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INEQUALITIES FOR CHAINS OF NORMALIZED SYMMETRIC SUMS

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ABSTRACT. In this paper we prove some inequalities between expressions of the following form:

$$\sum_{1\le i_1<\cdots<i_k\le n} \frac{a_{i_1}+\cdots+a_{i_k}}{a_1+\cdots+a_n-(a_{i_1}+\cdots+a_{i_k})},$$

where $a_1,\cdots,a_n$ are positive numbers and $k,n\in\mathbb{N}$, $k<n$.
Using the results in [1] which show that $\binom{n}{k}\cdot\frac{n-k}{k}$ give a lower bound for the expressions above, we norm them and obtain the chain $A(1),\ A(2),\ \ldots,\ A(n-1),\ A(n)$, whose terms are defined as

$$A(k)=\frac{\sum_{1\le i_1<\cdots<i_k\le n}\frac{a_{i_1}+\cdots+a_{i_k}}{a_1+\cdots+a_n-(a_{i_1}+\cdots+a_{i_k})}}{\binom{n}{k}\cdot\frac{n-k}{k}}.$$

We prove then some inequalities between the terms of this chain.
Particular cases of the results obtained in this paper represent refinements of some classical inequalities due to Nesbit[7], Peixoto [8] and to Mitrinović [5].
The results in this work are also closely related to the inequalities between complemental expressions obtained in [1].

Key words and phrases: Symmetric inequalities, Cyclic inequalities, Inequalities for sums.

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1. INTRODUCTION AND NOTATIONS

Let n and k be natural numbers, such that $n\ge 2$ and $1\le k\le n$.

Denote by $\mathcal{I}=\{\{i_1,\ldots,i_k\}|\ 1\le i_1<\cdots<i_k\le n\}$, (the subsets of $\{1,\ldots,n\}$ which have k elements in an increasing order). We also consider

$$S_I=a_{i_1}+\cdots+a_{i_k},\quad I=\{i_1,\ldots,i_k\},$$

$$S=a_1+\cdots+a_n.$$

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Denote by

$$E(k)=\sum_{I\in\mathcal{I}}\frac{S-S_I}{S_I}. \tag{1.1}$$

Our goal is to obtain certain inequalities for expressions which involve $E(k)$. For different values of n and cardinals of the set I one obtains some classical results. For proofs of these examples and further applications, see [6] or [1].

For $k=1$ and $n=3$ we obtain the result of Nesbit [7] (see e.g. [3], [4]),

$$\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_1}+\frac{a_3}{a_1+a_2}\geq\frac{3}{2}. \tag{1.2}$$

For $k=1$, we obtain the result of Peixoto [8] (see e.g. [6]),

$$\frac{a_1}{S-a_1}+\cdots+\frac{a_n}{S-a_n}\geq\frac{n}{n-1}. \tag{1.3}$$

For arbitrary naturals n,k provided that $1\leq k\leq n-1$, we get the result of Mitrinović [5] (see e.g. [6]),

$$\frac{a_1+a_2+\cdots+a_k}{a_{k+1}+\cdots+a_n}+\frac{a_2+a_3+\cdots+a_{k+1}}{a_{k+2}+\cdots+a_n+a_1}+\cdots+\frac{a_n+a_1+\cdots+a_{k-1}}{a_k+\cdots+a_{n-1}}\geq\frac{nk}{n-k}. \tag{1.4}$$

These results are examples of cyclic inequalities.

An extension of these results to the symmetric form is given by

$$E(k)\geq\frac{k}{n-k}\binom{n}{k}. \tag{1.5}$$

For different proofs of (1.5) one can see [1, Theorem 2] and [2].

The above inequality suggests considering the expressions $A(k)$ which are defined as:

$$A(k)=\frac{E(k)}{\binom{n}{k}\cdot\frac{n-k}{k}}=\frac{\sum_{I\in\mathcal{I}}\frac{S-S_I}{S_I}}{\binom{n}{k}\cdot\frac{n-k}{k}}. \tag{1.6}$$

This is in fact a normalization of $E(k)$.

We would like to find some inequalities in the chain of expressions $A(1),\ A(2),\ \ldots,\ A(n-1),\ A(n)$.

In [1, Theorem 1] it is proved that for these expressions the inequality

$$A(k)\geq A(n-k) \tag{1.7}$$

holds for a natural $k\leq[\frac{n}{2}]$.

A natural question to ask is: For which values of $k\in\{1,\ldots,n-1\}$ does the inequality

$$A(k)\geq A(k+1) \tag{1.8}$$

hold?

In this paper we prove that (1.8) holds for $1\leq k\leq[\frac{n}{2}]-1$.

Since the inequality (1.3) may be formulated as $A(n-1)\geq A(n)=1$, another value of k for which (1.8) holds is $k=n-1$.

