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SUBORDINATION RESULTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract

In this paper, we derive several interesting subordination results for certain class of analytic functions defined by the linear operator $\mathcal{L}(a,c)f(z)$ which introduced and studied by Carlson and Shaffer [2].

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{1.2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n. \tag{1.3}$$

Define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\},\ z \in \Delta), \tag{1.4}$$

where $(\lambda)_n$ is the Pochhammer symbol given, in terms of Gamma functions,

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n \in \mathbb{N} : \{1, 2, \dots\}. \end{cases} \tag{1.5}$$

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [2] introduced a linear operator $\mathcal{L}(a, c) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}(a, c)f(z) := \phi(a, c; z) * f(z), \tag{1.6}$$

or, equivalently, by

$$\mathcal{L}(a, c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \qquad (z \in \Delta).$$

Note that $\mathcal{L}(1, 1)f(z) = f(z)$, $\mathcal{L}(2, 1)f(z) = zf'(z)$ and $\mathcal{L}(3, 1)f(z) = zf'(z)+ \frac{1}{2}z^2 f''(z)$.

For $-1 \leq \alpha < 1$, $\beta \geq 0$, we let $\mathcal{L}(a, c; \alpha, \beta)$ consist of functions f in $\mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{\frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - (a-1)\right\} > \beta \left|\frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a\right| + \alpha, \qquad (z \in \Delta) \tag{1.7}$$

The family $\mathcal{L}(a, c; \alpha, \beta)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\mathcal{L}(1, 1; \alpha, 0)$, we obtain the family of starlike functions of order α $(0 \leq \alpha < 1)$ and $\mathcal{L}(2, 1; \alpha, 0)$ is the family of convex functions of order α $(0 \leq \alpha < 1)$. For $\mathcal{L}(1, 1; 0, \beta)$ and $\mathcal{L}(2, 1; 0, \beta)$, we obtain the class of uniformly β- starlike functions and uniformly β- convex functions, respectively, introduced by Kanas and Winsiowska

([3],[4]) (see also the work of Kanas and Srivastava [5], Goodman ([7],[8]), Rønning ([10],[11]), Ma and Minda [9] and Gangadharan et al. [6]).

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.1 (Subordination Principle). *Let $g(z)$ be analytic and univalent in Δ. If $f(z)$ is analytic in Δ, $f(0) = g(0)$, and $f(\Delta) \subset g(\Delta)$, then we see that the function $f(z)$ is subordinate to $g(z)$ in Δ, and we write $f(z) \prec g(z)$.*

Definition 1.2 (Subordinating Factor Sequence). *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic , univalent and convex in Δ, we have the subordination given by*

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z) \qquad (z \in \Delta,\ a_1 = 1). \tag{1.8}$$

Lemma 1.1 ([14]). *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \qquad (z \in \Delta). \tag{1.9}$$

Lemma 1.2. *If*

$$\sum_{n=2}^{\infty} \sigma_n(a, c; \alpha, \beta)\, |a_n| \le 1 - \alpha \tag{1.10}$$

where, for convenience,

$$\sigma_n(a,c;\alpha,\beta) := \frac{(1+\beta)(a)_n + [1-\alpha-a(1+\beta)](a)_{n-1}}{(c)_{n-1}} \tag{1.11}$$

$$(-1 \leq \alpha < 1;\ \beta \geq 0,\ n \geq 2),$$

then $f(z) \in \mathcal{L}(a,c;\alpha,\beta)$.

Proof. It suffices to show that

$$\beta \left| \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right| - \operatorname{Re}\left\{ \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right\} \leq 1-\alpha.$$

We have

$$\begin{aligned}
\beta \left| \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right| &- \operatorname{Re}\left\{ \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right\} \\
&\leq (1+\beta)\left| \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right| \\
&\leq \frac{(1+\beta)\sum_{n=2}^{\infty}\left(\frac{a(a+1)_{n-1}-a(a)_{n-1}}{(c)_{n-1}}\right)|a_n|\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\,|z|^{n-1}} \\
&\leq \frac{(1+\beta)\sum_{n=2}^{\infty}\left(\frac{(a)_n-a(a)_{n-1}}{(c)_{n-1}}\right)|a_n|}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.
\end{aligned}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty} \frac{(1+\beta)(a)_n + [1-\alpha-a(1+\beta)](a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1-\alpha$$

and the proof is complete. $\square$

Let $\mathcal{L}^{\star}(a,c;\alpha,\beta)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (1.10). We note that $\mathcal{L}^{\star}(a,c;\alpha,\beta) \subseteq \mathcal{L}(a,c;\alpha,\beta)$.

2. Main Theorem

Employing the techniques used earlier by Srivastava and Attiya [13], Attiya [1] and Singh [12], we state and prove the following theorem.

Theorem 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^\star(a,c;\alpha,\beta)$ where $-1 \leq \alpha < 1$; $\beta \geq 0$; $a > 0$; $c > 0$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then*

$$\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}(f*g)(z) \prec g(z) \qquad (z\in\Delta;\ g\in\mathcal{K}), \tag{2.1}$$

and

$$\operatorname{Re}(f(z)) > -\frac{1-\alpha+\sigma_2(a,c;\alpha,\beta)}{\sigma_2(a,c;\alpha,\beta)}, \qquad (z\in\Delta). \tag{2.2}$$

The constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$ is the best estimate.

