AN INEQUALITY AND ITS q-ANALOGUE

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Abstract: In this paper, we establish a new inequality and its q-analogue by means of the Gould-Hsu inversions, the Carlitz inversions and the Grüss inequality.

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1. Introduction and Some Known Results

q-series, which are also called basic hypergeometric series, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. In this paper, first we establish an inequality by means of the Gould-Hsu inversions, and then we obtain a q-analogue of the inequality.

We first state some notations and known results which will be used in the next sections. It is supposed in this paper that $0 < q < 1$. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.2}$$

The following inverse series relations are due to Gould-Hsu [4]:

Theorem 1.1. *Let $\{a_i\}$ and $\{b_j\}$ be two real or complex sequences such that the polynomials defined by*

$$\begin{cases} \psi(x,n) = \prod_{k=0}^{n-1}(a_k + xb_k), \ (n = 1, 2, \dots), \\ \psi(x,0) = 1, \end{cases}$$

differ from zero for any non-negative integer x. Then we have the following inverse

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series relations

$$(1.3)\qquad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k,n) g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + k b_k}{\psi(n,\, k+1)} f(k), \end{cases}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Carlitz [2] gave the following q-analogue of the Gould-Hsu inverse series relations:

Theorem 1.2. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two real or complex sequences such that the polynomials defined by*

$$\begin{cases} \phi(x,n) = \prod_{k=0}^{n-1} (a_k + q^x b_k), \ (n = 1, 2, \dots), \\ \phi(x,0) = 1, \end{cases}$$

differ from zero for $x = q^n$ *with* n *being non-negative integers. Then we have the following inverse series relations*

$$(1.4)\qquad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(k,n) g(k); \\ g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_k + q^k b_k}{\phi(n;\, k+1)} f(k). \end{cases}$$

We also need the following inequality, which is well known in the literature as the Grüss inequality [5]:

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Theorem 1.3. *We have*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right| \leq \frac{(M-m)(N-n)}{4}, \tag{1.5}$$

provided that $f, g : [a,b] \to \mathbb{R}$ *are integrable on* $[a,b]$ *and* $m \leq f(x) \leq M, n \leq g(x) \leq N$ *for all* $x \in [a,b]$, *where* m, M, n, N *are given constants.*

The discrete version of the Grüss inequality can be stated as:

Theorem 1.4. *If* $a \leq a_i \leq A$ *and* $b \leq b_i \leq B$ *for* $i = 1, 2, \ldots, n$, *then we have*

$$\left|\frac{1}{n}\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i \cdot \frac{1}{n}\sum_{i=1}^n b_i\right| \leq \frac{(A-a)(B-b)}{4}, \tag{1.6}$$

where a, A, a_i, b, B, b_i *are real numbers.*

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2. A New Inequality

In this section we obtain an inequality about series by using both the Gould-Hsu inversions and the Grüss inequality.

Theorem 2.1. *Suppose* $0 \leq a \leq f(k) \leq A$, $g(k) = \sum_{i=0}^{k} \binom{k}{i} f(i)$, $k = 1, 2, \dots, n$, *then the following inequality holds*

$$\left|(n+1)\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}^2 f(k)g(k) - f(n)g(n)\right| \leq 3(n+1)^2 2^{n-3} A\binom{n}{k_0}\left[A\binom{n}{k_0} - a\right], \tag{2.1}$$

where $k_0 = [\frac{n-1}{2}]$, $[x]$ *denotes the greatest integer less than or equal* x *.*

Proof. Letting $a_i = -1, b_i = 0$ in (1.3), we have

$$\begin{cases} f(n) = \sum\limits_{k=0}^{n} (-1)^{n+k}\binom{n}{k} g(k), \\ g(n) = \sum\limits_{k=0}^{n} \binom{n}{k} f(k). \end{cases} \tag{2.2}$$

Since $0 \leq a \leq f(k) \leq A$, we obtain

$$a \cdot \sum_{i=0}^{k}\binom{k}{i} \leq g(k) = \sum_{i=0}^{k}\binom{k}{i} f(i) \leq A \cdot \sum_{i=0}^{k}\binom{k}{i}.$$

Substituting $\sum_{i=0}^{k}\binom{k}{i} = 2^k$ into the above inequality we get

$$a \cdot 2^k \leq g(k) \leq A \cdot 2^k, \qquad k = 0, 1, \dots, n. \tag{2.3}$$

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On the other hand, we know that

