Journal of Inequalities in Pure and Applied Mathematics

DIRECT APPROXIMATION THEOREMS FOR DISCRETE TYPE OPERATORS

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volume 7, issue 5, article 163, 2006.

Received 16 July, 2006; accepted 10 October, 2006.

Communicated by: Z. Ditzian

ISSN (electronic): 1443-5756
189-06

Abstract

In the present paper we prove direct approximation theorems for discrete type operators

$$(L_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x)\lambda_{n,k}(f),$$

$f \in C[0,\infty)$, $x \in [0,\infty)$ using a modified $K-$functional. As applications we give direct theorems for Baskakov type operators, Szász-Mirakjan type operators and Lupaş operator.

2000 Mathematics Subject Classification: 41A36, 41A25.
Key words: Direct approximation theorem, $K-$functional, Ditzian-Totik modulus of smoothness.

Contents

1. Introduction

We introduce the following discrete type operators L_n, $n \in \{1, 2, 3, \ldots\}$, defined by

$$(L_n f)(x) \equiv L_n(f, x) = \sum_{k=0}^{\infty} u_{n,k}(x) \lambda_{n,k}(f), \tag{1.1}$$

where $f \in C[0, \infty)$, $x \geq 0$, $u_{n,k} \in C[0, \infty)$ with $u_{n,k} \geq 0$ on $[0, \infty)$ and $\lambda_{n,k} : C[0, \infty) \to \mathbb{R}$ are linear positive functionals, $k \in \{0, 1, 2, \ldots\}$.

The purpose of this paper is to establish sufficient conditions with the aim of obtaining direct local and global approximation theorems for (1.1). In [3] Ditzian gave the following interesting estimate:

$$|B_n(f, x) - f(x)| \leq C \omega^2_{\varphi^\lambda} \left(f, \frac{1}{\sqrt{n}} \varphi^{1-\lambda}(x) \right), \tag{1.2}$$

where

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0, 1],\ x \in [0, 1]$$

is the Bernstein-polynomial, $C > 0$ is an absolute constant and $\varphi(x) = \sqrt{x(1 - x)}$. This estimate unifies the classical estimate for $\lambda = 0$ and the norm estimate for $\lambda = 1$. Guo et al. in [7] proved a similar estimate to (1.2) for the Baskakov operator. For the more general operator (1.1) we shall give a result similar to the estimate (1.2) and to the result established in [7].

To formulate the main results we need some notations: let $C_B[0,\infty)$ be the space of all bounded continuous functions on $[0,\infty)$ with the norm $\|f\| = \sup_{x\geq 0}|f(x)|$. Furthermore, let

$$\omega_\varphi^\lambda(f,t) = \sup_{0<h\leq t}\ \sup_{x\pm h\varphi^\lambda(x)\in[0,\infty)} |f(x+h\varphi^\lambda(x)) - 2f(x) + f(x-h\varphi^\lambda(x))|$$

be the second order modulus of smoothness of Ditzian-Totik and let

$$\overline{K}_{\varphi^\lambda}(f,t) = \inf\left\{\|f-g\| + t\|\varphi^{2\lambda}g''\| + t^{2/(2-\lambda)}\|g''\| : g'', \varphi^{2\lambda}g'' \in C_B[0,\infty)\right\}$$

be the corresponding modified weighted $K-$functional, where $\lambda \in [0,1]$ and $\varphi : [0,\infty) \to \mathbb{R}$ is an admissible weight function (cf. [4, Section 1.2]) such that $\varphi^2(x) \sim x^\lambda$ as $x \to 0+$ and $\varphi^2(x) \sim x^\lambda$ as $x \to \infty$, respectively. Then, in view of [4, p.24, Theorem 3.1.2] we have

$$\overline{K}_{\varphi^\lambda}(f,t^2) \sim \omega^2_{\varphi^\lambda}(f,t) \tag{1.3}$$

($x \sim y$ means that there exists an absolute constant $C > 0$ such that $C^{-1}y \leq x \leq Cy$). Throughout this paper $C_1, C_2, \ldots, C_6$ denote positive constants and $C > 0$ is an absolute constant which can be different at each occurrence.

