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ON GRÜSS LIKE INTEGRAL INEQUALITIES VIA POMPEIU'S MEAN VALUE THEOREM

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ABSTRACT. In the present note we establish two new integral inequalities similar to that of the Grüss integral inequality via Pompeiu's mean value theorem.

Key words and phrases: Grüss like integral inequalities, Pompeiu's mean value theorem, Lagrange's mean value theorem, Differentiable, Properties of modulus.

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1. INTRODUCTION

In 1935 G. Grüss [4] proved the following integral inequality (see also [5, p. 296]):

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq \frac{1}{4}(P-p)(Q-q), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ such that

$$p \leq f(x) \leq P, \qquad q \leq g(x) \leq Q,$$

for all $x \in [a,b]$, where p, P, q, Q are constants.

The inequality (1.1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, see [1], [3], [5] – [10] and the references cited therein. The main aim of this note is to establish two new integral inequalities similar to the inequality (1.1) by using a variant of Lagrange's mean value theorem, now known as the Pompeiu's mean value theorem [11] (see also [12, p. 83] and [2]).

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2. Statement of Results

In what follows, $\mathbb{R}$ and $'$ denote the set of real numbers and derivative of a function respectively. For continuous functions $p, q : [a,b] \to \mathbb{R}$ which are differentiable on (a,b), we use the notations

$$G[p,q] = \int_a^b p(x)\, q(x)\, dx - \frac{1}{b^2-a^2}\left[\left(\int_a^b p(x)\, dx\right)\left(\int_a^b xq(x)\, dx\right) + \left(\int_a^b q(x)\, dx\right)\left(\int_a^b xp(x)\, dx\right)\right],$$

$$H[p,q] = \int_a^b p(x)\, q(x)\, dx - \frac{3}{b^3-a^3}\left(\int_a^b xp(x)\, dx\right)\left(\int_a^b xq(x)\, dx\right),$$

to simplify the details of presentation and define $\|p\|_\infty = \sup_{t\in[a,b]} |p(t)|$.

In the proofs of our results we make use of the following theorem, which is a variant of the well known Lagrange's mean value theorem given by Pompeiu in [11] (see also [2, 12]).

Theorem 2.1 (Pompeiu). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$ there exists a point c in (x_1,x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(c) - cf'(c).$$

Our main result is given in the following theorem.

Theorem 2.2. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then*

$$|G[f,g]| \leq \|f - lf'\|_\infty \int_a^b |g(x)| \left|\frac{1}{2} - \frac{x}{a+b}\right| dx + \|g - lg'\|_\infty \int_a^b |f(x)| \left|\frac{1}{2} - \frac{x}{a+b}\right| dx, \tag{2.1}$$

where $l(t) = t$, $t \in [a,b]$.

A slight variant of Theorem 2.2 is embodied in the following theorem.

Theorem 2.3. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then*

$$|H[f,g]| \leq \|f - lf'\|_\infty \|g - lg'\|_\infty |M|, \tag{2.2}$$

where $l(t) = t$, $t \in [a,b]$ and

$$M = (b-a)\left\{1 - \frac{3}{4}\cdot\frac{(a+b)^2}{a^2+ab+b^2}\right\}. \tag{2.3}$$

3. Proofs of Theorems 2.2 and 2.3

From the hypotheses of Theorems 2.2 and 2.3 and using Theorem 2.1 for $t \neq x$, $x, t \in [a,b]$, there exist points c and d between x and t such that

$$t\, f(x) - x\, f(t) = [f(c) - cf'(c)](t - x), \tag{3.1}$$

$$t\, g(x) - x\, g(t) = [g(d) - dg'(d)](t - x). \tag{3.2}$$

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

$$2\,t\,f\left(x\right)g\left(x\right)-x\,g\left(x\right)f\left(t\right)-x\,f\left(x\right)g\left(t\right)=\left[f\left(c\right)-cf'\left(c\right)\right]\left(t-x\right)g\left(x\right)+\left[g\left(d\right)-dg'\left(d\right)\right]\left(t-x\right)f\left(x\right). \tag{3.3}$$

Integrating both sides of (3.3) with respect to t over $[a,b]$ we have

$$\begin{aligned}\left(b^2-a^2\right)f\left(x\right)g\left(x\right)-x\,g\left(x\right)\int_a^b f\left(t\right)dt-x\,f\left(x\right)\int_a^b g\left(t\right)dt &= \left[f\left(c\right)-cf'\left(c\right)\right]\left\{\frac{b^2-a^2}{2}g\left(x\right)-x\,g\left(x\right)\left(b-a\right)\right\}\\&\quad+\left[g\left(d\right)-dg'\left(d\right)\right]\left\{\frac{b^2-a^2}{2}f\left(x\right)-x\,f\left(x\right)\left(b-a\right)\right\}.\end{aligned} \tag{3.4}$$