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n!/(k+1)!(n-k-1)!}{n!/(k)!(n-k)!} = \frac{n-k}{k+1},$$

consequently

$$\begin{cases} \frac{\binom{n}{k+1}}{\binom{n}{k}} \geq 1 & \text{when } k \leq k_0, \\ \frac{\binom{n}{k+1}}{\binom{n}{k}} \leq 1, & \text{when } k \geq k_0, \end{cases}$$

where $k_0 = [\frac{n-1}{2}]$. So, we get

$$1 \leq \binom{n}{k} \leq \binom{n}{k_0}, \qquad k = 0, 1, \ldots, n. \tag{2.4}$$

Let $A_k = \binom{n}{k} f(k)$ and $B_k = (-1)^{n+k} \binom{n}{k} g(k)$, then

$$a \leq A_k \leq A \binom{n}{k_0}. \tag{2.5}$$

From (2.3) and (2.4), we know that

$$\begin{cases} 0 \leq B_k \leq 2^n A \binom{n}{k_0} & \text{if } n-k \text{ is even}, \\ -2^{n-1} A \binom{n}{k_0} \leq B_k \leq 0, & \text{if } n-k \text{ is odd}. \end{cases}$$

So, for $k = 1, 2, \ldots, n$, we have

$$-2^{n-1} A \binom{n}{k_0} \leq B_k \leq 2^n A \binom{n}{k_0}. \tag{2.6}$$

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Combining (1.6), (2.5) and (2.6) we obtain

$$\left|\frac{1}{n+1}\sum_{i=0}^{n}A_iB_i-\left(\frac{1}{n+1}\sum_{i=0}^{n}A_i\right)\cdot\left(\frac{1}{n+1}\sum_{i=0}^{n}B_i\right)\right| \le \frac{\left(A\binom{n}{k_0}-a\right)\left(2^nA\binom{n}{k_0}+2^{n-1}A\binom{n}{k_0}\right)}{4},$$

which can be written as

$$\left|\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}^2 f(k)g(k) - \left(\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}f(k)\right)\cdot\left(\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}g(k)\right)\right| \le \frac{\left(A\binom{n}{k_0}-a\right)\left(2^nA\binom{n}{k_0}+2^{n-1}A\binom{n}{k_0}\right)}{4}.$$

Substituting (2.2) into the above inequality, we get (2.1). □

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3. A q-Analogue of the Inequality

In this section we give a q-analogue of the inequality (2.1) by means of the Carlitz inversions. First, we have the following lemma.

Lemma 3.1. *Suppose* $0 \leq f(k) \leq A$ *and* $g(k) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} f(i)$, *then for any* $k = 1, 2, \ldots, n$, *we have*

$$0 \leq g(k) \leq A \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}. \tag{3.1}$$

Proof. It is obvious that $g(k) \geq 0$. If $k \leq n_1 \leq n_2$, then we have

$$\begin{bmatrix} n_2 \\ k \end{bmatrix} = \frac{1-q^{n_1+1}}{1-q^{n_1+1-k}} \cdot \frac{1-q^{n_1+2}}{1-q^{n_1+2-k}} \cdots \frac{1-q^{n_2}}{1-q^{n_2-k}} \begin{bmatrix} n_1 \\ k \end{bmatrix}.$$

Since

$$\frac{1-q^{n_1+1}}{1-q^{n_1+1-k}} \cdot \frac{1-q^{n_1+2}}{1-q^{n_1+2-k}} \cdots \frac{1-q^{n_2}}{1-q^{n_2-k}} \geq 1,$$

we get

$$\begin{bmatrix} n_2 \\ k \end{bmatrix} \geq \begin{bmatrix} n_1 \\ k \end{bmatrix}.$$

Consequently,

$$g(k) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} f(i) \leq \sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix} f(i) \leq \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} f(i).$$

$\square$

The main result of this section is the following theorem.

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Theorem 3.2. *Suppose* $0 \le a \le f(k) \le A$, $g(k) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} f(i)$, $k = 1, 2, \dots, n$, *then the following inequality holds*

$$\left| (n+1) \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n \\ i \end{bmatrix}^2 q^{\binom{n-k}{2}} f(k)g(k) - f(n)g(n) \right| \le \frac{A(n+1)^2}{4} \begin{bmatrix} n \\ k_0 \end{bmatrix} \left(A \begin{bmatrix} n \\ k_0 \end{bmatrix} - a \right) \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} + \sum_{i=0}^{n-1} \begin{bmatrix} n-1 \\ i \end{bmatrix} \right), \tag{3.2}$$

where $k_0 = [\frac{n-1}{2}]$, $[x]$ *denotes the greatest integer less than or equal* x.