Proof. Let $f(z) \in \mathcal{L}^\star(a,c;\alpha,\beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned} &\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}(f*g)(z) \\ &\qquad = \frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}\left(z+\sum_{n=2}^{\infty} a_n c_n z^n\right). \end{aligned} \tag{2.3}$$

Thus, by Definition 1.2, the assertion of our theorem will hold if the sequence

$$\left\{\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} \frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]} a_n z^n\right\} > 0 \qquad (z \in \Delta). \tag{2.5}$$

Now

$$\begin{aligned}
\operatorname{Re}&\left\{1 + \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} \sum_{n=1}^{\infty} a_n z^n\right\} \\
&= \operatorname{Re}\left\{1 + \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} z \right. \\
&\qquad\left. + \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} \sum_{n=1}^{\infty} \sigma_2(a,c;\alpha,\beta) a_n z^n\right\} \\
&\geq 1 - \left\{\frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} r \right. \\
&\qquad\left. - \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} \sum_{n=1}^{\infty} \sigma_n(a,c;\alpha,\beta) a_n r^n\right\}.
\end{aligned}$$

Since $\sigma_n(a,c;\alpha,\beta)$ is an increasing function of n $(n \geq 2)$

$$\begin{aligned}
1 - &\left\{\frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} r \right. \\
&\qquad\left. - \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} \sum_{n=1}^{\infty} \sigma_n(a,c;\alpha,\beta) a_n r^n\right\}
\end{aligned}$$

$$> 1 - \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} r - \frac{1-\alpha}{1-\alpha+\sigma_2(a,c;\alpha,\beta)} r \qquad (|z| = r)$$
$$> 0.$$

Thus (2.5) holds true in Δ. This proves the inequality (2.1). The inequality (2.2) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ in (2.1). To prove the sharpness of the constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$, we consider the function $f_0(z) \in \mathcal{L}^\star(a,c;\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\alpha}{\sigma_2(a,c;\alpha,\beta)} z^2 \qquad (-1 \leq \alpha < 1;\ \beta \geq 0). \tag{2.6}$$

Thus from (2.1), we have

$$\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]} f_0(z) \prec \frac{z}{1-z}. \tag{2.7}$$

It can easily verified that

$$\min\left\{ \operatorname{Re}\left(\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]} f_0(z) \right) \right\} = -\frac{1}{2} \qquad (z \in \Delta), \tag{2.8}$$

This shows that the constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$ is best possible. $\square$

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^\star(1,1;\alpha,\beta)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)]\, |a_n| \leq 1 - \alpha \tag{2.9}$$

then

$$\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}(f*g)(z) \prec g(z) \tag{2.10}$$
$$(-1 \leq \alpha < 1;\ \beta \geq 0;\ z \in \Delta;\ g \in \mathcal{K})$$

and

$$\operatorname{Re}(f(z)) > -\frac{\beta+3-2\alpha}{\beta+2-\alpha}, \qquad (z \in \Delta). \tag{2.11}$$

The constant $\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}$ *is the best estimate.*

Corollary 2.3. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{L}^{\star}(1,1;\alpha,0)$ *and satisfy the condition*

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| \leq 1-\alpha, \tag{2.12}$$

then

$$\frac{2-\alpha}{6-4\alpha}(f*g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.13}$$

and

$$\operatorname{Re}(f(z)) > -\frac{3-2\alpha}{2-\alpha}, \qquad (z \in \Delta). \tag{2.14}$$

The constant $\frac{2-\alpha}{6-4\alpha}$ *is the best estimate.*

Putting $\alpha = 0$ in Corollary 2.3, we obtain

Corollary 2.4 ([12]). *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^{\star}(1,1;0,0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n\,|a_n| \leq 1 \tag{2.15}$$

then

$$\frac{1}{3}(f*g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.16}$$

and

$$\mathrm{Re}(f(z)) > -\frac{3}{2}, \qquad (z \in \Delta). \tag{2.17}$$

The constant $1/3$ is the best estimate.

Corollary 2.5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^{\star}(2,1;\alpha,\beta)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\,|a_n| \leq 1-\alpha, \tag{2.18}$$

then

$$\frac{\beta+2-\alpha}{2\beta+5-3\alpha}(f*g)(z) \prec g(z) \tag{2.19}$$
$$(-1 \leq \alpha < 1;\ \beta \geq 0;\ z \in \Delta;\ g \in \mathcal{K})$$

and

$$\operatorname{Re}(f(z)) > -\frac{2\beta+5-3\alpha}{2(\beta+2-\alpha)}, \qquad (z \in \Delta). \tag{2.20}$$

The constant $\frac{\beta+2-\alpha}{2\beta+5-3\alpha}$ *is the best estimate.*

Corollary 2.6. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{L}^{\star}(2,1;\alpha,0)$ *and satisfy the condition*

$$\sum_{n=2}^{\infty} n(n-\alpha)\,|a_n| \le 1-\alpha, \tag{2.21}$$

then

$$\frac{2-\alpha}{5-3\alpha}(f*g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.22}$$

and

$$\operatorname{Re}(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)}, \qquad (z \in \Delta). \tag{2.23}$$

The constant $\frac{2-\alpha}{5-3\alpha}$ *is the best estimate.*

Putting $\alpha = 0$ in Corollary 2.6, we obtain

Corollary 2.7. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{L}^{\star}(2,1;0,0)$ *and satisfy the condition*

$$\sum_{n=2}^{\infty} n^2\,|a_n| \le 1 \tag{2.24}$$

then

$$\frac{2}{5}(f * g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.25}$$

and

$$\operatorname{Re}(f(z)) > \frac{-5}{4}, \qquad (z \in \Delta). \tag{2.26}$$

The constant $2/5$ is the best estimate.

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