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YOUNG'S INTEGRAL INEQUALITY ON TIME SCALES REVISITED

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ABSTRACT. A more complete Young's integral inequality on arbitrary time scales (unbounded above) is presented.

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1. INTRODUCTION

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time-scale calculus was first accomplished by Hilger in his Ph.D. thesis [4]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculuses, and in extending our understanding to a new, more general and overarching theory.

The purpose of this note is to illustrate this new understanding by extending a continuous result, Young's inequality [3, 6], to arbitrary time scales. Throughout this note a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to calculus on time scales, see Bohner and Peterson [1].

2. REVISITING YOUNG'S INEQUALITY ON TIME SCALES

Recently Wong, Yeh, Yu, and Hong [5] presented a version of Young's inequality on time scales. An important subplot in the story of Young's inequality includes an if and only if clause concerning an actual equality; this is missing in the formulation in [5]. Moreover, in [5] the authors implicitly assume that the integrand function in the proposed integral inequality is delta differentiable, an unnecessarily strong assumption. In this note we will rectify these shortcomings by presenting a more complete version of Young's inequality on time scales with standard assumptions on the integrand function. To this end, let $\mathbb{T}$ be any time scale (unbounded above) that contains 0. Then we have the following extension of Young's inequality to arbitrary time scales, whose statement and proof are quite different from that found in [5].

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Theorem 2.1 (Young's Inequality I). *Let $\mathbb{T}$ be any time scale (unbounded above) with $0 \in \mathbb{T}$. Further, suppose that $f : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is a real-valued function satisfying*

(1) $f(0) = 0$;
(2) *f is continuous on $[0, \infty)_{\mathbb{T}}$, right-dense continuous at 0;*
(3) *f is strictly increasing on $[0, \infty)_{\mathbb{T}}$ such that $f(\mathbb{T})$ is also a time scale.*

Then for any $a \in [0, \infty)_{\mathbb{T}}$ and $b \in [0, \infty) \cap f(\mathbb{T})$, we have

$$\int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^b f^{-1}(y)\Delta y + \int_0^b f^{-1}(y)\nabla y \geq 2ab, \tag{2.1}$$

with equality if and only if $b = f(a)$.

Proof. The proof is modelled after the one given on $\mathbb{R}$ in [2]. Note that f is both delta and nabla integrable by the continuity assumption in (ii). For simplicity, define

$$F(a, b) := \int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^b f^{-1}(y)\Delta y + \int_0^b f^{-1}(y)\nabla y - 2ab.$$

Then, the inequality to be shown is just $F(a, b) \geq 0$.

(I). We will first show that

$$F(a, b) \geq F(a, f(a)), \quad a \in [0, \infty)_{\mathbb{T}}, \quad b \in [0, \infty) \cap f(\mathbb{T}),$$

with equality if and only if $b = f(a)$. For any such a and b, we have

$$\begin{aligned} F(a, b) - F(a, f(a)) &= \int_{f(a)}^b \left[f^{-1}(y) - a\right] \Delta y + \int_{f(a)}^b \left[f^{-1}(y) - a\right] \nabla y \\ &= \int_b^{f(a)} \left[a - f^{-1}(y)\right] \Delta y + \int_b^{f(a)} \left[a - f^{-1}(y)\right] \nabla y. \end{aligned}$$

There are two cases to consider. The first case is $b \geq f(a)$. Here, whenever $y \in [f(a), b] \cap f(\mathbb{T})$, we have $f^{-1}(b) \geq f^{-1}(y) \geq f^{-1}(f(a)) = a$. Consequently,

$$F(a, b) - F(a, f(a)) = \int_{f(a)}^b \left[f^{-1}(y) - a\right] \Delta y + \int_{f(a)}^b \left[f^{-1}(y) - a\right] \nabla y \geq 0.$$

Since $f^{-1}(y) - a$ is continuous and strictly increasing for $y \in [f(a), b] \cap f(\mathbb{T})$, equality will hold if and only if $b = f(a)$. The second case is $b \leq f(a)$. Here, whenever $y \in [b, f(a)] \cap f(\mathbb{T})$, we have $f^{-1}(b) \leq f^{-1}(y) \leq f^{-1}(f(a)) = a$. Consequently,

$$F(a, b) - F(a, f(a)) = \int_b^{f(a)} \left[a - f^{-1}(y)\right] \Delta y + \int_b^{f(a)} \left[a - f^{-1}(y)\right] \nabla y \geq 0.$$

Since $a - f^{-1}(y)$ is continuous and strictly decreasing for $y \in [b, f(a)] \cap f(\mathbb{T})$, equality will hold if and only if $b = f(a)$.