Proof. Letting $a_i = -1, b_i = 0$ in (1.4) we get

$$\begin{cases} f(n) = \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} g(k), \\ g(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} f(k). \end{cases} \tag{3.3}$$

Using the lemma, we have

$$a \cdot \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \le g(k) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} f(i) \le A \cdot \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}. \tag{3.4}$$

On the other hand, we notice that

$$\frac{\begin{bmatrix} n \\ k+1 \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}} = \frac{(q;q)_n/(q;q)_{k+1}(q;q)_{n-k-1}}{(q;q)_n/(q;q)_k(q;q)_{n-k}} = \frac{1-q^{n-k}}{1-q^{k+1}},$$

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consequently

$$\begin{cases} \dfrac{\left[{n \atop k+1}\right]}{\left[{n \atop k}\right]} \geq 1, & \text{when } k \leq k_0, \\ \dfrac{\left[{n \atop k+1}\right]}{\left[{n \atop k}\right]} \leq 1, & \text{when } k \geq k_0, \end{cases}$$

where $k_0 = [\frac{n-1}{2}]$. So, we have

$$1 \leq \begin{bmatrix} n \\ k \end{bmatrix} \leq \begin{bmatrix} n \\ k_0 \end{bmatrix}, \qquad k = 0, 1, \ldots, n. \tag{3.5}$$

Let $A_k = \left[{n \atop k}\right] f(k)$ and $B_k = (-1)^{n+k} \left[{n \atop k}\right] q^{\binom{n-k}{2}} g(k)$, then

$$a \leq A_k \leq A \begin{bmatrix} n \\ k_0 \end{bmatrix}. \tag{3.6}$$

From (3.4) and (3.5), we know that

$$\begin{cases} 0 \leq B_k \leq A \left[{n \atop k_0}\right] \sum\limits_{i=0}^{n} \left[{n \atop i}\right], & \text{if } n-k \text{ is even}, \\ -A \left[{n \atop k_0}\right] \sum\limits_{i=0}^{n-1} \left[{n-1 \atop i}\right] \leq B_k \leq 0, & \text{if } n-k \text{ is odd}. \end{cases}$$

So, for $k = 1, 2, \ldots, n$, we get

$$-A \begin{bmatrix} n \\ k_0 \end{bmatrix} \sum_{i=0}^{n-1} \begin{bmatrix} n-1 \\ i \end{bmatrix} \leq B_k \leq A \begin{bmatrix} n \\ k_0 \end{bmatrix} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}. \tag{3.7}$$

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Combining (1.6), (3.6) and (3.7) we obtain

$$\left|\frac{1}{n+1}\sum_{i=0}^{n}A_iB_i-\left(\frac{1}{n+1}\sum_{i=0}^{n}A_i\right)\cdot\left(\frac{1}{n+1}\sum_{i=0}^{n}B_i\right)\right|$$
$$\leq\frac{1}{4}\left(A\begin{bmatrix}n\\k_0\end{bmatrix}-a\right)\left(A\begin{bmatrix}n\\k_0\end{bmatrix}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}+A\begin{bmatrix}n\\k_0\end{bmatrix}\sum_{i=0}^{n-1}\begin{bmatrix}n-1\\i\end{bmatrix}\right),$$

which can be written as

$$\left|\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{n+k}\begin{bmatrix}n\\k\end{bmatrix}^2q^{\binom{n-k}{2}}f(k)g(k)\right.$$
$$\left.-\left(\frac{1}{n+1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}f(k)\right)\left(\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{n+k}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}g(k)\right)\right|$$
$$\leq\frac{A}{4}\begin{bmatrix}n\\k_0\end{bmatrix}\left(A\begin{bmatrix}n\\k_0\end{bmatrix}-a\right)\left(\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}+\sum_{i=0}^{n-1}\begin{bmatrix}n-1\\i\end{bmatrix}\right).$$

Substituting (3.3) into the above inequality, we get (3.2).

From [3], we know

$$\lim_{q\to1}\begin{bmatrix}n\\i\end{bmatrix}=\binom{n}{i}.$$

Let $q\to1$ in both sides of the inequality (3.2) to get

$$\left|(n+1)\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}^2f(k)g(k)-f(n)g(n)\right|$$

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$$\leq \frac{A(n+1)^2}{4}\binom{n}{k_0}\left[A\binom{n}{k_0}-a\right]\left[\sum_{i=0}^{n}\binom{n}{2}+\sum_{i=0}^{n-1}\binom{n-1}{2}\right]$$
$$=\frac{A(n+1)^2}{4}\binom{n}{k_0}\left[A\binom{n}{k_0}-a\right][2^n+2^{n-1}]=3(n+1)^2 2^{n-3}A\binom{n}{k_0}\left[A\binom{n}{k_0}-a\right],$$

which is the inequality (2.1). So the inequality (3.2) is the q-analogue of the inequality (2.1). $\square$

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