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NEW OSTROWSKI TYPE INEQUALITIES VIA MEAN VALUE THEOREMS

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Abstract

The main aim of the present note is to establish two new Ostrowski type inequalities by using the mean value theorems.

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Contents

1. Introduction

The well known Ostrowski's inequality [5] can be stated as follows (see also [4, p. 468]).

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \tag{1.1}$$

for all $x \in [a, b]$.

In the past few years inequality (1.1) has received considerable attention from many researchers and a number of papers have appeared in the literature, which deal with alternative proofs, various generalizations, numerous variants and applications. A survey of some of the earlier and recent developments related to the inequality (1.1) can be found in [4] and [1] and the references given therein (see also [2], [3], [6] – [8]). The main purpose of the present note is to establish two new Ostrowski type inequalities using the well known Cauchy's mean value theorem and a variant of the Lagrange's mean value theorem given by Pompeiu in [9] (see also [10, p. 83] and [3]).

2. Statement of Results

In the proofs of our results we make use of the well known Cauchy's mean value theorem and the following variant of the Lagrange's mean value theorem given by Pompeiu in [9] (see also [3, 10]).

Theorem A. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point c in (x_1,x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(c) - cf'(c).$$

Our main results are given in the following theorems.

Theorem 2.1. *Let $f,g,h : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, $a<b$; $a,b \in \mathbb{R}$ and differentiable on (a,b) and $w : [a,b] \to [0,\infty)$ be an integrable function such that $\int_a^b w(y)\,dy > 0$. If $h'(t) \neq 0$ for each $t \in (a,b)$, then*

$$\begin{aligned} &\left| f(x)\,g(x) \right. \\ &\qquad \left. - \frac{1}{2\int_a^b w(y)\,dy}\left[f(x)\int_a^b w(y)\,g(y)\,dy + g(x)\int_a^b w(y)\,f(y)\,dy \right] \right| \\ &\qquad \leq \frac{1}{2}\left\{ \left\| \frac{f'}{h'} \right\|_\infty |g(x)| + \left\| \frac{g'}{h'} \right\|_\infty |f(x)| \right\} \left\{ h(x) - \frac{\int_a^b w(y)\,h(y)\,dy}{\int_a^b w(y)\,dy} \right\}. \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$, *where*

$$\left\|\frac{f'}{h'}\right\|_\infty = \sup_{t\in(a,b)}\left|\frac{f'(t)}{h'(t)}\right| < \infty, \qquad \left\|\frac{g'}{h'}\right\|_\infty = \sup_{t\in(a,b)}\left|\frac{g'(t)}{h'(t)}\right| < \infty.$$

Theorem 2.2. *Let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, $a < b$; $a, b \in \mathbb{R}$ *and differentiable on* (a,b) *with* $[a,b]$ *not containing* 0 *and* $w : [a,b] \to [0,\infty)$ *an integrable function such that* $\int_a^b yw(y)dy > 0$. *Then*

$$(2.2)\quad \Bigg| f(x)\,g(x) - \frac{1}{2\int_a^b yw(y)\,dy}\left[xf(x)\int_a^b w(y)\,g(y)\,dy + xg(x)\int_a^b w(y)\,f(y)\,dy\right]\Bigg| \le \frac{1}{2}\left\{\|f - lf'\|_\infty |g(x)| + \|g - lg'\|_\infty |f(x)|\right\}\left|1 - \frac{x\int_a^b w(y)\,dy}{\int_a^b yw(y)\,dy}\right|,$$

for all $x \in [a,b]$, *where* $l(t) = t$, $t \in [a,b]$ *and*

$$\|f - lf'\|_\infty = \sup_{t\in[a,b]} |f(t) - tf'(t)| < \infty,$$

$$\|g - lg'\|_\infty = \sup_{t\in[a,b]} |g(t) - tg'(t)| < \infty.$$

3. Proofs of Theorems 2.1 and 2.2

Let $x, y \in [a, b]$ with $y \neq x$. From the hypotheses of Theorem 2.1 and applying Cauchy's mean value theorem to the pairs of functions f, h and g, h there exist points c and d between x and y such that

$$f(x) - f(y) = \frac{f'(c)}{h'(c)} \{h(x) - h(y)\}, \tag{3.1}$$

$$g(x) - g(y) = \frac{g'(d)}{h'(d)} \{h(x) - h(y)\}. \tag{3.2}$$

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding we get

$$\begin{aligned} &2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ &\qquad = \frac{f'(c)}{h'(c)} g(x)\{h(x) - h(y)\} + \frac{g'(d)}{h'(d)} f(x)\{h(x) - h(y)\}. \end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $w(y)$ and integrating the resulting identity with respect to y over $[a, b]$ we have

$$\begin{aligned} &2\left(\int_a^b w(y)\,dy\right) f(x)g(x) \\ &\qquad - g(x)\int_a^b w(y)f(y)\,dy - f(x)\int_a^b w(y)g(y)\,dy \end{aligned} \tag{3.4}$$

$$= \frac{f'(c)}{h'(c)} g(x) \left\{ \left(\int_a^b w(y)\,dy \right) h(x) - \int_a^b w(y)\,h(y)\,dy \right\}$$
$$+ \frac{g'(d)}{h'(d)} f(x) \left\{ \left(\int_a^b w(y)\,dy \right) h(x) - \int_a^b w(y)\,h(y)\,dy \right\}.$$

