

SOLUTION OF ONE CONJECTURE ON INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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Abstract: In this paper, we prove one conjecture presented in the paper [V. Cîrtoaje, *On some inequalities with power-exponential functions*, J. Inequal. Pure Appl. Math. 10 (2009) no. 1, Art. 21. <http://jipam.vu.edu.au/article.php?sid=1077>].

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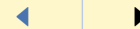
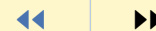
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1. Introduction

In the paper [1], V. Cîrtoaje posted 5 conjectures on inequalities with power-exponential functions. In this paper, we prove Conjecture 4.6.

Conjecture 4.6. *Let r be a positive real number. The inequality*

$$(1.1) \quad a^{rb} + b^{ra} \leq 2$$

holds for all nonnegative real numbers a and b with $a + b = 2$, if and only if $r \leq 3$.

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2. Proof of Conjecture 4.6

First, we prove the necessary condition. Put $a = 2 - \frac{1}{x}$, $b = \frac{1}{x}$, $r = 3x$ for $x > 1$. Then we have

$$(2.1) \quad a^{rb} + b^{ra} > 2.$$

In fact,

$$\left(2 - \frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^{3x(2-\frac{1}{x})} = 8 - \frac{12}{x} + \frac{6}{x^2} - \frac{1}{x^3} + \left(\frac{1}{x}\right)^{6x-3}$$

and if we show that $\left(\frac{1}{x}\right)^{6x-3} > -6 + \frac{12}{x} - \frac{6}{x^2} + \frac{1}{x^3}$ then the inequality (2.1) will be fulfilled for all $x > 1$. Put $t = \frac{1}{x}$, then $0 < t < 1$. The inequality (2.1) becomes

$$t^{\frac{6}{t}} > t^3(t^3 - 6t^2 + 12t - 6) = t^3\beta(t),$$

where $\beta(t) = t^3 - 6t^2 + 12t - 6$. From $\beta'(t) = 3(t-2)^2$, $\beta(0) = -6$, and from that there is only one real $t_0 = 0.7401$ such that $\beta(t_0) = 0$ and we have that $\beta(t) \leq 0$ for $0 \leq t \leq t_0$. Thus, it suffices to show that $t^{\frac{6}{t}} > t^3\beta(t)$ for $t_0 < t < 1$. Rewriting the previous inequality we get

$$\alpha(t) = \left(\frac{6}{t} - 3\right) \ln t - \ln(t^3 - 6t^2 + 12t - 6) > 0.$$

From $\alpha(1) = 0$, it suffices to show that $\alpha'(t) < 0$ for $t_0 < t < 1$, where

$$\alpha'(t) = -\frac{6}{t^2} \ln t + \left(\frac{6}{t} - 3\right) \frac{1}{t} - \frac{3t^2 - 12t + 12}{t^3 - 6t^2 + 12t - 6}.$$



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$\alpha'(t) < 0$ is equivalent to

$$\gamma(t) = 2 \ln t - 2 + t + \frac{t^2(t-2)^2}{t^3 - 6t^2 + 12t - 6} > 0.$$

From $\gamma(1) = 0$, it suffices to show that $\gamma'(t) < 0$ for $t_0 < t < 1$, where

$$\begin{aligned} \gamma'(t) &= \frac{(4t^3 - 12t^2 + 8t)(t^3 - 6t^2 + 12t - 6) - (t^4 - 4t^3 + 4t^2)(3t^2 - 12t + 12)}{(t^3 - 6t^2 + 12t - 6)^2} \\ &\quad + \frac{2}{t} + 1 \\ &= \frac{t^6 - 12t^5 + 56t^4 - 120t^3 + 120t^2 - 48t}{(t^3 - 6t^2 + 12t - 6)^2} + \frac{2}{t} + 1. \end{aligned}$$

$\gamma'(t) < 0$ is equivalent to

$$p(t) = 2t^7 - 22t^6 + 92t^5 - 156t^4 + 24t^3 + 240t^2 - 252t + 72 < 0.$$

From

$$p(t) = 2(t-1)(t^6 - 10t^5 + 36t^4 - 42t^3 - 30t^2 + 90t - 36),$$

it suffices to show that

$$(2.2) \quad q(t) = t^6 - 10t^5 + 36t^4 - 42t^3 - 30t^2 + 90t - 36 > 0.$$

Since $q(0.74) = 5.893$, $q(1) = 9$ it suffices to show that $q''(t) < 0$ and (2.2) will be proved. Indeed, for $t_0 < t < 1$, we have

$$\begin{aligned} q''(t) &= 2(15t^4 - 100t^3 + 216t^2 - 126t - 30) \\ &< 2(40t^4 - 100t^3 + 216t^2 - 126t - 30) \\ &= 4(t-1)(20t^3 - 30t^2 + 78t + 15) \\ &< 4(t-1)(-30t^2 + 78t) < 0. \end{aligned}$$



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This completes the proof of the necessary condition.