(II). We will next show that $F(a, f(a)) = 0$. For brevity, put $\delta(a) = F(a, f(a))$, that is

$$\delta(a) := \int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^{f(a)} f^{-1}(y)\Delta y + \int_0^{f(a)} f^{-1}(y)\nabla y - 2af(a).$$

First, assume a is a right-scattered point. Then

$$\begin{aligned}\delta^\sigma(a)-\delta(a) &= [\sigma(a)-a]f(a)+[\sigma(a)-a]f^\sigma(a)+[f^\sigma(a)-f(a)]\,f^{-1}(f(a))\\ &\qquad +[f^\sigma(a)-f(a)]\,f^{-1}\left(f^\sigma(a)\right)-2\left[\sigma(a)f^\sigma(a)-af(a)\right]\\ &= [\sigma(a)-a]\left[f(a)+f^\sigma(a)\right]+[f^\sigma(a)-f(a)]\left[a+\sigma(a)\right]\\ &\qquad -2\left[\sigma(a)f^\sigma(a)-af(a)\right]\\ &= 0.\end{aligned}$$

Therefore, if a is a right-scattered point, then $\delta^\Delta(a)=0$. Next, assume a is a right-dense point. Let $\{a_n\}_{n\in\mathbb{N}}\subset[a,\infty)_\mathbb{T}$ be a decreasing sequence converging to a. Then

$$\begin{aligned}\delta(a_n)-\delta(a) &= \int_a^{a_n} f(t)\Delta t+\int_a^{a_n} f(t)\nabla t+\int_{f(a)}^{f(a_n)} f^{-1}(y)\Delta y\\ &\qquad +\int_{f(a)}^{f(a_n)} f^{-1}(y)\nabla y-2a_nf(a_n)+2af(a)\\ &= \int_a^{a_n}\left[f(t)-f(a_n)\right]\Delta t+\int_a^{a_n}\left[f(t)-f(a_n)\right]\nabla t\\ &\qquad +\int_{f(a)}^{f(a_n)}\left[f^{-1}(y)-a\right]\Delta y+\int_{f(a)}^{f(a_n)}\left[f^{-1}(y)-a\right]\nabla y.\end{aligned}$$

Since the functions f and f^{-1} are strictly increasing,

$$\begin{aligned}\delta(a_n)-\delta(a) &\geq \int_a^{a_n}\left[f(a)-f(a_n)\right]\Delta t+\int_a^{a_n}\left[f(a)-f(a_n)\right]\nabla t\\ &\qquad +\int_{f(a)}^{f(a_n)}\left[f^{-1}(f(a))-a\right]\Delta y+\int_{f(a)}^{f(a_n)}\left[f^{-1}(f(a))-a\right]\nabla y\\ &= 2(a_n-a)\left[f(a)-f(a_n)\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\delta(a_n)-\delta(a) &\leq \int_a^{a_n}\left[f(a_n)-f(a_n)\right]\Delta t+\int_a^{a_n}\left[f(a_n)-f(a_n)\right]\nabla t\\ &\qquad +\int_{f(a)}^{f(a_n)}\left[f^{-1}(f(a_n))-a\right]\Delta y+\int_{f(a)}^{f(a_n)}\left[f^{-1}(f(a_n))-a\right]\nabla y\\ &= 2\left[f(a_n)-f(a)\right](a_n-a).\end{aligned}$$

Therefore,

$$0=\lim_{n\to\infty}2\left[f(a)-f(a_n)\right]\leq\lim_{n\to\infty}\frac{\delta(a_n)-\delta(a)}{a_n-a}\leq\lim_{n\to\infty}2\left[f(a_n)-f(a)\right]=0.$$

It follows that $\delta^\Delta(a)$ exists, and $\delta^\Delta(a)=0$ for right-dense a as well. In other words, in either case, $\delta^\Delta(a)=0$ for $a\in(0,\infty)_\mathbb{T}$. As $\delta(0)=0$, by the uniqueness theorem for initial value problems, we have that $\delta(a)=0$ for all $a\in[0,\infty)_\mathbb{T}$.

