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SOME PROPERTIES FOR AN INTEGRAL OPERATOR DEFINED BY AL-OBOUDI DIFFERENTIAL OPERATOR

SERAP BULUT
KOCAELI UNIVERSITY
CIVIL AVIATION COLLEGE
ARSLANBEY CAMPUS
41285 İZMIT-KOCAELI
TURKEY
serap.bulut@kocaeli.edu.tr

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ABSTRACT. In this paper, we investigate some properties for an integral operator defined by Al-Oboudi differential operator.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

For $f \in \mathcal{A}$, Al-Oboudi [2] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0, \tag{1.3}$$

$$D^n f(z) = D_\delta(D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.5}$$

with $D^n f(0) = 0$.

190-08

When $\delta = 1$, we get Sălăgean's differential operator [10].

A function $f \in \mathcal{A}$ is said to be *starlike of order* α if it satisfies the inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U})$$

for some $0 \leq \alpha < 1$. We say that f is in the class $\mathcal{S}^*(\alpha)$ for such functions.

A function $f \in \mathcal{A}$ is said to be *convex of order* α if it satisfies the inequality:

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha \quad (z \in \mathbb{U})$$

for some $0 \leq \alpha < 1$. We say that f is in the class $\mathcal{K}(\alpha)$ if it is convex of order α in $\mathbb{U}$.

We note that $f \in \mathcal{K}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$.

In particular, the classes

$$\mathcal{S}^*(0) := \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) := \mathcal{K}$$

are familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

Now, we introduce two new classes $\mathcal{S}^n(\delta, \alpha)$ and $\mathcal{M}^n(\delta, \beta)$ as follows:

Let $\mathcal{S}^n(\delta, \alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left\{\frac{z\left(D^n f(z)\right)'}{D^n f(z)}\right\} > \alpha \quad (z \in \mathbb{U})$$

for some $0 \leq \alpha < 1$, $\delta \geq 0$, and $n \in \mathbb{N}_0$.

It is clear that

$$\mathcal{S}^0(\delta, \alpha) \equiv \mathcal{S}^*(\alpha) \equiv \mathcal{S}^n(0, \alpha), \qquad \mathcal{S}^n(0, 0) \equiv \mathcal{S}^*.$$

Let $\mathcal{M}^n(\delta, \beta)$ be the subclass of $\mathcal{A}$, consisting of the functions f, which satisfy the inequality

$$\operatorname{Re}\left\{\frac{z\left(D^n f(z)\right)'}{D^n f(z)}\right\} < \beta \quad (z \in \mathbb{U})$$

for some $\beta > 1$, $\delta \geq 0$, and $n \in \mathbb{N}_0$.

Also, let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$, consisting of the functions f, which satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \beta \quad (z \in \mathbb{U}).$$

It is obvious that

$$\mathcal{M}^0(\delta, \beta) \equiv \mathcal{M}(\beta) \equiv \mathcal{M}^n(0, \beta).$$

The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were studied by Uralegaddi et al. in [11], Owa and Srivastava in [9], and Breaz in [4].

Definition 1.1. Let $n, m \in \mathbb{N}_0$ and $k_i > 0$, $1 \leq i \leq m$. We define the integral operator $I_n(f_1, \ldots, f_m) : \mathcal{A}^m \to \mathcal{A}$

$$I_n(f_1, \ldots, f_m)(z) := \int_0^z \left(\frac{D^n f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^n f_m(t)}{t}\right)^{k_m} dt, \quad (z \in \mathbb{U}),$$

where $f_i \in \mathcal{A}$ and D^n is the Al-Oboudi differential operator.

Remark 1.

(i) For $n = 0$, we have the integral operator

$$I_0(f_1, \ldots, f_m)(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{f_m(t)}{t}\right)^{k_m} dt$$

introduced in [5]. More details about $I_0(f_1, \ldots, f_m)$ can be found in [3] and [4].

(ii) For $n = 0$, $m = 1$, $k_1 = 1$, $k_2 = \cdots = k_m = 0$ and $D^0 f_1(z) := D^0 f(z) = f(z) \in \mathcal{A}$, we have the integral operator of Alexander

$$I_0(f)(z) := \int_0^z \frac{f(t)}{t} dt$$

introduced in [1].

(iii) For $n = 0$, $m = 1$, $k_1 = k \in [0, 1]$, $k_2 = \cdots = k_m = 0$ and $D^0 f_1(z) := D^0 f(z) = f(z) \in \mathcal{S}$, we have the integral operator

$$I(f)(z) := \int_0^z \left(\frac{f(t)}{t} \right)^k dt$$

studied in [8].

(iv) If $k_i \in \mathbb{C}$ for $1 \leq i \leq m$, then we have the integral operator $I_n(f_1, \ldots, f_m)$ studied in [7].

In this paper, we investigate some properties for the operators I_n on the classes $\mathcal{S}^n(\delta, \alpha)$ and $\mathcal{M}^n(\delta, \beta)$.

2. SOME PROPERTIES FOR I_n ON THE CLASS $\mathcal{S}^n(\delta, \alpha)$

Theorem 2.1. *Let $f_i \in \mathcal{S}^n(\delta, \alpha_i)$ for $1 \leq i \leq m$ with $0 \leq \alpha_i < 1$, $\delta \geq 0$ and $n \in \mathbb{N}_0$. Also let $k_i > 0$, $1 \leq i \leq m$. If*

$$\sum_{i=1}^m k_i(1 - \alpha_i) \leq 1,$$

then $I_n(f_1, \ldots, f_m) \in \mathcal{K}(\lambda)$ with $\lambda = 1 + \sum_{i=1}^m k_i(\alpha_i - 1)$.