We prove the sufficient condition. Put $a = 1 - x$ and $b = 1 + x$, where $0 < x < 1$. Since the desired inequality is true for $x = 0$ and for $x = 1$, we only need to show that

$$(2.3) \quad (1 - x)^{r(1+x)} + (1 + x)^{r(1-x)} \leq 2 \quad \text{for } 0 < x < 1, 0 < r \leq 3.$$

Denote $\varphi(x) = (1 - x)^{r(1+x)} + (1 + x)^{r(1-x)}$. We show that $\varphi'(x) < 0$ for $0 < x < 1$, $0 < r \leq 3$ which gives that (2.3) is valid ($\varphi(0) = 2$).

$$\varphi'(x) = (1-x)^{r(1+x)} \left(r \ln(1-x) - r \frac{1+x}{1-x} \right) + (1+x)^{r(1-x)} \left(r \frac{1-x}{1+x} - r \ln(1+x) \right).$$

The inequality $\varphi'(x) < 0$ is equivalent to

$$(2.4) \quad \left(\frac{1+x}{1-x} \right)^r \left(\frac{1-x}{1+x} - \ln(1+x) \right) \leq (1-x^2)^{rx} \left(\frac{1+x}{1-x} - \ln(1-x) \right).$$

If $\delta(x) = \frac{1-x}{1+x} - \ln(1+x) \leq 0$, then (2.4) is evident. Since $\delta'(x) = -\frac{2}{(1+x)^2} - \frac{1}{1+x} < 0$ for $0 \leq x < 1$, $\delta(0) = 1$ and $\delta(1) = -\ln 2$, we have $\delta(x) > 0$ for $0 \leq x < x_0 \cong 0.4547$. Therefore, it suffices to show that $h(x) \geq 0$ for $0 \leq x \leq x_0$, where

$$h(x) = rx \ln(1-x^2) - r \ln \left(\frac{1+x}{1-x} \right) + \ln \left(\frac{1+x}{1-x} - \ln(1-x) \right) - \ln \left(\frac{1-x}{1+x} - \ln(1+x) \right).$$

We show that $h'(x) \geq 0$ for $0 < x < x_0$, $0 < r \leq 3$. Then from $h(0) = 0$ we obtain



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$h(x) \geq 0$ for $0 < x \leq x_0$ and it implies that the inequality (2.4) is valid.

$$h'(x) = r \ln(1 - x^2) - 2r \frac{1 + x^2}{1 - x^2} + \frac{3 - x}{(1 - x)(1 + x - (1 - x) \ln(1 - x))} + \frac{3 + x}{(1 + x)(1 - x - (1 + x) \ln(1 + x))}.$$

Put $A = \ln(1 + x)$ and $B = \ln(1 - x)$. The inequality $h'(x) \geq 0$, $0 < x < x_0$ is equivalent to

$$(2.5) \quad r(2x^2 + 2 - (1 - x^2)(A + B)) \leq \frac{3 - 2x - x^2}{1 - x - (1 + x)A} + \frac{3 + 2x - x^2}{1 + x - (1 - x)B}.$$

Since $2x^2 + 2 - (1 - x^2)(A + B) > 0$ for $0 < x < 1$, it suffices to prove that

$$(2.6) \quad 3(2x^2 + 2 - (1 - x^2)(A + B)) \leq \frac{3 - 2x - x^2}{1 - x - (1 + x)A} + \frac{3 + 2x - x^2}{1 + x - (1 - x)B}$$

and then the inequality (2.5) will be fulfilled for $0 < r \leq 3$. The inequality (2.6) for $0 < x < x_0$ is equivalent to

$$(2.7) \quad 6x^2 - 6x^4 - (9x^4 + 13x^3 + 5x^2 + 7x + 6)A - (9x^4 - 13x^3 + 5x^2 - 7x + 6)B \\ - (3x^4 + 6x^3 - 6x - 3)A^2 - (3x^4 - 6x^3 + 6x - 3)B^2 \\ - (12x^4 - 12)AB - (3x^4 - 6x^2 + 3)AB(A + B) \leq 0.$$

It is easy to show that the following Taylor's formulas are valid for $0 < x < 1$:

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad B = - \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1},$$

$$A^2 = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) x^{n+1}, \quad B^2 = \sum_{n=1}^{\infty} \frac{2}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) x^{n+1},$$

$$AB = - \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sum_{i=1}^{2n+1} \frac{(-1)^{i+1}}{i} \right) x^{2n+2}.$$