2. Main Results

Our first theorem is the following:

Theorem 2.1. *Let* $(L_n)_{n\geq 1}$ *be defined as in (1.1) satisfying*

(i) $L_n(1,x) = 1, \quad x \geq 0;$

(ii) $L_n(t,x) = x, \quad x \geq 0;$

(iii) $L_n(t^2,x) \leq x^2 + C_1 n^{-1}\varphi^2(x), \quad x \geq 0;$

(iv) $\|L_n f\| \leq C_2\|f\|, \quad f \in C_B[0,\infty);$

(v) $L_n\left(\left|\int_x^t |t-u|\frac{du}{\varphi^{2\lambda}(u)}\right|, x\right) \leq C_3 n^{-1}\varphi^{2(1-\lambda)}(x), \quad x \in [1/n,\infty)$ *and*

(vi) $n^{-1}\varphi^2(x) \leq C_4\left(n^{-1}\varphi^{2(1-\lambda)}(x)\right)^{2/(2-\lambda)}, \quad x \in [0,1/n).$

Then for every $f \in C_B[0,\infty)$, $n \in \{1,2,3,\dots\}$ *and* $x \geq 0$ *one has*

$$|(L_n f)(x) - f(x)| \leq \max\{1+C_2, C_3, C_1C_4\} \cdot \overline{K}_{\varphi^\lambda}\left(f, n^{-1}\varphi^{2(1-\lambda)}(x)\right).$$

Proof. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \qquad t \geq 0$$

and the assumptions (i), (ii), (iii), (v) and (vi) we obtain

$$|(L_n g)(x) - g(x)| \leq \left|L_n\left(\int_x^t (t-u)g''(u)du, x\right)\right| \tag{2.1}$$

$$\le L_n\left(\left|\int_x^t |t-u|\cdot|g''(u)|du\right|, x\right)$$
$$\le L_n\left(\left|\int_x^t |t-u|\cdot\frac{du}{\varphi^{2\lambda}(u)}\right|, x\right)\cdot\|\varphi^{2\lambda}g''\|$$
$$\le \frac{C_3}{n}\cdot\varphi^{2(1-\lambda)}(x)\cdot\|\varphi^{2\lambda}g''\|,$$

where $x\in[1/n,\infty)$, and

$$\begin{aligned}|(L_ng)(x)-g(x)| &\le L_n\left(\left|\int_x^t |t-u|\cdot|g''(u)|du\right|, x\right)\\ &\le L_n\left((t-x)^2,x\right)\cdot\|g''\|\\ &\le C_1\frac{\varphi^2(x)}{n}\cdot\|g''\|\\ &\le C_1C_4\left(\frac{1}{n}\cdot\varphi^{2(1-\lambda)}(x)\right)^{2/(2-\lambda)}\cdot\|g''\|,\end{aligned}\tag{2.2}$$

where $x\in[0,1/n)$.

In conclusion, by (2.1) and (2.2),

$$\begin{aligned}|(L_ng)(x)-g(x)| \le \max\{C_3,C_1C_4\}\cdot\Bigg\{&\frac{1}{n}\cdot\varphi^{2(1-\lambda)}(x)\cdot\|\varphi^{2\lambda}g''\|\\ &+\left(\frac{1}{n}\cdot\varphi^{2(1-\lambda)}(x)\right)^{2/(2-\lambda)}\cdot\|g''\|\Bigg\}\end{aligned}\tag{2.3}$$

