on an implicit Lagrangian function because the proof of Theorem 5.1 relies on the property (4.2). Our algorithm is applicable to the Yamada, Yamashita and Fukushima function and any NCP function possessing the same properties as the Yamada, Yamashita, and Fukushima function.

5. CONCLUDING REMARKS

We considered a nonlinear complementarity problem corresponding to H-differentiable functions, with an associated nonnegative NCP function Φ and a merit function $\Psi(\bar{x}) := \sum_{i=1}^n \Phi_i(\bar{x})$ and showed that under certain regularity conditions the global/local minimum or a stationary point of Ψ is a solution of NCP(f).

For nonlinear complementarity problems based on the implicit Lagrangian function or/and the Yamada, Yamashita, and Fukushima function, our results recover/extend various results stated for nonlinear complementarity problems when the underlying functions are continuously differentiable (locally Lipschitzian, semismooth, and directionally differentiable). Our results are applicable to any nonnegative NCP function satisfying Lemma 3.4, but for simplicity, we consider the Yamada, Yamashita, and Fukushima function and the implicit Lagrangian function. Indeed, as far as the author is aware, solving nonlinear complementarity problems on the basis of the Yamada, Yamashita, and Fukushima function is considered new when the underlying functions are H-differentiable.

It worth noting that an H-differentiable function need not be locally Lipschitzian/ directionally differentiable; hence the approaches taken in [9], [19] are not applicable to NCP(f) when the underlying functions are merely H-differentiable.

We note here that similar methodologies under H-differentiability can be carried out for the following NCP functions:

(1)
$$\phi_1(a,b) := \frac{1}{2}\min{}^2\{a,b\}.$$

(2)
$$\phi_2(a,b) := \frac{1}{2}[(ab)^2 + \min{}^2\{0,a\} + \min{}^2\{0,b\}].$$

(3)
$$\phi_3(a,b) := \frac{1}{2}\left[\phi_\beta(a,b)\right]^2 = \frac{1}{2}\left[a+b-\sqrt{(a-b)^2+\beta ab}\right]^2$$

ϕ_1 and ϕ_2 were introduced by Kanzow [20]. The NCP function ϕ_β was proposed by Kanzow and Kleinmichel [21].

REFERENCES

[1] F.H. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983 (reprinted by SIAM, Philadelphia, PA, 1990).

[2] R.W. COTTLE, J.-S. PANG AND R.E. STONE, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[3] F. FACCHINEI AND C. KANZOW, On unconstrained and constrained stationary points of the implicit Lagrangian, *Journal of Optimization Theory and Applications*, **92** (1997), 99–115.

[4] F. FACCHINEI AND J. SOARES, A new merit function for nonlinear complementarity problems and related algorithm, *SIAM Journal on Optimization*, **7** (1997), 225–247.

[5] M.C. FERRIS AND J.-S. PANG, Engineering and economic applications of complementarity problems, *SIAM Review*, **39** (1997), 669–713.

[6] M.C. FERRIS AND D. RALPH, Projected gradient methods for nonlinear complementarity problems via normal maps, *Recent Advances in Nonsmooth Optimization*, D.Z. Du, L.Qi, and R.S. Womersley (Eds.), World Scientific Publishers, Singapore, Republic of Singapore (1995), 57–87.

[7] A. FISCHER, A new constrained optimization reformulation for complementarity problems, *Journal of Optimization Theory and Applications*, **97** (1998), 105–117.

[8] A. FISCHER, An NCP-function and its use for the solution of complementarity problems, *Recent Advances in Nonsmooth Optimization*, D.Z. Du, L.Qi, and R.S. Womersley (Eds.), World Scientific Publishers, Singapore, Republic of Singapore (1995), 88–105.

[9] A. FISCHER, Solution of monotone complementarity problems with locally Lipschitzian functions, *Mathematical Programming*, **76** (1997), 513–532.

[10] A. FISCHER, V. JEYAKUMAR AND D.T. LUC, Solution point characterizations and convergence analysis of a descent algorithm for nonsmooth continuous complementarity problems, *Journal of Optimization Theory and Applications*, **110** (2001), 493–513.

[11] M. FUKUSHIMA, Merit functions for variational inequality and complementarity problems, *Nonlinear Optimization and Applications*, G. Di Pillo and F. Giannessi (Eds.), Plenum Press, (1996), 155–170.

[12] C. GEIGER AND C. KANZOW, On the resolution of monotone complementarity problems, *Computational Optimization*, **5** (1996), 155–173.

