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A DISCRETE VERSION OF AN OPEN PROBLEM AND SEVERAL ANSWERS

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Abstract. In this article, a discrete version of an open problem in [Q. A. Ngô, D. D. Thang, T. T. Dat, and D. A. Tuan, *Notes on an integral inequality*, J. Inequal. Pure Appl. Math. **7** (2006), no. 4, Art. 120; Available online at http://jipam.vu.edu.au/article.php?sid=737] is posed and several answers are provided.

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1. Introduction

In [4], some integral inequalities were obtained and the following open problem was posed.

Open Problem 1. *Let f be a continuous function on $[0,1]$ satisfying the following condition*

$$\int_x^1 f(t)\,\mathrm{d}t \ge \int_x^1 t\,\mathrm{d}t \tag{1.1}$$

for $x \in [0,1]$. Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(t)\,\mathrm{d}t \ge \int_0^1 t^\beta f^\alpha(t)\,\mathrm{d}t \tag{1.2}$$

hold for α and β?

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In [1], some affirmative answers to Open Problem 1 and the reversed inequality of (1.2) were given.

In [3], an abstract version of Open Problem 1 was posed, respective answers to these two open problems were presented, and the results in [1] were extended.

Now we would like to further pose the following discrete version of the open problems in [1, 3] as follows.

Open Problem 2. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences satisfying $x_1 \ge x_2 \ge \dots \ge x_n$, $y_1 \ge y_2 \ge \dots \ge y_n$ and*

$$\sum_{i=1}^{m} x_i \le \sum_{i=1}^{m} y_i \tag{1.3}$$

for $1 \le m \le n$. Under what conditions does the inequality

$$\sum_{i=1}^{n} x_i^\alpha y_i^\beta \le \sum_{i=1}^{n} y_i^{\alpha+\beta} \tag{1.4}$$

hold for α and β?

In the next sections, we shall establish several answers to Open Problem 2.

2. LEMMAS

In order to establish several answers to Open Problem 2, the following lemmas are necessary.

Lemma 2.1. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \dots, y_n\}$ be two real sequences. Then*

$$\sum_{i=1}^{n} x_i y_i = x_{n+1} \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \sum_{j=1}^{i} y_j (x_i - x_{i+1}). \tag{2.1}$$

Proof. Identity (2.1) follows from standard straightforward arguments. □

Lemma 2.2 ([2, p. 17])**.** *Let a and b be positive numbers with $a + b = 1$. Then*

$$ax + by \ge x^a y^b \tag{2.2}$$

is valid for positive numbers x and y.

Lemma 2.3. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences satisfying $x_1 \ge x_2 \ge \dots \ge x_n$, $y_1 \ge y_2 \ge \dots \ge y_n$ and inequality* (1.3)*. Then*

$$\sum_{i=1}^{m} x_i^\alpha \le \sum_{i=1}^{m} y_i^\alpha \tag{2.3}$$

holds for $\alpha \ge 1$ and $1 \le m \le n$.

Proof. Let x_{n+1} be a positive number such that $x_{n+1} \leq x_n$. From Lemma 2.1 and using inequality (1.3), it is easy to see that, for $\alpha = 2$ and $1 \leq m \leq n$,

$$\begin{aligned}\sum_{i=1}^{m} x_i y_i &= x_{m+1}\sum_{i=1}^{m} y_i + \sum_{i=1}^{m}\sum_{j=1}^{i} y_j(x_i - x_{i+1}) \\ &\geq x_{m+1}\sum_{i=1}^{m} x_i + \sum_{i=1}^{m}\sum_{j=1}^{i} x_j(x_i - x_{i+1}) \\ &= \sum_{i=1}^{m} x_i^2\end{aligned}$$

which implies that

$$\sum_{i=1}^{m} y_i^2 \geq 2\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i^2 \geq \sum_{i=1}^{m} x_i^2. \tag{2.4}$$

