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ON SOME CLASSES OF ANALYTIC FUNCTIONS

KHALIDA INAYAT NOOR AND M.A. SALIM

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
COLLEGE OF SCIENCE,
UNITED ARAB EMIRATES UNIVERSITY
P.O. BOX 17551, AL-AIN,
UNITED ARAB EMIRATES.
KhalidaN@uaeu.ac.ae

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ABSTRACT. We define some classes of analytic functions related with the class of functions with bounded boundary rotation and study these classes with reference to certain integral operators.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let $C, S^\star, K$ and S be the subclasses of $\mathcal{A}$ which are respectively convex, starlike, close-to-convex and univalent in E. It is known that $C \subset S^\star \subset K \subset S$. In [1], Kaplan showed that $f \in K$ if, and only if, for $z \in E,\ 0 \leq \theta_1 < \theta_2 \leq 2\pi,\ 0 < r < 1$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta > -\pi, \quad z = re^{i\theta}.$$

Let $V_k (k \geq 2)$ be the class of locally univalent functions $f \in \mathcal{A}$ that map E conformally onto a domain whose boundary rotation is at most $k\pi$. It is well known that $V_2 \equiv C$ and $V_k \subset K$ for $2 \leq k \leq 4$.

Definition 1.1. Let $f \in \mathcal{A}$ and $f'(z) \neq 0$. Then $f \in T_k(\lambda),\ k \geq 2,\ 0 \leq \lambda < 1$ if there exists a function $g \in V_k$ such that, for $z \in E$

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \lambda.$$

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The class $T_k(0) = T_k$ was considered in [2, 3] and $T_2(0) = K$, the class of close-to-convex functions.

Definition 1.2. Let $f \in \mathcal{A}$ and $\frac{f(z)f'(z)}{z} \neq 0,\ z \in E$. Then $f \in T_k(a, \gamma, \lambda)$, $\operatorname{Re} a \geq 0,\ 0 \leq \gamma \leq 1$ if, and only if, there exists a function $g \in T_k(\lambda)$ such that

$$zf'(z) + af(z) = (a+1)z(g'(z))^{\gamma}, \quad z \in E. \tag{1.1}$$

We note that $T_k(0, 1, \lambda) = T_k(\lambda)$ and $T_2(0, 1, \lambda) = K(\lambda) \subset K$, and it follows that $f \in T_k(a, \gamma, \lambda)$ if, and only if, there exists $F \in T_k(\infty, \gamma, \lambda)$ such that

$$f(z) = \frac{a+1}{z^a} \int_0^z t^{a-1} F(t) dt.$$

2. Preliminary Results

Lemma 2.1 ([2])**.** *Let $f \in \mathcal{A}$. Then, for $0 \leq \theta_1 < \theta_2 \leq 2\pi,\ \ z = re^{i\theta},\ \ 0 < r < 1$, $f \in T_k$ if and only if*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{zf'(z))'}{f'(z)}\right\} d\theta > -\frac{k}{2}\pi.$$

Lemma 2.2. *Let $q(z)$ be analytic in E and of the form $q(z) = 1 + b_1 z + \cdots$ for $|z| = r \in (0,1)$. Then, for $a, c_1, \theta_1, \theta_2$ with $a \geq 1,\ \operatorname{Re}(c_1) \geq 0,\ 0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{q(z) + \frac{azq'(z)}{c_1 a + q(z)}\right\} d\theta > -\beta_1\pi; \qquad (0 < \beta_1 \leq 1)$$

implies

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z) d\theta > -\beta_1\pi, \qquad z = re^{i\theta}.$$

This result is a direct consequence of the one proved in [4] *for $\beta_1 = 1$.*

From (1.1) and Lemma 2.1, we can easily have the following:

Lemma 2.3. *A function $f \in T_k(\infty, \gamma, \lambda)$ if and only if, it may be represented as $f(z) = p(z) \cdot zG'(z)$, where $G \in V_k$ and $\operatorname{Re} p(z) > \lambda,\ z \in E$.*

Proof. Since $f \in T_k(\infty, \gamma, \lambda)$, we have

$$\begin{aligned} f(z) &= z(g'(z))^{\gamma}, \qquad g \in T_k(\lambda) \\ &= z\left[G_1'(z)p_1(z)\right]^{\gamma},\ \ G_1 \in V_k,\ \ \operatorname{Re} p_1(z) > \lambda \\ &= zG'(z).p(z), \end{aligned}$$

where $G'(z) = (G_1'(z))^{\gamma} \in V_k$ and $p(z) = (p_1(z))^{\gamma},\ \ \operatorname{Re} p(z) > \lambda$, since $0 \leq \gamma \leq 1$.

The converse case follows along similar lines. □

Using Lemma 2.1 and Lemma 2.3, we have:

Lemma 2.4.