Rewriting (3.4) we have

$$\begin{aligned} (3.5)\quad & f(x)\,g(x) \\ & - \frac{1}{2\int_a^b w(y)\,dy} \left[f(x) \int_a^b w(y)\,g(y)\,dy + g(x) \int_a^b w(y)\,f(y)\,dy \right] \\ & = \frac{1}{2} \frac{f'(c)}{h'(c)} g(x) \left\{ h(x) - \frac{\int_a^b w(y)\,h(y)\,dy}{\int_a^b w(y)\,dy} \right\} \\ & \quad + \frac{1}{2} \frac{g'(d)}{h'(d)} f(x) \left\{ h(x) - \frac{\int_a^b w(y)\,h(y)\,dy}{\int_a^b w(y)\,dy} \right\}. \end{aligned}$$

From (3.5) and using the properties of modulus we have

$$(3.6)\quad \left| f(x)\,g(x) - \frac{1}{2\int_a^b w(y)\,dy} \left[f(x) \int_a^b w(y)\,g(y)\,dy + g(x) \int_a^b w(y)\,f(y)\,dy \right] \right|$$

$$\leq \frac{1}{2}\left\|\frac{f'}{h'}\right\|_{\infty}|g(x)|\left|h(x)-\frac{\int_a^b w(y)h(y)\,dy}{\int_a^b w(y)\,dy}\right| + \frac{1}{2}\left\|\frac{g'}{h'}\right\|_{\infty}|f(x)|\left|h(x)-\frac{\int_a^b w(y)h(y)\,dy}{\int_a^b w(y)\,dy}\right|.$$

Rewriting (3.6) we get the desired inequality in (2.1) and the proof of Theorem 2.1 is complete.

From the hypotheses of Theorem 2.2 and applying Theorem A for any $y \neq x$, $x, y \in [a, b]$, there exist points c and d between x and y such that

$$yf(x) - xf(y) = [f(c) - cf'(c)](y - x), \tag{3.7}$$

$$yg(x) - xg(y) = [g(d) - dg'(d)](y - x). \tag{3.8}$$

Multiplying both sides of (3.7) and (3.8) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

$$\begin{aligned} &2yf(x)g(x) - xg(x)f(y) - xf(x)g(y) \\ &\quad = [f(c) - cf'(c)](y - x)g(x) + [g(d) - dg'(d)](y - x)f(x). \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.9) by $w(y)$ and integrating the resulting identity with respect to y over $[a,b]$ we have

$$2\left(\int_a^b yw(y)\,dy\right)f(x)g(x) - xg(x)\int_a^b w(y)f(y)\,dy - xf(x)\int_a^b w(y)g(y)\,dy = [f(c)-cf'(c)]g(x)\left\{\int_a^b yw(y)\,dy - x\int_a^b w(y)\,dy\right\} + [g(d)-dg'(d)]f(x)\left\{\int_a^b yw(y)\,dy - x\int_a^b w(y)\,dy\right\}. \tag{3.10}$$

Rewriting (3.10) we get

$$f(x)g(x) - \frac{1}{2\int_a^b yw(y)\,dy}\left[xf(x)\int_a^b w(y)g(y)\,dy + xg(x)\int_a^b w(y)f(y)\,dy\right] = \frac{1}{2}[f(c)-cf'(c)]g(x)\left\{1-\frac{x\int_a^b w(y)\,dy}{\int_a^b yw(y)\,dy}\right\} + \frac{1}{2}[g(d)-dg'(d)]f(x)\left\{1-\frac{x\int_a^b w(y)\,dy}{\int_a^b yw(y)\,dy}\right\}. \tag{3.11}$$

From (3.11) and using the properties of modulus we have

$$
\begin{aligned}
(3.12)\quad & \Bigg| f(x)\, g(x) \\
&\quad - \frac{1}{2\int_a^b y w(y)\, dy}\left[x f(x) \int_a^b w(y)\, g(y)\, dy + x g(x) \int_a^b w(y)\, f(y)\, dy \right] \Bigg| \\
&\leq \frac{1}{2}\left\| f - l f' \right\|_\infty |g(x)| \left| 1 - \frac{x \int_a^b w(y)\, dy}{\int_a^b y w(y)\, dy} \right| \\
&\quad + \frac{1}{2}\left\| g - l g' \right\|_\infty |f(x)| \left| 1 - \frac{x \int_a^b w(y)\, dy}{\int_a^b y w(y)\, dy} \right| .
\end{aligned}
$$

Rewriting (3.12) we get the required inequality in (2.2). The proof of Theorem 2.2 is complete.

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