for $x \geq 0$. Using (iv) and (2.3) we get

$$\begin{aligned}
&|(L_nf)(x) - f(x)| \\
&\leq |L_n(f-g,x) - (f-g)(x)| + |(L_ng)(x) - g(x)| \\
&\leq (C_2+1)\|f-g\| + \max\{C_3, C_1C_4\} \\
&\qquad \cdot \left\{ \frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \cdot \|\varphi^{2\lambda} g''\| + \left(\frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \right)^{2/(2-\lambda)} \cdot \|g''\| \right\} \\
&\leq \max\{1+C_2, C_3, C_1C_4\} \cdot \{\|f-g\| \\
&\qquad + \left(n^{-1/2} \cdot \varphi^{1-\lambda}(x)\right)^2 \cdot \|\varphi^{2\lambda} g''\| + \left(n^{-1/2} \cdot \varphi^{1-\lambda}(x)\right)^{4/(2-\lambda)} \cdot \|g''\| \right\}.
\end{aligned}$$

Now taking the infimum on the right-hand side over g and using the definition of $\overline{K}_{\varphi^\lambda}(f, n^{-1}\varphi^{2(1-\lambda)}(x))$ we get the assertion of the theorem. □

Corollary 2.2. *Under the assumptions of Theorem 2.1 and for arbitrary $f \in C_B[0,\infty)$, $n \in \{1,2,3,\dots\}$ and $x \geq 0$ we have the estimate*

$$|(L_nf)(x) - f(x)| \leq C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right).$$

Proof. It is an immediate consequence of Theorem 2.1 and (1.3). □

Remark 1. *In Corollary 2.2 the case $\lambda = 0$ gives the local estimate and for $\lambda = 1$ we obtain a global estimate.*

3. Applications

The applications are in connection with Baskakov type operators, Szász - Mirakjan type operators and the Lupaş operator. To be more precise, we shall study the following operators:

$$(L_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x)\lambda_{n,k}(f), \quad v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)};$$

$$(L_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x)\lambda_{n,k}(f), \quad s_{n,k}(x) = e^{-nx} \cdot \frac{(nx)^k}{k!},$$

and their generalizations:

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\lambda_{n,k}(f),$$

$$v_{n,k}^{(\alpha)}(x) = \binom{n+k-1}{k} \frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=1}^{n}(1+j\alpha)}{\prod_{r=1}^{n+k}(x+1+r\alpha)}, \quad \alpha \geq 0;$$

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\lambda_{n,k}(f),$$

$$s_{n,k}^{(\alpha)}(x) = (1+n\alpha)^{-x/\alpha} \frac{nx(nx+n\alpha)\cdots(nx+n(k-1)\alpha)}{k!(1+n\alpha)^k}, \quad \alpha \geq 0$$

(the parameter α may depend only on the natural number n), and the Lupaş operator [8] defined by

$$(\tilde{L}_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{nx(nx+1)\cdots(nx+k-1)}{2^k k!} f\left(\frac{k}{n}\right).$$

For different values of $\lambda_{n,k}$ we obtain the following explicit forms of the above operators:

1) *the Baskakov operator* [2]

$$(V_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right);$$

2) *the generalized Baskakov operator* [5]

$$(V_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right);$$

3) *the modified Agrawal and Thamer operator* [1]

$$(L_{1,n} f)(x) = v_{n,0}(x) f(0) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt;$$

4) *the generalized Agrawal and Thamer type operator*

$$(L_{1,n}^{(\alpha)} f)(x) = v_{n,0}^{(\alpha)}(x) f(0) + \sum_{k=1}^{\infty} v_{n,k}^{(\alpha)}(x) \frac{1}{B(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt;$$

5) *Szász - Mirakjan operator* [12]

$$(S_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right);$$

6) *Mastroianni operator* [9]

$$(S_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right);$$

7) *Phillips operator* [10], [11]

$$(L_{2,n} f)(x) = s_{n,0}(x) f(0) + n \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt;$$

8) *the generalized Phillips operator*

$$(L_{2,n}^{(\alpha)} f)(x) = s_{n,0}^{(\alpha)}(x) f(0) + n \sum_{k=1}^{\infty} s_{n,k}^{(\alpha)}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt;$$