A further step in our research is made by using (1.8) to find some more inequalities between the left and right side of the chain $\{A(k)\}_{k=\overline{1,n}}$. We obtain results of the kind

$$A(k)\geq A(n-l) \tag{1.9}$$

for $1\leq k\leq l\leq[\frac{n}{2}]$.

At the conclusion of our paper we present some ideas which may lead to new results.

2. MAIN RESULTS

In this section we present the inequalities that we discussed in the introduction.

Theorem 2.1. *Let n and k be natural numbers, such that $n \geq 2$ and $1 \leq k \leq \left[\frac{n}{2}\right] - 1$.*

Denote by $\mathcal{I} = \{\{i_1, \dots, i_k\} | 1 \leq i_1 < \dots < i_k \leq n\}$. Denote then by $\mathcal{J} = \{\{i_1, \dots, i_{k+1}\} | 1 \leq i_1 < \dots < i_{k+1} \leq n\}$.

Considering the positive numbers $a_1, \dots, a_n$, the next inequality holds:

$$\frac{\sum_{I \in \mathcal{I}} \frac{S - S_I}{S_I}}{\binom{n}{k} \cdot \frac{n-k}{k}} \geq \frac{\sum_{J \in \mathcal{J}} \frac{S - S_J}{S_J}}{\binom{n}{k+1} \cdot \frac{n-(k+1)}{k+1}}.$$

This also may be written as $A(k) \geq A(k+1)$.

A result that gives some more information over the k's for which inequality (1.8) holds is the following:

Theorem 2.2. *Let n be a natural number such that $n \geq 2$. Then inequality (1.8) holds for $k = n - 1$.*

Combining the inequalities given in [1] and Theorem 2.1 we obtain the following result:

Theorem 2.3. *Let n and k, l be natural numbers, such that $n \geq 2$ and $1 \leq k \leq l \leq \left[\frac{n}{2}\right]$. Then the following inequality*

$$A(k) \geq A(n - l)$$

holds.

3. PROOFS

In this section we give the proofs of the results mentioned above. The proofs do not require complicated notions. The main idea is to write a sum of $k+1$ terms as a symmetric sum of some sums containing k terms.

Proof of Theorem 2.1. We begin with the proof of the inequality

$$A(k) \geq A(k+1), \tag{3.1}$$

for the case when $1 \leq k \leq \left[\frac{n}{2}\right] - 1$. The idea is to decompose sums of "bigger" sets into symmetric sums of "smaller" sets.

We write

$$E(k) = \sum_{I \in \mathcal{I}} \frac{S - S_I}{S_I} = \sum_{I \in \mathcal{I}} \frac{\sum_{j \notin I} a_j}{S_I}, \tag{3.2}$$

and note that $\sharp\{j \in \{1, \dots, n\} | \; j \notin I\} = n - k \geq k$.

We write $\sum_{j \notin I} a_j$ as a symmetric sum containing all possible sums of k distinct terms, which do not contain indices in I. Each such sum of k terms appears once.

Example. In the case $n = 5$, $k = 2$ we have:

$$a_1 + a_2 + a_3 = \frac{(a_1 + a_2) + (a_1 + a_3) + (a_2 + a_3)}{2}.$$

In the general case we write, for example, the sum of the first $n - k$ terms:

$$a_1 + \dots + a_{n-k} = \frac{(a_1 + \dots + a_k) + \dots + (a_{n-2k+1} + \dots + a_{n-k})}{\alpha}. \tag{3.3}$$

Clearly in the right member, a_1 appears for $\binom{n-k-1}{k-1}$ times, so $\alpha = \binom{n-k-1}{k-1}$. It is now easy to see that we may write

$$\sum_{j \notin I} a_j = \frac{\sum_{J \in \mathcal{I}} S_J}{\binom{n-k-1}{k-1}}, \tag{3.4}$$

where $J = \{j_1, \ldots, j_k\}$, with $I \cap J = \emptyset$.

We write (3.4) as

$$S - S_I = \sum_{\substack{J \in \mathcal{I} \\ J \cap I = \emptyset}} \frac{S_J}{\binom{n-k-1}{k-1}}. \tag{3.5}$$

We obtain

$$E(k) = \sum_{I \in \mathcal{I}} \frac{1}{\binom{n-k-1}{k-1}} \sum_{\substack{J \in \mathcal{I} \\ J \cap I = \emptyset}} \frac{S_J}{S_I},$$

that is,

$$E(k) = \frac{1}{\binom{n-k-1}{k-1}} \sum_{J \in \mathcal{I}} S_J \sum_{\substack{I \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_I}.$$