As an overall result, we have that

$$F(a,b)\geq F(a,f(a))=0,$$

with equality if and only if $b=f(a)$, as claimed. □

Theorem 2.2 (Young's Inequality II). *Let $\mathbb{T}$ be any time scale (unbounded above) with $0\in\mathbb{T}$. Further, suppose that $f:[0,\infty)_\mathbb{T}\to\mathbb{R}$ is a real-valued function satisfying*

(1) $f(0) = 0$;
(2) f *is continuous on* $[0,\infty)_{\mathbb{T}}$*, right-dense continuous at* 0;
(3) f *is strictly increasing on* $[0,\infty)_{\mathbb{T}}$ *such that* $f(\mathbb{T})$ *is also a time scale.*

Then for any $a \in [0,\infty)_{\mathbb{T}}$ *and* $b \in [0,\infty) \cap f(\mathbb{T})$*, we have*

$$\int_0^a [f(t) + f(\sigma(t))]\,\Delta t + \int_0^b \left[f^{-1}(y) + f^{-1}(\sigma(y))\right] \Delta y \geq 2ab, \tag{2.2}$$

with equality if and only if $b = f(a)$.

Proof. For a continuous function g and $a \in [0,\infty)_{\mathbb{T}}$, define the function

$$G(a) := \int_0^a g(t)\Delta t + \int_0^a g(t)\nabla t - \int_0^a [g(t) + g(\sigma(t))]\,\Delta t.$$

Then $G(0) = 0$, and

$$G^{\Delta}(a) := g(a) + g(\sigma(a)) - [g(a) + g(\sigma(a))] = 0.$$

Therefore $G \equiv 0$, and Theorem 2.2 follows from Theorem 2.1. □

Remark 2.3. Consider (2.2). If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and the theorem yields the classic Young inequality,

$$\int_0^a f(t)dt + \int_0^b f^{-1}(y)dy \geq ab.$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and the theorem yields Young's discrete inequality,

$$\sum_{t=0}^{a-1} [f(t) + f(t+1)] + \sum_{y \in [0,b) \cap f(\mathbb{Z})} \mu(y) \left[2f^{-1}(y) + 1\right] \geq 2ab,$$

since by the discrete nature of the expression, $f^{-1}(\sigma(y)) = \sigma\left(f^{-1}(y)\right) = f^{-1}(y) + 1$. If $\mathbb{T} = \mathbb{T}_r$, where $r > 1$ and $\mathbb{T}_r := \{0\} \cup \{r^z\}_{z \in \mathbb{Z}}$, then we have Young's quantum inequality

$$(r-1) \sum_{\tau=-\infty}^{\alpha-1} r^{\tau} \left[f(r^{\tau}) + f(r^{\tau+1})\right] + (r+1) \sum_{y \in [0,b) \cap f(\mathbb{T}_r)} \mu(y) f^{-1}(y) \geq 2ab,$$

where $a = r^{\alpha}$ and $t = r^{\tau}$ for $\alpha, \tau \in \mathbb{Z}$.

Corollary 2.4. *Assume* $\mathbb{T}$ *is any time scale with* $0 \in \mathbb{T}$*. Let* $p, q > 1$ *be real numbers with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for any* $a \in [0,\infty)_{\mathbb{T}}$ *and* $b \in [0,\infty)_{\mathbb{T}^*}$*, where* $\mathbb{T}^* := \{t^{p-1} : t \in \mathbb{T}\}$*, we have*

$$\int_0^a t^{p-1}\Delta t + \int_0^a t^{p-1}\nabla t + \int_0^b y^{q-1}\Delta y + \int_0^b y^{q-1}\nabla y \geq 2ab,$$

with equality if and only if $b = a^{p-1}$.

Proof. Let $f(t) := t^{p-1}$ for $t \in [0,\infty)_{\mathbb{T}}$. Then $f^{-1}(y) = y^{q-1}$ for $y \in \mathbb{T}^*$, and all the hypotheses of Theorem 2.1 are satisfied; the result follows. □

REFERENCES

[1] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston (2001).

[2] J.B. DIAZ AND F.T. METCALF, An analytic proof of Young's inequality, *American Math. Monthly*, **77** (1970), 603–609.

[3] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge, 1934.

[4] S. HILGER, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD. thesis, Universität Würzburg (1988).

[5] F.H. WONG, C.C. YEH, S.L. YU AND C.H. HONG, Young's inequality and related results on time scales, *Appl. Math. Letters*, **18** (2005), 983–988.

[6] W.H. YOUNG, On classes of summable functions and their Fourier series, *Proc. Royal Soc.*, Series (A), **87** (1912), 225–229.