9) *a new generalized Phillips type operator* [6] defined as follows: let $I = \{k_i : 0 = k_0 \leq k_1 \leq k_2 \leq \cdots\} \subseteq \{0, 1, 2, \ldots\}$. Then we can introduce the operators

$$(L_{3,n} f)(x) = \sum_{\substack{k=0 \\ k \in I}}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right) + \sum_{\substack{k=0 \\ k \notin I}}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt$$

and its generalization

$$(L_{3,n}^{(\alpha)}f)(x) = \sum_{\substack{k=0\\k\in I}}^{\infty} s_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) + \sum_{\substack{k=0\\k\notin I}}^{\infty} s_{n,k}^{(\alpha)}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt.$$

For the above enumerated operators we have the following theorem:

Theorem 3.1. *If* $f \in C_B[0,\infty)$, $x \geq 0$, $\varphi(x) = \sqrt{x(1+x)}$, $\lambda \in [0,1]$ *then*

a) $|(V_n f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 1;$

b) $|(V_n^\alpha f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 1,$
$\alpha = \alpha(n) \leq C_5/(4n), C_5 < 1;$

c) $|(L_{1,n} f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 9;$

d) $|(L_{1,n}^{(\alpha)} f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 9,$
$\alpha = \alpha(n) \leq C_6/(4n), C_6 < 1.$

For $f \in C_B[0,\infty)$, $x \geq 0$, $\varphi(x) = \sqrt{x}$, $\lambda \in [0,1]$ *and* $L_n \in \{S_n, L_{2,n}, L_{3,n}, \tilde{L}_n\}$ *resp.* $L_n^{(\alpha)} \in \{S_n^{(\alpha)}, L_{2,n}^{(\alpha)}, L_{3,n}^{(\alpha)}\}$ *we have*

e) $|(L_n f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 1;$

f) $\left|\left(L_n^{(\alpha)} f\right)(x) - f(x)\right| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 1,$
$\alpha = \alpha(n) \leq 1/n;$

g) $\left|\left(\tilde{L}_n f\right)(x) - f(x)\right| \leq C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi^{1-\lambda}(x)\right), \quad n \geq 1.$

Proof. First of all let us observe that we have the integral representation

$$\left(L_n^{(\alpha)} f\right)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} (L_n f)(\theta) d\theta, \tag{3.1}$$

where $0 < \alpha < 1$ and $\left(L_n, L_n^{(\alpha)}\right) \in \left\{\left(V_n, V_n^{(\alpha)}\right), \left(L_{1,n}, L_{1,n}^{(\alpha)}\right)\right\}$.

Analogously

$$\left(L_n^{(\alpha)} f\right)(x) = \frac{\left(\frac{1}{\alpha}\right)^{\frac{x}{\alpha}}}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^\infty e^{-\frac{\theta}{\alpha}} \theta^{\frac{x}{\alpha}-1} (L_n f)(\theta) d\theta, \tag{3.2}$$

where $\alpha > 0$, and $\left(L_n, L_n^{(\alpha)}\right) \in \left\{\left(S_n, S_n^{(\alpha)}\right), \left(L_{2,n}, L_{2,n}^{(\alpha)}\right), \left(L_{3,n}, L_{3,n}^{(\alpha)}\right)\right\}$.

The relations (3.1) and (3.2) can be proved with the same idea. For example, if $L_n^{(\alpha)} = V_n^{(\alpha)}$ and $L_n = V_n$ then

$$\begin{aligned}
&\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} V_n(f,\theta) d\theta \\
&= \sum_{k=0}^\infty \binom{n+k-1}{k} \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} \frac{\theta^k}{(1+\theta)^{n+k}} d\theta f\left(\frac{k}{n}\right) \\
&= \sum_{k=0}^\infty \binom{n+k-1}{k} \frac{B\left(\frac{x}{\alpha}+k, \frac{1}{\alpha}+n+1\right)}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} f\left(\frac{k}{n}\right) \\
&= \sum_{k=0}^\infty v_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) = V_n^{(\alpha)}(f,x).
\end{aligned}$$