We choose then an appropriate notation for our further study and rewrite the above expression as:

$$E(k) = \frac{1}{\binom{n-k-1}{k-1}} \cdot \sum_{I_1 \in \mathcal{I}} S_{I_1} \sum_{\substack{I_2 \in \mathcal{I} \\ I_1 \cap I_2 = \emptyset}} \frac{1}{S_{I_2}}.$$

In the same way as before, we obtain:

$$E(k+1) = \frac{1}{\binom{n-k-2}{k}} \cdot \sum_{J_1 \in \mathcal{J}} S_{J_1} \sum_{\substack{J_2 \in \mathcal{J} \\ J_1 \cap J_2 = \emptyset}} \frac{1}{S_{J_2}}.$$

As in the previous case, one has to take into account that a necessary condition for the expression of the sum to be the same, is to have $n - 2(k+1) \geq 0$, so the assumptions we have made about k are essential.

By the use of the the above results, inequality (3.1) becomes:

$$\frac{\sum_{I_1 \in \mathcal{I}} S_{I_1} \sum_{\substack{I_2 \in \mathcal{I} \\ I_1 \cap I_2 = \emptyset}} \frac{1}{S_{I_2}}}{\binom{n}{k} \cdot \frac{n-k}{k} \cdot \binom{n-k-1}{k-1}} \geq \frac{\sum_{J_1 \in \mathcal{J}} S_{J_1} \sum_{\substack{J_2 \in \mathcal{J} \\ J_1 \cap J_2 = \emptyset}} \frac{1}{S_{J_2}}}{\binom{n}{k+1} \cdot \frac{n-(k+1)}{k+1} \cdot \binom{n-k-2}{k}}. \tag{3.6}$$

Using classical formulas for the binomial coefficients we obtain

$$\frac{n-k}{k} \cdot \binom{n-k-1}{k-1} = \binom{n-k}{k},$$
$$\frac{n-(k+1)}{k+1} \cdot \binom{n-k-2}{k} = \binom{n-k-1}{k+1}.$$

Letting $\alpha = \frac{(k+1)^2}{(n-2k)(n-(2k+1))}$, simple computation shows that (3.6) is equivalent to:

$$\sum_{I_1 \in \mathcal{I}} S_{I_1} \sum_{\substack{I_2 \in \mathcal{I} \\ I_1 \cap I_2 = \emptyset}} \frac{1}{S_{I_2}} \geq \alpha \cdot \sum_{J_1 \in \mathcal{J}} S_{J_1} \sum_{\substack{J_2 \in \mathcal{J} \\ J_1 \cap J_2 = \emptyset}} \frac{1}{S_{J_2}}. \tag{3.7}$$

In order to prove (3.7), it is useful to write the sum of $S_J{}'s$, in terms of sums of $S_I{}'s$.

To this end, we remark first that

$$\sum_{J_1\in\mathcal{J}} S_{J_1} \sum_{\substack{J_2\in\mathcal{I}\\ J_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}} = \sum_{\substack{J_2\in\mathcal{J}\\ J_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}} \sum_{J_1\in\mathcal{J}} S_{J_1}. \tag{3.8}$$

This is clearly just changing the order of summation.

Consider a fixed set J_2 in $\mathcal{J}$. We obtain that (just count the number of terms):

$$\sum_{\substack{J_1\in\mathcal{J}\\ J_1\cap J_2=\emptyset}} S_{J_1} = \frac{\binom{n-k-1}{k+1}\cdot(k+1)}{n-k-1}\cdot(S-S_{J_2}). \tag{3.9}$$

Following the idea that led to (3.5), we obtain:

$$S - S_{J_2} = \frac{\sum_{\substack{I_1\in\mathcal{I}\\ I_1\cap J_2=\emptyset}} S_{I_1}}{\binom{n-k-2}{k-1}}. \tag{3.10}$$

Putting together (3.9) and (3.10) we get that:

$$\sum_{\substack{J_1\in\mathcal{J}\\ J_1\cap J_2=\emptyset}} S_{J_1} = \frac{n-2k-1}{k} \sum_{\substack{I_1\in\mathcal{I}\\ I_1\cap J_2=\emptyset}} S_{I_1}. \tag{3.11}$$

Using again the changing of the order in summation one obtains:

$$\sum_{I_1\in\mathcal{I}} S_{I_1} \sum_{\substack{J_2\in\mathcal{J}\\ I_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}} = \sum_{\substack{J_2\in\mathcal{J}\\ I_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}} \sum_{I_1\in\mathcal{I}} S_{I_1}. \tag{3.12}$$

With the notations we have just established, we are ready now to begin the proof of the transformed inequality (3.7), which now can be written as:

$$\sum_{I_1\in\mathcal{I}} S_{I_1} \sum_{\substack{I_2\in\mathcal{I}\\ I_1\cap I_2=\emptyset}} \frac{1}{S_{I_2}} \ge \alpha\cdot\frac{n-2k-1}{k}\cdot\sum_{I_1\in\mathcal{I}} S_{I_1} \sum_{\substack{J_2\in\mathcal{J}\\ I_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}}. \tag{3.13}$$

