A DISCRETE VERSION OF AN OPEN PROBLEM AND SEVERAL ANSWERS

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Abstract: In this article, a discrete version of an open problem in [Q. A. Ngô, D. D. Thang, T. T. Dat, and D. A. Tuan, *Notes on an integral inequality*, J. Inequal. Pure Appl. Math. **7** (2006), no. 4, Art. 120; Available online at `http://jipam.vu.edu.au/article.php?sid=737`] is posed and several answers are provided.

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1. Introduction

In [4], some integral inequalities were obtained and the following open problem was posed.

Open Problem 1. *Let f be a continuous function on $[0,1]$ satisfying the following condition*

$$\int_x^1 f(t)\,\mathrm{d}t \geq \int_x^1 t\,\mathrm{d}t \tag{1.1}$$

for $x \in [0,1]$. Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(t)\,\mathrm{d}t \geq \int_0^1 t^{\beta} f^{\alpha}(t)\,\mathrm{d}t \tag{1.2}$$

hold for α and β?

In [1], some affirmative answers to Open Problem 1 and the reversed inequality of (1.2) were given.

In [3], an abstract version of Open Problem 1 was posed, respective answers to these two open problems were presented, and the results in [1] were extended.

Now we would like to further pose the following discrete version of the open problems in [1, 3] as follows.

Open Problem 2. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i \tag{1.3}$$

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for $1 \leq m \leq n$. *Under what conditions does the inequality*

$$\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \leq \sum_{i=1}^{n} y_i^{\alpha+\beta} \tag{1.4}$$

hold for α *and* β*?*

In the next sections, we shall establish several answers to Open Problem 2.

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2. Lemmas

In order to establish several answers to Open Problem 2, the following lemmas are necessary.

Lemma 2.1. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \dots, y_n\}$ be two real sequences. Then*

$$\sum_{i=1}^{n} x_i y_i = x_{n+1} \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \sum_{j=1}^{i} y_j (x_i - x_{i+1}). \tag{2.1}$$

Proof. Identity (2.1) follows from standard straightforward arguments. □

Lemma 2.2 ([2, p. 17]). *Let a and b be positive numbers with $a + b = 1$. Then*

$$ax + by \geq x^a y^b \tag{2.2}$$

is valid for positive numbers x and y.

Lemma 2.3. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$ and inequality (1.3). Then*

$$\sum_{i=1}^{m} x_i^\alpha \leq \sum_{i=1}^{m} y_i^\alpha \tag{2.3}$$

holds for $\alpha \geq 1$ and $1 \leq m \leq n$.

Proof. Let x_{n+1} be a positive number such that $x_{n+1} \leq x_n$. From Lemma 2.1 and

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using inequality (1.3), it is easy to see that, for $\alpha = 2$ and $1 \leq m \leq n$,

$$\begin{aligned}\sum_{i=1}^{m} x_i y_i &= x_{m+1} \sum_{i=1}^{m} y_i + \sum_{i=1}^{m} \sum_{j=1}^{i} y_j (x_i - x_{i+1}) \\ &\geq x_{m+1} \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \sum_{j=1}^{i} x_j (x_i - x_{i+1}) \\ &= \sum_{i=1}^{m} x_i^2\end{aligned}$$

which implies that

$$\sum_{i=1}^{m} y_i^2 \geq 2 \sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i^2 \geq \sum_{i=1}^{m} x_i^2. \tag{2.4}$$

Suppose that inequality (2.3) holds for some integer $\alpha > 2$. Since $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ are two positive sequences, then

$$(y_i^\alpha - x_i^\alpha)(y_i - x_i) \geq 0$$

which leads to

$$\sum_{i=1}^{m} y_i^{\alpha+1} \geq \sum_{i=1}^{m} y_i^\alpha x_i + \sum_{i=1}^{m} y_i x_i^\alpha - \sum_{i=1}^{m} x_i^{\alpha+1} \tag{2.5}$$

for $1 \leq m \leq n$. Further, by virtue of Lemma 2.1, it follows that

$$\sum_{i=1}^{m} y_i^\alpha x_i = x_{m+1} \sum_{i=1}^{m} y_i^\alpha + \sum_{i=1}^{m} \sum_{j=1}^{i} y_j^\alpha (x_i - x_{i+1}) \tag{2.6}$$