Suppose that inequality (2.3) holds for some integer $\alpha > 2$. Since $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ are two positive sequences, then

$$(y_i^\alpha - x_i^\alpha)(y_i - x_i) \geq 0$$

which leads to

$$\sum_{i=1}^{m} y_i^{\alpha+1} \geq \sum_{i=1}^{m} y_i^\alpha x_i + \sum_{i=1}^{m} y_i x_i^\alpha - \sum_{i=1}^{m} x_i^{\alpha+1} \tag{2.5}$$

for $1 \leq m \leq n$. Further, by virtue of Lemma 2.1, it follows that

$$\begin{aligned}\sum_{i=1}^{m} y_i^\alpha x_i &= x_{m+1}\sum_{i=1}^{m} y_i^\alpha + \sum_{i=1}^{m}\sum_{j=1}^{i} y_j^\alpha(x_i - x_{i+1}) \\ &\geq x_{m+1}\sum_{i=1}^{m} x_i^\alpha + \sum_{i=1}^{m}\sum_{j=1}^{i} x_j^\alpha(x_i - x_{i+1}) \\ &= \sum_{i=1}^{m} x_i^{\alpha+1}.\end{aligned} \tag{2.6}$$

A similar argument also yields

$$\sum_{i=1}^{m} y_i x_i^\alpha \geq \sum_{i=1}^{m} x_i^{\alpha+1}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) gives inequality (2.3) for $\alpha + 1$.

By induction, this means that inequality (2.3) holds for all $\alpha \in \mathbb{N}$.

Let $[\alpha]$ denote the integral part of a real number $\alpha \geq 1$. By inequality (2.2) in Lemma 2.2, we have

$$\frac{[\alpha]}{\alpha} y_i^\alpha + \frac{\alpha - [\alpha]}{\alpha} x_i^\alpha \geq y_i^{[\alpha]} x_i^{\alpha - [\alpha]}. \tag{2.8}$$

Summing on both sides of (2.8) and utilizing Lemma 2.1, the conclusion obtained above for $\alpha \in \mathbb{N}$ yields

$$\begin{aligned}\frac{[\alpha]}{\alpha}\sum_{i=1}^{m} y_i^{\alpha} &\ge \sum_{i=1}^{m} y_i^{[\alpha]} x_i^{\alpha-[\alpha]} - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} \\ &= x_{m+1}^{\alpha-[\alpha]}\sum_{i=1}^{m} y_i^{[\alpha]} + \sum_{i=1}^{m}\sum_{j=1}^{i} y_j^{[\alpha]}\left(x_i^{\alpha-[\alpha]} - x_{i+1}^{\alpha-[\alpha]}\right) - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} \\ &\ge \sum_{i=1}^{m} x_i^{\alpha} - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} \\ &= \frac{[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha}.\end{aligned}$$

Since $\frac{[\alpha]}{\alpha} \ne 0$, the required result is proved. □

3. Several Answers to Open Problem 2

Now we establish several answers to Open Problem 2.

Theorem 3.1. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences such that $x_1 \ge x_2 \ge \cdots \ge x_n$, $y_1 \ge y_2 \ge \cdots \ge y_n$ and inequality* (1.3) *is satisfied. Then*

$$\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \le \sum_{i=1}^{n} y_i^{\alpha+\beta} \tag{3.1}$$

holds for $\alpha \ge 1$ and $\beta > 0$.

Proof. By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} &\le \left[\sum_{i=1}^{n}\left(x_i^{\alpha}\right)^{\frac{\alpha+\beta}{\alpha}}\right]^{\frac{\alpha}{\alpha+\beta}} \left[\sum_{i=1}^{n}\left(y_i^{\beta}\right)^{\frac{\alpha+\beta}{\beta}}\right]^{\frac{\beta}{\alpha+\beta}} \\ &\le \left(\frac{\sum_{i=1}^{n} x_i^{\alpha+\beta}}{\sum_{i=1}^{n} y_i^{\alpha+\beta}}\right)^{\frac{\alpha}{\alpha+\beta}} \sum_{i=1}^{n} y_i^{\alpha+\beta} \\ &\le \sum_{i=1}^{n} y_i^{\alpha+\beta}.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $\{x_{1,l}, x_{2,l}, \dots, x_{n,l}\}$ and $\{y_{1,l}, y_{2,l}, \dots, y_{n,l}\}$ for $n \in \mathbb{N}$, $k > 0$ and $1 \le l \le k$ be positive sequences such that $x_{1,l} \ge x_{2,l} \ge \cdots \ge x_{n,l}$, $y_{1,l} \ge y_{2,l} \ge \cdots \ge y_{n,l}$ and*

$$\sum_{i=1}^{m} x_{i,l} \le \sum_{i=1}^{m} y_{i,l}, \quad 1 \le m \le n, \quad 1 \le l \le k. \tag{3.2}$$