(i) *Let $f \in T_k(0, \gamma, \lambda)$. Then, with $z = re^{i\theta},\ 0 \leq \theta_1 < \theta - 2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\frac{k\gamma}{2}, \quad \textit{see also } [3].$$

(ii) *Let $f \in T_k(\infty, \gamma, \lambda)$. Then, for $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta > -\frac{k\gamma}{2}.$$

3. MAIN RESULTS

Theorem 3.1. *For* $0 < \alpha < \frac{1}{1-\lambda+\lambda\beta},\ \ 0 < \beta < \frac{\lambda}{1-\lambda},\ \ 0 \leq \lambda < \frac{1}{2}$ *and* $f, g \in T_k(\infty, \gamma, \lambda),\ z \in E$, *let*

$$F(z) = \left[\left(\beta + \frac{1}{\alpha}\right) z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2}(f(t))^\beta g(t)dt\right]^{\frac{1}{1+\beta}}. \tag{3.1}$$

Then F_1, *with* $F = zF_1'$ *and* $0 < \gamma < 1,\ k \leq \frac{2}{\gamma}$, *is close-to-convex and hence univalent in* E.

Proof. We can write (3.1) as

$$(F(z))^{\beta+1} = \left(\beta + \frac{1}{\alpha}\right) z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2}(f(t))^\beta g(t)dt. \tag{3.2}$$

Let

$$\frac{zF'(z)}{F(z)} = \frac{(zF_1'(z))'}{F_1'(z)} = (1-\lambda)H(z) + \lambda, \tag{3.3}$$

where $H(z)$ is analytic in E and $H(z) = 1 + c_1 z + c_2 z^2 + \cdots$.

We differentiate (3.2) logarithmically to obtain

$$(\beta+1)\frac{zF'(z)}{F(z)} = \left(1 - \frac{1}{\alpha}\right) + \frac{z^{\frac{1}{\alpha}-1}(f(z))^\beta g(z)}{\int_0^z t^{\frac{1}{\alpha}-2}(f(z))^\beta(t)g(t)dt}.$$

Using (3.2) and differentiating again, we have after some simplifications,

$$(1-\lambda)zH'\frac{\int_0^z t^{\frac{1}{\alpha}-2}(f(t))^\beta g(t)dt}{z^{\frac{1}{\alpha}-1}(f(z))^\beta g(z)} + (1-\lambda)H(z) = \frac{\beta}{1+\beta}\cdot\frac{zf'(z)}{f(z)} + \frac{1}{\beta+1}\cdot\frac{zg'(z)}{g(z)} - \lambda.$$

Now

$$\frac{z^{\frac{1}{\alpha}-1}(f(z))^\beta g(z)}{\int_0^z t^{\frac{1}{\alpha}-2}(f(t))^\beta g(t)dt} = \left(\frac{1}{\alpha} - 1\right) + (1+\beta)\frac{zF'(z)}{F(z)}.$$

Hence

$$-\lambda + \frac{\beta}{1+\beta}\cdot\frac{zf'(z)}{f(z)} + \frac{1}{\beta+1}\cdot\frac{zg'(z)}{g(z)} = (1-\lambda)H(z) + \frac{(1-\lambda)zH'(z)}{(1-\lambda)(1+\beta)H(z) + (\frac{1}{\alpha}-1) + \lambda(1+\beta)}$$

and we have

$$\frac{1}{1-\lambda}\left[\frac{\beta}{1+\beta}\left(\frac{zf'(z)}{f(z)} - \lambda\right) + \frac{1}{1+\beta}\left(\frac{zg'(z)}{g(z)} - \lambda\right)\right] = H(z) + \frac{\frac{1}{(1+\beta)(1-\lambda)}zH'(z)}{H(z) + \left[\frac{(\frac{1}{\alpha}-1)}{(1+\beta)(1-\lambda)} + \frac{\lambda}{1-\lambda}\right]}. \tag{3.4}$$

Since $f, g \in T_k(\infty, \gamma, \lambda)$, so with $z = re^{i\theta},\ 0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\frac{\beta}{1+\beta}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{1}{1-\lambda}\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right\} d\theta + \frac{1}{1+\beta}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{1}{1-\lambda}\left(\frac{zg'(z)}{g(z)} - \lambda\right)\right\} d\theta > \frac{-k\gamma}{2}\pi,$$

and, therefore,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[H(z) + \frac{\frac{1}{(1+\beta)(1-\lambda)} zH'(z)}{H(z) + \left\{\frac{(\frac{1}{\alpha}-1)}{(1+\beta)(1-\lambda)} + \frac{\lambda}{1-\lambda}\right\}}\right] d\theta > \frac{-k\gamma}{2}\pi.$$

Now using Lemma 2.2 with $a = \frac{1}{(1+\beta)(1-\lambda)} \geq 1,\ c_1 = \left\{\left(\frac{1}{\alpha} - 1\right) + (1+\beta)\lambda\right\} \geq 0$, we obtain the required result. ☐

Theorem 3.2. *Let $f, g \in T_k(\infty, \gamma, \lambda),\ \alpha, c, \delta$ and ν be positively real, $0 < \alpha \leq \frac{1}{1-\lambda},\ c > \alpha(1-\lambda),\ (\nu + \delta) = \alpha$. Then the function F defined by*

$$[F(z)]^{\alpha} = cz^{\alpha - c}\int_0^z t^{(c-\delta-\nu)-1}(f(t))^{\delta}(g(t))^{\nu} dt \tag{3.5}$$

belongs to $T_k(\infty, \gamma, \lambda)$ for $k \leq \frac{2}{\gamma},\ 0 < \gamma < 1$.