Since

$$A^2 + B^2 = \sum_{n=1,3,5,\dots} \frac{4}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) x^{n+1}$$

and

$$\frac{4}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) \leq \frac{4}{n+1} \left(1 + \frac{n-1}{2} \right) = 2,$$

we have

$$\begin{aligned} A^2 + B^2 &= \sum_{n=1,3,5,\dots} \frac{4}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) x^{n+1} \\ &= 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \sum_{n=7,9,\dots} \frac{4}{n+1} \left(\sum_{i=1}^n \frac{1}{i} \right) x^{n+1} \\ &< 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + 2 \sum_{n=7,9,\dots} x^{n+1} \\ &= 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \frac{2x^8}{1-x^2}. \end{aligned}$$

From this and from the previous Taylor's formulas we have

$$(2.8) \quad A + B > -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4} \left(\frac{x^8}{1-x^2} \right),$$



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$$(2.9) \quad A - B > 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7,$$

$$(2.10) \quad A^2 + B^2 < 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \frac{2x^8}{1-x^2},$$

$$(2.11) \quad A^2 - B^2 < -2x^3 - \frac{5}{3}x^5,$$

$$(2.12) \quad AB < -x^2 - \frac{5}{12}x^4 \quad \text{for } 0 < x < 1.$$

Now, having in view (2.12) and the obvious inequality $A + B < 0$, to prove (2.7) it suffices to show that

$$\begin{aligned} &6x^2 - 6x^4 - (6 + 5x^2 + 9x^4)(A + B) + (7x + 13x^3)(B - A) + (3 - 3x^4)(A^2 + B^2) \\ &+ (6x - 6x^3)(A^2 - B^2) - (12 - 12x^4) \left(x^2 + \frac{5}{12}x^4 \right) \\ &+ \left(x^2 + \frac{5}{12}x^4 \right) (3 - 6x^2 + 3x^4)(A + B) \leq 0. \end{aligned}$$

By using the inequalities (2.10), (2.11), the previous inequality will be proved if we



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show that

$$\begin{aligned} & 6x^2 - 6x^4 - (6 + 5x^2 + 9x^4)(A + B) + (7x + 13x^3)(B - A) \\ & + (3 - 3x^4) \left(2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \frac{2x^8}{1-x^2} \right) - (6x - 6x^3) \left(2x^3 + \frac{5}{3}x^5 \right) \\ & - (12 - 12x^4) \left(x^2 + \frac{5}{12}x^4 \right) + \left(x^2 + \frac{5}{12}x^4 \right) (3 - 6x^2 + 3x^4)(B + A) \leq 0, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} (2.13) \quad & -\frac{35}{2}x^4 + \frac{377}{30}x^6 + \frac{19}{2}x^8 - \frac{137}{30}x^{10} + 6(x^8 + x^{10}) \\ & - (A + B) \left(6 + 2x^2 + \frac{55}{4}x^4 - \frac{1}{2}x^6 - \frac{5}{4}x^8 \right) \\ & + (7x + 13x^3)(B - A) \leq 0. \end{aligned}$$

To prove (2.13) it suffices to show

$$\begin{aligned} (2.14) \quad & -8x^2 - \frac{259}{6}x^4 + \frac{357}{20}x^6 + \frac{1841}{120}x^8 + \frac{337}{420}x^{10} - \frac{19}{24}x^{12} - \frac{5}{12}x^{14} \\ & + \frac{x^8}{1-x^2} \left(\frac{3}{2} + \frac{1}{2}x^2 + \frac{55}{16}x^4 - \frac{1}{8}x^6 - \frac{5}{16}x^8 \right) < 0. \end{aligned}$$

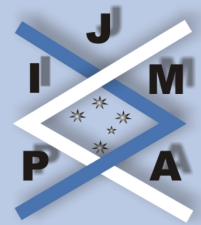
It follows from (2.8) and (2.9). Since $0 < x < \frac{1}{2}$ we have $\frac{1}{1-x^2} < \frac{4}{3}$. If we show

$$\begin{aligned} \varepsilon(x) = & -8x^2 - \frac{259}{6}x^4 + \frac{357}{20}x^6 + \frac{1841}{120}x^8 + \frac{337}{420}x^{10} - \frac{19}{24}x^{12} \\ & - \frac{5}{12}x^{14} + 2x^8 + \frac{2}{3}x^{10} + \frac{55}{12}x^{12} - \frac{1}{6}x^{14} - \frac{5}{12}x^{16} < 0, \end{aligned}$$

then the inequality (2.14) will be proved. From $x^6 < x^4$, $x^8 < x^4$, $x^{10} < x^4$ and $x^{12} < x^4$, we obtain that

$$\varepsilon(x) < -8x^2 - \frac{19}{7}x^4 - \frac{7}{12}x^{14} - \frac{5}{12}x^{16} < 0.$$

This completes the proof.



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