Proof. By (1.5), for $1 \leq i \leq m$, we have

$$\frac{D^n f_i(z)}{z} = 1 + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_{j,i} z^{j-1}, \quad (n \in \mathbb{N}_0)$$

and

$$\frac{D^n f_i(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$.

On the other hand, we obtain

$$I_n(f_1, \ldots, f_m)'(z) = \left(\frac{D^n f_1(z)}{z} \right)^{k_1} \cdots \left(\frac{D^n f_m(z)}{z} \right)^{k_m}$$

for $z \in \mathbb{U}$. This equality implies that

$$\ln I_n(f_1, \ldots, f_m)'(z) = k_1 \ln \frac{D^n f_1(z)}{z} + \cdots + k_m \ln \frac{D^n f_m(z)}{z}$$

or equivalently

$$\ln I_n(f_1, \ldots, f_m)'(z) = k_1 [\ln D^n f_1(z) - \ln z] + \cdots + k_m [\ln D^n f_m(z) - \ln z].$$

By differentiating the above equality, we get

$$\frac{I_n(f_1, \ldots, f_m)''(z)}{I_n(f_1, \ldots, f_m)'(z)} = \sum_{i=1}^m k_i \left[\frac{(D^n f_i(z))'}{D^n f_i(z)} - \frac{1}{z} \right].$$

Thus, we obtain

$$\frac{zI_n(f_1,\ldots,f_m)''(z)}{I_n(f_1,\ldots,f_m)'}+1=\sum_{i=1}^{m}k_i\frac{z\left(D^nf_i(z)\right)'}{D^nf_i(z)}-\sum_{i=1}^{m}k_i+1.$$

This relation is equivalent to

$$\operatorname{Re}\left\{\frac{zI_n(f_1,\ldots,f_m)''(z)}{I_n(f_1,\ldots,f_m)'}+1\right\}=\sum_{i=1}^{m}k_i\operatorname{Re}\left\{\frac{z\left(D^nf_i(z)\right)'}{D^nf_i(z)}\right\}-\sum_{i=1}^{m}k_i+1.$$

Since $f_i\in\mathcal{S}^n(\delta,\alpha_i)$, we get

$$\operatorname{Re}\left\{\frac{zI_n(f_1,\ldots,f_m)''(z)}{I_n(f_1,\ldots,f_m)'}+1\right\}>\sum_{i=1}^{m}k_i\alpha_i-\sum_{i=1}^{m}k_i+1=1+\sum_{i=1}^{m}k_i(\alpha_i-1).$$

So, the integral operator $I_n(f_1,\ldots,f_m)$ is convex of order λ with $\lambda=1+\sum_{i=1}^{m}k_i(\alpha_i-1)$. □

Corollary 2.2. *Let $f_i\in\mathcal{S}^n(\delta,\alpha)$ for $1\le i\le m$ with $0\le\alpha<1$, $\delta\ge0$ and $n\in\mathbb{N}_0$. Also let $k_i>0$, $1\le i\le m$. If*

$$\sum_{i=1}^{m}k_i\le\frac{1}{1-\alpha},$$

then $I_n(f_1,\ldots,f_m)\in\mathcal{K}(\rho)$ with $\rho=1+(\alpha-1)\sum_{i=1}^{m}k_i$.

Proof. In Theorem 2.1, we consider $\alpha_1=\cdots=\alpha_m=\alpha$. □

Corollary 2.3. *Let $f\in\mathcal{S}^n(\delta,\alpha)$ with $0\le\alpha<1$, $\delta\ge0$ and $n\in\mathbb{N}_0$. Also let $0<k\le1/(1-\alpha)$. Then the function*

$$I_n(f)(z)=\int_0^z\left(\frac{D^nf(t)}{t}\right)^k dt$$

is in $\mathcal{K}(1+k(\alpha-1))$.

Proof. In Corollary 2.2, we consider $m=1$ and $k_1=k$. □

Corollary 2.4. *Let $f\in\mathcal{S}^n(\delta,\alpha)$. Then the integral operator*

$$I_n(f)(z)=\int_0^z\left(D^nf(t)/t\right)dt\in\mathcal{K}(\alpha).$$

Proof. In Corollary 2.3, we consider $k=1$. □

3. SOME PROPERTIES FOR I_n ON THE CLASS $\mathcal{M}^n(\delta,\beta)$

Theorem 3.1. *Let $f_i\in\mathcal{M}^n(\delta,\beta_i)$ for $1\le i\le m$ with $\beta_i>1$. Then $I_n(f_1,\ldots,f_m)\in\mathcal{N}(\lambda)$ with $\lambda=1+\sum_{i=1}^{m}k_i(\beta_i-1)$ and $k_i>0$, $(1\le i\le m)$.*

Proof. Proof is similar to the proof of Theorem 2.1. □

Remark 2. For $n=0$, we have Theorem 2.1 in [4].

Corollary 3.2. *Let $f_i\in\mathcal{M}^n(\delta,\beta)$ for $1\le i\le m$ with $\beta>1$. Then $I_n(f_1,\ldots,f_m)\in\mathcal{N}(\rho)$ with $\rho=1+(\beta-1)\sum_{i=1}^{m}k_i$ and $k_i>0$, $(1\le i\le m)$.*

Corollary 3.3. *Let $f\in\mathcal{M}^n(\delta,\beta)$ with $\beta>1$. Then the integral operator*

$$I_n(f)(z)=\int_0^z\left(\frac{D^nf(t)}{t}\right)^k dt\in\mathcal{N}(1+k(\beta-1))$$

and $k>0$.

Corollary 3.4. *Let* $f \in \mathcal{M}^n(\delta, \beta)$ *with* $\beta > 1$. *Then the integral operator*

$$I_n(f)(z) = \int_0^z \frac{D^n f(t)}{t} dt \in \mathcal{N}(\beta).$$

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