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$$\geq x_{m+1}\sum_{i=1}^{m} x_i^{\alpha} + \sum_{i=1}^{m}\sum_{j=1}^{i} x_j^{\alpha}(x_i - x_{i+1}) = \sum_{i=1}^{m} x_i^{\alpha+1}.$$

A similar argument also yields

$$\sum_{i=1}^{m} y_i x_i^{\alpha} \geq \sum_{i=1}^{m} x_i^{\alpha+1}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) gives inequality (2.3) for $\alpha+1$.

By induction, this means that inequality (2.3) holds for all $\alpha \in \mathbb{N}$.

Let $[\alpha]$ denote the integral part of a real number $\alpha \geq 1$. By inequality (2.2) in Lemma 2.2, we have

$$\frac{[\alpha]}{\alpha} y_i^{\alpha} + \frac{\alpha-[\alpha]}{\alpha} x_i^{\alpha} \geq y_i^{[\alpha]} x_i^{\alpha-[\alpha]}. \tag{2.8}$$

Summing on both sides of (2.8) and utilizing Lemma 2.1, the conclusion obtained above for $\alpha \in \mathbb{N}$ yields

$$\begin{aligned}
\frac{[\alpha]}{\alpha}\sum_{i=1}^{m} y_i^{\alpha} &\geq \sum_{i=1}^{m} y_i^{[\alpha]} x_i^{\alpha-[\alpha]} - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} \\
&= x_{m+1}^{\alpha-[\alpha]}\sum_{i=1}^{m} y_i^{[\alpha]} + \sum_{i=1}^{m}\sum_{j=1}^{i} y_j^{[\alpha]}\left(x_i^{\alpha-[\alpha]} - x_{i+1}^{\alpha-[\alpha]}\right) - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} \\
&\geq \sum_{i=1}^{m} x_i^{\alpha} - \frac{\alpha-[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha} = \frac{[\alpha]}{\alpha}\sum_{i=1}^{m} x_i^{\alpha}.
\end{aligned}$$

Since $\frac{[\alpha]}{\alpha} \neq 0$, the required result is proved. □

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3. Several Answers to Open Problem 2

Now we establish several answers to Open Problem 2.

Theorem 3.1. *For $n \in \mathbb{N}$, let $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ be two positive sequences such that $x_1 \ge x_2 \ge \dots \ge x_n$, $y_1 \ge y_2 \ge \dots \ge y_n$ and inequality (1.3) is satisfied. Then*

$$\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \le \sum_{i=1}^{n} y_i^{\alpha+\beta} \tag{3.1}$$

holds for $\alpha \ge 1$ and $\beta > 0$.

Proof. By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} &\le \left[\sum_{i=1}^{n} (x_i^{\alpha})^{\frac{\alpha+\beta}{\alpha}}\right]^{\frac{\alpha}{\alpha+\beta}} \left[\sum_{i=1}^{n} (y_i^{\beta})^{\frac{\alpha+\beta}{\beta}}\right]^{\frac{\beta}{\alpha+\beta}} \\ &\le \left(\frac{\sum_{i=1}^{n} x_i^{\alpha+\beta}}{\sum_{i=1}^{n} y_i^{\alpha+\beta}}\right)^{\frac{\alpha}{\alpha+\beta}} \sum_{i=1}^{n} y_i^{\alpha+\beta} \le \sum_{i=1}^{n} y_i^{\alpha+\beta}.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $\{x_{1,l}, x_{2,l}, \dots, x_{n,l}\}$ and $\{y_{1,l}, y_{2,l}, \dots, y_{n,l}\}$ for $n \in \mathbb{N}$, $k > 0$ and $1 \le l \le k$ be positive sequences such that $x_{1,l} \ge x_{2,l} \ge \dots \ge x_{n,l}$, $y_{1,l} \ge y_{2,l} \ge \dots \ge y_{n,l}$ and*

$$\sum_{i=1}^{m} x_{i,l} \le \sum_{i=1}^{m} y_{i,l}, \quad 1 \le m \le n, \quad 1 \le l \le k. \tag{3.2}$$