Now, integrating both sides of (3.4) with respect to x over $[a,b]$ we have

$$\begin{aligned}&\left(b^2-a^2\right)\int_a^b f\left(x\right)g\left(x\right)dx\\&\quad-\left(\int_a^b f\left(t\right)dt\right)\left(\int_a^b xg\left(x\right)dx\right)-\left(\int_a^b g\left(t\right)dt\right)\left(\int_a^b xf\left(x\right)dx\right)\\&\quad=\left[f\left(c\right)-cf'\left(c\right)\right]\left\{\frac{\left(b^2-a^2\right)}{2}\int_a^b g\left(x\right)dx-\left(b-a\right)\int_a^b xg\left(x\right)dx\right\}\\&\qquad+\left[g\left(d\right)-dg'\left(d\right)\right]\left\{\frac{\left(b^2-a^2\right)}{2}\int_a^b f\left(x\right)dx-\left(b-a\right)\int_a^b xf\left(x\right)dx\right\}.\end{aligned} \tag{3.5}$$

Rewriting (3.5) we have

$$\begin{aligned}G\left[f,g\right]&=\left[f\left(c\right)-cf'\left(c\right)\right]\int_a^b g\left(x\right)\left\{\frac{1}{2}-\frac{x}{a+b}\right\}dx\\&\quad+\left[g\left(d\right)-dg'\left(d\right)\right]\int_a^b f\left(x\right)\left\{\frac{1}{2}-\frac{x}{a+b}\right\}dx.\end{aligned} \tag{3.6}$$

Using the properties of modulus, from (3.6) we have

$$\begin{aligned}\left|G\left[f,g\right]\right|&\le\left\|f-lf'\right\|_\infty\int_a^b\left|g\left(x\right)\right|\left|\frac{1}{2}-\frac{x}{a+b}\right|dx\\&\quad+\left\|g-lg'\right\|_\infty\int_a^b\left|f\left(x\right)\right|\left|\frac{1}{2}-\frac{x}{a+b}\right|dx.\end{aligned}$$

This completes the proof of Theorem 2.2.

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$t^2f\left(x\right)g\left(x\right)-\left(xf\left(x\right)\right)\left(tg\left(t\right)\right)-\left(xg\left(x\right)\right)\left(tf\left(t\right)\right)+x^2f\left(t\right)g\left(t\right)=\left[f\left(c\right)-cf'\left(c\right)\right]\left[g\left(d\right)-dg'\left(d\right)\right]\left(t-x\right)^2. \tag{3.7}$$

Integrating both sides of (3.7) with respect to t over $[a,b]$ we have

$$\frac{(b^3-a^3)}{3} f(x)\,g(x) - xf(x)\int_a^b tg(t)\,dt - xg(x)\int_a^b tf(t)\,dt + x^2\int_a^b f(t)\,g(t)\,dt = [f(c)-cf'(c)]\,[g(d)-dg'(d)]\left\{\frac{(b^3-a^3)}{3} - x\left(b^2-a^2\right) + x^2(b-a)\right\}. \tag{3.8}$$

Now, integrating both sides of (3.8) with respect to x over $[a,b]$ we have

$$\begin{aligned}\frac{(b^3-a^3)}{3}\int_a^b f(x)\,g(x)\,dx &- \left(\int_a^b xf(x)\,dx\right)\left(\int_a^b tg(t)\,dt\right)\\ &- \left(\int_a^b xg(x)\,dx\right)\left(\int_a^b tf(t)\,dt\right) + \frac{(b^3-a^3)}{3}\int_a^b f(t)\,g(t)\,dt\\ &= [f(c)-cf'(c)]\,[g(d)-dg'(d)]\\ &\quad\times\left\{\frac{(b^3-a^3)}{3}(b-a) - \left(b^2-a^2\right)\frac{(b^2-a^2)}{2} + (b-a)\frac{(b^3-a^3)}{3}\right\}.\end{aligned} \tag{3.9}$$

Rewriting (3.9) we have

$$H[f,g] = [f(c)-cf'(c)]\,[g(d)-dg'(d)]\,M. \tag{3.10}$$

Using the properties of modulus, from (3.10) we have

$$|H[f,g]| \le \|f-lf'\|_\infty \|g-lg'\|_\infty |M|.$$

The proof of Theorem 2.3 is complete.

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