[13] M.S. GOWDA, A note on H-differentiable functions, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland 21250, November, 1998.

[14] M.S. GOWDA, Inverse and implicit function theorems for H-differentiable and semismooth functions, *Optimization Methods and Software*, **19** (2004), 443–461.

[15] M.S. GOWDA AND G. RAVINDRAN, Algebraic univalence theorems for nonsmooth functions, *Journal of Mathematical Analysis and Applications*, **252** (2000), 917–935.

[16] J.-S. PANG, Complementarity problems, in *Handbook on Global Optimization*, R. Horst and P. Pardalos (Eds.), Kluwer Academic publishers, Boston, 1994.

[17] P.T. HARKER AND J.-S. PANG, Finite dimension variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Mathematical Programming*, **48** (1990), 161–220.

[18] J. JEYAKUMAR AND D.T. LUC, Approximate Jacobians matrices for nonsmooth continuous maps and C^1-optimization, *SIAM Journal on Control and Optimization*, **36** (1998), 1815–1832.

[19] H. JIANG, Unconstrained minimization approaches to nonlinear complementarity problems, *Journal of Global Optimization*, **9** (1996), 169–181.

[20] C. KANZOW, Nonlinear complementarity as unconstrained optimization, *J. Optimization Theory and Applications*, **88** (1996), 139–155.

[21] C. KANZOW AND H. KLEINMICHEL, A new class of semismooth Newton-type methods for nonlinear complementarity problems, *Computational Optimization and Applications*, **11** (1998), 227–251.

[22] T.D. LUCA, F. FACCHINEI, AND C. KANZOW, A semismooth equation approach to the solution of nonlinear complementarity problems, *Mathematical Programming*, **75** (1996), 407–439.

[23] O.L. MANGASARIAN AND M.V. SOLODOV, Nonlinear complementarity as unconstrained and constrained minimization, *Mathematical Programming*, **62** (1993), 277–297.

[24] R. MIFFLIN, Semismooth and semiconvex functions in constrained optimization, *SIAM Journal on Control and Optimization*, **15** (1977), 952–972.

[25] J.J. MORÉ, Global methods for nonlinear complementarity problems, *Mathematics of Operations Research*, **21** (1996), 589–614.

[26] J.J. MORÉ AND W.C. RHEINBOLDT, On P- and S- functions and related classes of N-dimensional nonlinear mappings, *Linear Algebra and its Applications*, **6** (1973), 45–68.

[27] L. QI, Convergence analysis of some algorithms for solving nonsmooth equations, *Mathematics of Operations Research*, **18** (1993), 227–244.

[28] L. QI, C-Differentiability, C-differential operators and generalized Newton methods, Research Report, School of Mathematics, The University of New South Wales, Australia, January 1996.

[29] L. QI AND J. SUN, A nonsmooth version of Newton's method, *Mathematical Programming*, **58** (1993), 353–367.

[30] R.T. ROCKAFELLAR AND R.J.-B. WETS, *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften, 317, Springer-Verlag, Berlin, Germany, 1998.

[31] Y. SONG, M.S. GOWDA AND G. RAVINDRAN, On characterizations of $\mathbf{P}$- and $\mathbf{P}_0$-properties in nonsmooth functions, *Mathematics of Operations Research*, **25** (2000), 400–408.

[32] Y. SONG, M.S. GOWDA AND G. RAVINDRAN, On some properties of **P**-matrix sets, *Linear Algebra and its Applications*, **290** (1999), 246–273.

[33] M.A. TAWHID, On the local uniqueness of solutions of variational inequalities under H-differentiability, *Journal of Optimization Theory and Applications*, **113** (2002), 149–164.

[34] M.A. TAWHID AND M.S. GOWDA, On two applications of H-differentiability to optimization and complementarity problems, *Computational Optimization and Applications*, **17** (2000), 279–299.

[35] K. YAMADA, N. YAMASHITA AND M. FUKUSHIMA, A new derivative-free descent method for the nonlinear complementarity problem, *Nonlinear Optimization and Related Topics*, (Erice, 1998), 463–487, Appl. Optim., 36, Kluwer Acad. Publ., Dordrecht, 2000.

[36] N. YAMASHITA AND M. FUKUSHIMA, On stationary points of the implicit Lagrangian for nonlinear complementarity problems, *J. Optimization Theory and Applications*, **84** (1995), 653–663.