Consider a fixed $I_1\in\mathcal{I}$ and prove that the following inequality holds:

$$\sum_{\substack{I_2\in\mathcal{I}\\ I_1\cap I_2=\emptyset}} \frac{1}{S_{I_2}} \ge \frac{(k+1)^2}{k(n-2k)}\cdot \sum_{\substack{J_2\in\mathcal{J}\\ I_1\cap J_2=\emptyset}} \frac{1}{S_{J_2}}. \tag{3.14}$$

We write

$$S_J = \frac{S(J,j_1)+\cdots+S_(J,j_{k+1})}{k},$$

where

$$J=(j_1,\dots,j_{k+1});\quad S(J,j_i)=S_J-a_{j_i};\quad i=\overline{1,k+1}.$$

This leads to

$$\frac{1}{S_J} = \frac{k}{S(J,j_1)+\cdots+S(J,j_{k+1})} \le \frac{k}{(k+1)^2}\cdot\left(\frac{1}{S(J,j_1)}+\cdots+\frac{1}{S(J,j_{k+1})}\right). \tag{3.15}$$

(we have used the inequality $(x_1+\cdots+x_n)\cdot(\frac{1}{x_1}+\cdots+\frac{1}{x_n})\ge n^2$, for positive numbers $x_1,\dots,x_n$.).

We just have to prove now that by summing the inequalities from (3.15), we get (3.14).

This is just an easy counting problem. It is enough to prove that the following identity

$$\sum_{\substack{I_2\in\mathcal{I}\\ I_1\cap I_2=\emptyset}} \frac{1}{S_{I_2}} = \frac{1}{(n-2k)} \cdot \sum_{\substack{J\in\mathcal{J}\\ I_1\cap J=\emptyset}} \left(\frac{1}{S(J,j_1)} + \cdots + \frac{1}{S(J,j_{k+1})} \right). \tag{3.16}$$

holds.

Taking a fixed $I_1 \in \mathcal{I}$ in the left side of (3.16) this term will appear in the right side if and only if I_1 is contained in J. But because $J \cap I_1 = \emptyset$, $I_1,\ I_2$ have k fixed elements and $|J| = k+1$, it follows that $J \setminus I_1$ consists of one of the remaining $(n-2k)$ elements of the set $n \setminus (I_1 \cup I_2)$. (the reunion $(I_1 \cup I_2)$ has exactly $2k$ elements, since the two sets are disjoint).

This shows that (3.16) holds and ends the proof of (1.8) for $k \le \left[\frac{n}{2}\right] - 1$. The proof is complete. $\square$

Proof of Theorem 2.2. This is nothing else than the inequality (1.3) which is due to Peixoto. $\square$

Proof of Theorem 2.3. A direct proof of this result may not be a very pleasant task, but by applying the results we have obtained, it is straight forward. First we apply inequality (1.8) for $(l-k)$ times and obtain

$$A(k) \ge A(k+1) \ge \cdots \ge A(l). \tag{3.17}$$

We may then apply the inequality (1.7) which gives

$$A(l) \ge A(n-l). \tag{3.18}$$

Combining inequalities (3.17) and (3.18) we finally obtain that (1.9) holds, which ends the proof. $\square$

4. FURTHER RESULTS

In the case when n is odd the following extension holds:

Theorem 4.1. *Let n be a natural number such that $n \ge 2$ and consider the positive numbers $a_1, \dots, a_n$. The following inequality holds:*

$$A\left(\left[\frac{n}{2}\right]\right) \ge A\left(\left[\frac{n}{2}\right] + 1\right). \tag{4.1}$$

Proof. Since in this case we have $\left[\frac{n}{2}\right] = n - \left(\left[\frac{n}{2}\right] + 1\right)$, we may just apply (1.7) in the case when $k = \left[\frac{n}{2}\right]$. We are done. $\square$

The method we have used may give the possibility of extending the results given in Theorem 2.3 up to the following inequality:

Theorem 4.2. *Let n and k, l be natural numbers, such that $n \ge 2$ and $1 \le k, l \le \left[\frac{n}{2}\right]$. Then the following inequality*

$$A(k) \ge A(n-l)$$

holds.

This result emphasizes that in the chain of expressions $A(1), \dots, A(n)$ any term in the left side is greater than or equal to any member in the right side. The left and right side are taken by considering $A(\left[\frac{n}{2}\right])$ as the *middle* element.

Even with this improvement, by using our method one cannot obtain any inequality between the elements in the right side, other than the one where $k = n-1$.

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