Then

$$\sum_{i=1}^{n}\prod_{l=1}^{k} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} \le \sum_{i=1}^{n}\prod_{l=1}^{k} y_{i,l}^{\alpha_l+\beta_l} \tag{3.3}$$

for $\alpha_l \ge 1$ and $\beta_l > 0$, $1 \le l \le k$.

Proof. As in the proof of Lemma 2.3, let $x_{n+1,l}$ be positive numbers such that $x_{n+1,l} \le x_{n,l}$ for $1 \le l \le k$. By Lemma 2.1 and Theorem 3.1, it is shown that

$$\begin{aligned}\sum_{i=1}^{n}\prod_{l=1}^{k} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} &= \prod_{l=1}^{k-1} x_{n+1,l}^{\alpha_l} y_{n+1,l}^{\beta_l} \sum_{i=1}^{n} x_{i,k}^{\alpha_k} y_{i,k}^{\beta_k} \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{i} x_{j,k}^{\alpha_k} y_{j,k}^{\beta_k} \left(\prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} - \prod_{l=1}^{k-1} x_{i+1,l}^{\alpha_l} y_{i+1,l}^{\beta_l}\right) \\ &\le \prod_{l=1}^{k-1} x_{n+1,l}^{\alpha_l} y_{n+1,l}^{\beta_l} \sum_{i=1}^{n} y_{i,k}^{\alpha_k+\beta_k} \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{i} y_{j,k}^{\alpha_k+\beta_k} \left(\prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} - \prod_{l=1}^{k-1} x_{i+1,l}^{\alpha_l} y_{i+1,l}^{\beta_l}\right) \\ &= \sum_{i=1}^{n} y_{j,k}^{\alpha_k+\beta_k} \prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} \le \cdots \le \sum_{i=1}^{n}\prod_{l=1}^{k} y_{i,l}^{\alpha_l+\beta_l}.\end{aligned}$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences with the properties that $x_1 \ge x_2 \ge \cdots \ge x_n$, $y_1 \ge y_2 \ge \cdots \ge y_n$ and inequality* (1.3) *is satisfied. Then*

$$\sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\beta_1} \le \sum_{i=1}^{n} y_i^{\alpha} x_i^{\beta} \tag{3.4}$$

if $\alpha \ge \alpha_1 \ge 1$, $\beta > 0$ and $\beta + \alpha = \beta_1 + \alpha_1$.

Proof. Let x_{n+1} be a positive number such that $x_{n+1} \le x_n$. By Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}\sum_{i=1}^{n} y_i^{\alpha} x_i^{\beta} &= x_{n+1}^{\beta} \sum_{i=1}^{n} y_i^{\alpha} + \sum_{i=1}^{n}\sum_{j=1}^{i} y_j^{\alpha}\left(x_i^{\beta} - x_{i+1}^{\beta}\right) \\ &\ge\ x_{n+1}^{\beta} \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\alpha-\alpha_1} + \sum_{i=1}^{n}\sum_{j=1}^{i} y_j^{\alpha_1} x_j^{\alpha-\alpha_1}\left(x_i^{\beta} - x_{i+1}^{\beta}\right) \\ &= \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\alpha-\alpha_1+\beta} = \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\beta_1}\end{aligned}$$

which completes the proof of Theorem 3.3. □

REFERENCES

[1] L. BOUGOFFA, Note on an open problem, *J. Inequal. Pure Appl. Math.*, **8**(2) (2007), Art. 58. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=871`].

[2] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1952.

[3] Y. MIAO AND F. QI, *Another answer to an open problem*, submitted.

[4] Q.A. NGÔ, D.D. THANG, T.T. DAT AND D.A. TUAN, Notes on an integral inequality, *J. Inequal. Pure Appl. Math.*, **7**(4) (2006), Art. 120. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=737`].