The statements of our theorem follow from Corollary 2.2 if we verify the conditions $(i) - (vi)$. It is easy to show that each operator preserves the linear functions and

$$
\begin{aligned}
(3.3)\qquad V_n((t-x)^2,x) &= \frac{1}{n}x(1+x),\\
V_n^{(\alpha)}((t-x)^2,x) &= \frac{x(1+x)}{(1-\alpha)n} + \frac{\alpha x(1+x)}{1-\alpha} \le \frac{5}{3n}x(1+x),\\
L_{1,n}((t-x)^2,x) &= \frac{2x(1+x)}{n-1} \le \frac{4}{n}x(1+x),\\
L_{1,n}^{(\alpha)}((t-x)^2,x) &= \frac{2x(1+x)}{(1-\alpha)(n-1)} + \frac{\alpha x(1+x)}{1-\alpha} \le \frac{17}{3n}x(1+x),\\
S_n((t-x)^2,x) &= \frac{1}{n}x,\\
S_n^{(\alpha)}((t-x)^2,x) &= \left(\alpha + \frac{1}{n}\right)x + \alpha x \le \frac{3}{n}x,\\
L_{2,n}((t-x)^2,x) &= \frac{2}{n}x,\\
L_{2,n}^{(\alpha)}((t-x)^2,x) &= \frac{2}{n}x + \alpha x \le \frac{3}{n}x,\\
L_{3,n}((t-x)^2,x) &\le \frac{2}{n}x,\\
L_{3,n}^{(\alpha)}((t-x)^2,x) &\le \frac{2}{n}x + \alpha x \le \frac{3}{n}x \qquad \text{(see [6, p. 179])},\\
\tilde{L}_n((t-x)^2,x) &= \frac{2}{n}x,
\end{aligned}
$$

which imply (i), (ii) and (iii). The condition (iv) can be obtained from the integral representations (3.1) – (3.2) and the definition of $\tilde{L}_n$.

For (v) we have in view of [4, p.140, Lemma 9.6.1] that

$$\left|\int_x^t |t-u|\frac{du}{\varphi^{2\lambda}(u)}\right| = \left|\int_x^t |t-u|\frac{du}{u^\lambda(1+u)^\lambda}\right| \le \frac{(t-x)^2}{x^\lambda}\cdot\left(\frac{1}{(1+x)^\lambda}+\frac{1}{(1+t)^\lambda}\right) \tag{3.4}$$

or

$$\left|\int_x^t |t-u|\frac{du}{\varphi^{2\lambda}(u)}\right| = \left|\int_x^t |t-u|\frac{du}{u^\lambda}\right| \le \frac{(t-x)^2}{x^\lambda}. \tag{3.5}$$

Because L_n is a linear positive operator, therefore either (3.4) and (3.3) or (3.5) and (3.3) imply

$$L_n\left(\left|\int_x^t |t-u|\frac{du}{\varphi^{2\lambda}(u)}\right|, x\right) \le \frac{17}{3n}\cdot\frac{x(1+x)}{x^\lambda(1+x)^\lambda}+\frac{1}{x^\lambda}L_n\left((t-x)^2(1+t)^{-\lambda}, x\right),$$

and

$$L_n\left(\left|\int_x^t |t-u|\frac{du}{\varphi^{2\lambda}(u)}\right|, x\right) \le \frac{3}{n}\cdot\frac{x}{x^\lambda} = \frac{3}{n}x^{1-\lambda},$$

respectively. Thus we have to prove the estimation

$$L_n\left((t-x)^2(1+t)^{-\lambda}, x\right) \le \frac{C}{n}\cdot\frac{x(1+x)}{(1+x)^\lambda}, \qquad x\in[1/n,\infty) \tag{3.6}$$

for each Baskakov type operator defined in this section.