Proof. First we show that there exists an analytic function F satisfying (3.5).

Let

$$\begin{aligned} G(z) &= z^{-(\nu+\delta)}(f(z))^{\delta}(g(z))^{\nu} \\ &= 1 + d_1 z + d_2 z^2 + \cdots \end{aligned}$$

and choose the branches which equal 1 when $z = 0$. For

$$K(z) = z^{(c-\nu-\delta)-1}(f(z))^{\delta}(g(z))^{\nu} = z^{c-1}G(z),$$

we have

$$L(z) = \frac{c}{z^c}\int_0^z K(t)dt = 1 + \frac{c}{1+c}d_1 z + \cdots .$$

Hence L is well-defined and analytic in E.

Now let

$$F(z) = \left[z^{\alpha}L(z)\right]^{\frac{1}{\alpha}} = z\left[L(z)\right]^{\frac{1}{\alpha}},$$

where we choose the branch of $[L(z)]^{\frac{1}{\alpha}}$ which equals 1 when $z = 0$. Thus F is analytic in E and satisfies (3.5).

Set

$$\frac{zF'(z)}{F(z)} = (1-\lambda)h(z) + \lambda, \tag{3.6}$$

and let

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= (1-\lambda)h_1(z) + \lambda \\ \frac{zg'(z)}{g(z)} &= (1-\lambda)h_2(z) + \lambda. \end{aligned}$$

Now, from (3.5), we have

$$z^{(c-\alpha)}\left[F(z)\right]^{\alpha}\left[(c-\alpha)+\alpha\frac{zF'(z)}{F(z)}\right]=c\left[z^{(c-\delta-\nu)-1}(f(z))^{\delta}(g(z))^{\nu}\right]. \tag{3.7}$$

We differentiate (3.7) logarithmically and use (3.6) to obtain

$$\begin{aligned}\alpha(1-\lambda)\left[h(z)+\frac{zh'(z)}{(c-\alpha)+\alpha\{\lambda+(1-\lambda)h(z)\}}\right]+(\delta+\nu-\alpha)\\ =\delta\frac{zf'(z)}{f(z)}+\nu\frac{zg'(z)}{g(z)}-\alpha\lambda\\ =\delta\left[\frac{zf'(z)}{f(z)}-\lambda\right]+\nu\left[\frac{zg'(z)}{g(z)}-\lambda\right].\end{aligned}$$

This gives us

$$\begin{aligned}&\left[h(z)+\frac{zh'(z)}{(c-\alpha)+\alpha\{\lambda+(1-\lambda)h(z)\}}\right]\\ &\qquad=\frac{\delta}{\alpha(1-\lambda)}\left[\frac{zf'(z)}{f(z)}-\lambda\right]+\frac{\nu}{\alpha(1-\lambda)}\left[\frac{zg'(z)}{g(z)}-\lambda\right].\end{aligned}$$

Since $f,g\in T_k(\infty,\gamma,\lambda)$, we have, for $0\le\theta_1<\theta_2\le 2\pi$, $z=re^{i\theta}$,

$$\begin{aligned}\int_{\theta_1}^{\theta_2}\operatorname{Re}&\left[h(z)+\frac{zh'(z)}{(c-\alpha)+\alpha\{\lambda+(1-\lambda)h(z)\}}\right]d\theta\\ &=\left[\frac{\delta}{\alpha}\int_{\theta_1}^{\theta_2}\operatorname{Re}h_1(z)d\theta+\frac{\nu}{\alpha}\int_{\theta_1}^{\theta_2}\operatorname{Re}h_2(z)d\theta\right]\\ &>\frac{\delta}{\alpha}\left(-\frac{\gamma k}{2}\pi\right)+\frac{\nu}{\alpha}\left(-\frac{\gamma k}{2}\pi\right)\\ &=\frac{\delta+\nu}{\alpha}\left(-\frac{\gamma k}{2}\pi\right)=-\frac{\gamma k}{2}\pi,\end{aligned}$$

where we have used Lemma 2.4.

Now using Lemma 2.2 with $a=\frac{1}{\alpha(1-\lambda)}>1$, for $\alpha<\frac{1}{1-\lambda}$ and

$$c_1=c-\alpha+\alpha\lambda=c-\alpha(1-\lambda)\ge 0,$$

we obtain the required result. □

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