1. By Hölder's inequality and [4, p.128, Lemma 9.4.3 and p.141, Lemma 9.6.2] we have

$$V_n((t-x)^2(1+t)^{-\lambda},x) \le \{V_n((t-x)^4,x)\}^{\frac{1}{2}} \cdot \{V_n((1+t)^{-4},x)\}^{\frac{\lambda}{4}}$$
$$\le C(n^{-2}x^2(1+x)^2)^{\frac{1}{2}} \cdot ((1+x)^{-4})^{\frac{\lambda}{4}}$$
$$= \frac{C}{n} \cdot \frac{x(1+x)}{(1+x)^{\lambda}},$$

where $x \in [1/n, \infty)$;

2. Using

$$V_n((t-x)^4,x) = \frac{3}{n^2}\left(1+\frac{2}{n}\right) \cdot x^2(1+x)^2 + \frac{1}{n^3} \cdot x(1+x),$$

(3.1) and [4, p.141, Lemma 9.6.2] we obtain

$$V_n^{(\alpha)}((t-x)^4,x) \tag{3.7}$$
$$= \frac{3}{n^2}\left(1+\frac{2}{n}\right) \cdot \frac{x(x+\alpha)(x+1)(x+1-\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)} + \frac{1}{n^3} \cdot \frac{x(x+1)}{1-\alpha}$$
$$\le \frac{3}{n^2}\left(1+\frac{2}{n}\right) \cdot \frac{5}{4} \cdot \frac{1}{(1-C_5)^4} \cdot x^2(1+x)^2 + \frac{1}{n^2} \cdot \frac{1}{(1-C_5)^2} \cdot x^2(1+x)^2$$
$$\le C\left(n^{-1}x(1+x)\right)^2,$$

where $x \in [1/n, \infty)$, and

$$V_n^{(\alpha)}((1+t)^{-4}, x) \tag{3.8}$$
$$\le \frac{C}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} \cdot \frac{d\theta}{(1+\theta)^4}$$
$$= C\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}{(1+x+\alpha)(1+x+2\alpha)(1+x+3\alpha)(1+x+4\alpha)}$$
$$\le C(1+x)^{-4},$$

where $x \in [0, \infty)$, $\alpha = \alpha(n) \le C_5/(4n)$, $n \ge 1$, $C_5 < 1$. Therefore the Hölder inequality, (3.7) and (3.8) imply (3.6) for $L_n = V_n^{(\alpha)}$;

3. We have

$$L_{1,n}((t-x)^2(1+t)^{-\lambda}, x) \tag{3.9}$$
$$\le \left\{L_{1,n}((t-x)^4, x)\right\}^{\frac{1}{2}} \cdot \left\{L_{1,n}((1+t)^{-4}, x)\right\}^{\frac{\lambda}{4}}.$$

By direct computation we get

$$L_{1,n}((t-x)^4, x) \tag{3.10}$$
$$= v_{n,0}(x)x^4$$
$$+ \sum_{k=1}^{\infty} v_{n,k}(x)\frac{1}{B(k,n+1)}\int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k+1}}(t-x)^4 dt$$
$$= v_{n,0}(x)x^4 + \sum_{k=1}^{\infty} v_{n,k}(x)\frac{1}{B(k,n+1)}\{B(k+4,n-3)$$
$$- 4xB(k+3,n-2) + 6x^2B(k+2,n-1)$$

$$- 4x^3 B(k+1,n) + x^4 B(k,n+1)\}$$

$$= v_{n,0}(x)x^4 + \sum_{k=1}^{\infty} v_{n,k}(x) \left\{ \frac{k(k+1)(k+2)(k+3)}{n(n-1)(n-2)(n-3)} \right.$$

$$\left. - 4x \cdot \frac{k(k+1)(k+2)}{n(n-1)(n-2)} + 6x^2 \cdot \frac{k(k+1)}{n(n-1)} - 4x^3 \cdot \frac{k}{n} + x^4 \right\}$$

$$= \frac{(12n+84)x^4 + (24n+168)x^3 + (12n+108)x^2 + 11x}{(n-1)(n-2)(n-3)}.$$

Hence, for $x \in [1/n, \infty)$ and $n \geq 9$ one has

(3.11) $$L_{1,n}((t-x)^4, x)$$

$$\leq \frac{(12n+84)x^4 + (24n+168)x^3 + (12n+108)x^2 + 11nx^2}{(n-1)(n-2)(n-3)}$$

$$\leq \frac{C}{n^2} x^2 (1+x)^2.$$

Further,

(3.12) $$L_{1,n}((1+t)^{-4}, x)$$

$$= v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k,n+1)}$$

$$\times \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} \cdot \frac{dt}{(1+t)^4}$$

$$= v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{B(k,n+5)}{B(k,n+1)}$$

$$= v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{(n+1)(n+2)(n+3)(n+4)}{(n+k+1)(n+k+2)(n+k+3)(n+k+4)}$$

$$= v_{n-4,0}(x) \cdot \frac{1}{(1+x)^4} + \sum_{k=1}^{\infty} \frac{v_{n-4,k}(x)}{(1+x)^4}$$

$$\cdot \frac{(n+k-4)(n+k-3)(n+k-2)(n+k-1)}{(n+k+1)(n+k+2)(n+k+3)(n+k+4)}$$

$$\cdot \frac{(n+1)(n+2)(n+3)(n+4)}{(n-4)(n-3)(n-2)(n-1)}$$

$$\leq 16(1+x)^{-4},$$

where $n \geq 9$. Now (3.9), (3.11) and (3.12) imply (3.6) for $L_n = L_{1,n}$;

4. Using (3.1), Hölder's inequality, (3.10) and (3.12) we have

$$L_{1,n}^{(\alpha)}((t-x)^2(1+t)^{-\lambda}, x)$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} L_{1,n}((t-x)^2(1+t)^{-\lambda}, \theta) d\theta$$

$$\leq \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} L_{1,n}((t-x)^4, \theta) d\theta \right)^{\frac{1}{2}}$$

$$\cdot \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}+1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} L_{1,n}((1+t)^{-4}, \theta) d\theta \right)^{\frac{\lambda}{4}}$$

$$\le C\left(\frac{1}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}+1\right)}\int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}}\left[L_{1,n}((t-\theta)^4,\theta)+(\theta-x)^4\right]d\theta\right)^{\frac{1}{2}}$$

$$\cdot\left(\frac{B\left(\frac{x}{\alpha},\frac{1}{\alpha}+5\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}+1\right)}\right)^{\frac{\lambda}{4}}$$

$$\le C\left(\frac{1}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}+1\right)}\cdot\frac{1}{n^2}\int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}}(\theta^4+\theta^3+\theta^2+n^{-1}\theta)d\theta\right.$$

$$\left.+\int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}}(\theta-x)^4d\theta\right)^{\frac{1}{2}}$$

$$\cdot\left(\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}{(1+x+\alpha)(1+x+2\alpha)(1+x+3\alpha)(1+x+4\alpha)}\right)^{\frac{\lambda}{4}}$$

$$\le C\left[n^{-2}(x^4+x^3+x^2)+n^{-2}(6\alpha^3+2\alpha^2+\alpha+n^{-1})x+(18\alpha^3+3\alpha^2)x^4\right.$$

$$\left.+(36\alpha^3+6\alpha^2)x^3+(24\alpha^3+3\alpha^2)x^2+6\alpha^3x\right]^{\frac{1}{2}}\cdot(1+x)^{-\lambda}$$

$$\le \frac{C}{n}\cdot\frac{x(1+x)}{(1+x)^\lambda}$$

for $x\in[1/n,\infty)$, $n\ge 9$, $\alpha=\alpha(n)\le C_6/(4n)$, $C_6<1$.

Condition (vi) follows by direct computation if $\varphi^2(x)=x(1+cx)$, $c\in\{0,1\}$ and $x\in[0,1/n)$. Thus the theorem is proved. □

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