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Then

$$\sum_{i=1}^{n}\prod_{l=1}^{k} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} \le \sum_{i=1}^{n}\prod_{l=1}^{k} y_{i,l}^{\alpha_l+\beta_l} \tag{3.3}$$

for $\alpha_l \ge 1$ *and* $\beta_l > 0$, $1 \le l \le k$.

Proof. As in the proof of Lemma 2.3, let $x_{n+1,l}$ be positive numbers such that $x_{n+1,l} \le x_{n,l}$ for $1 \le l \le k$. By Lemma 2.1 and Theorem 3.1, it is shown that

$$\begin{aligned}
\sum_{i=1}^{n}\prod_{l=1}^{k} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} &= \prod_{l=1}^{k-1} x_{n+1,l}^{\alpha_l} y_{n+1,l}^{\beta_l} \sum_{i=1}^{n} x_{i,k}^{\alpha_k} y_{i,k}^{\beta_k} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{i} x_{j,k}^{\alpha_k} y_{j,k}^{\beta_k} \left(\prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} - \prod_{l=1}^{k-1} x_{i+1,l}^{\alpha_l} y_{i+1,l}^{\beta_l} \right) \\
&\le \prod_{l=1}^{k-1} x_{n+1,l}^{\alpha_l} y_{n+1,l}^{\beta_l} \sum_{i=1}^{n} y_{i,k}^{\alpha_k+\beta_k} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{i} y_{j,k}^{\alpha_k+\beta_k} \left(\prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} - \prod_{l=1}^{k-1} x_{i+1,l}^{\alpha_l} y_{i+1,l}^{\beta_l} \right) \\
&= \sum_{i=1}^{n} y_{j,k}^{\alpha_k+\beta_k} \prod_{l=1}^{k-1} x_{i,l}^{\alpha_l} y_{i,l}^{\beta_l} \le \cdots \le \sum_{i=1}^{n}\prod_{l=1}^{k} y_{i,l}^{\alpha_l+\beta_l}.
\end{aligned}$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3. *For* $n \in \mathbb{N}$, *let* $\{x_1, x_2, \dots, x_n\}$ *and* $\{y_1, y_2, \dots, y_n\}$ *be two positive sequences with the properties that* $x_1 \ge x_2 \ge \cdots \ge x_n$, $y_1 \ge y_2 \ge \cdots \ge y_n$ *and*

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inequality (1.3) *is satisfied. Then*

$$\sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\beta_1} \le \sum_{i=1}^{n} y_i^{\alpha} x_i^{\beta} \tag{3.4}$$

if $\alpha \ge \alpha_1 \ge 1$, $\beta > 0$ *and* $\beta + \alpha = \beta_1 + \alpha_1$.

Proof. Let x_{n+1} be a positive number such that $x_{n+1} \le x_n$. By Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}\sum_{i=1}^{n} y_i^{\alpha} x_i^{\beta} &= x_{n+1}^{\beta} \sum_{i=1}^{n} y_i^{\alpha} + \sum_{i=1}^{n} \sum_{j=1}^{i} y_j^{\alpha} (x_i^{\beta} - x_{i+1}^{\beta}) \\ &\ge x_{n+1}^{\beta} \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\alpha-\alpha_1} + \sum_{i=1}^{n} \sum_{j=1}^{i} y_j^{\alpha_1} x_j^{\alpha-\alpha_1} (x_i^{\beta} - x_{i+1}^{\beta}) \\ &= \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\alpha-\alpha_1+\beta} = \sum_{i=1}^{n} y_i^{\alpha_1} x_i^{\beta_1}\end{aligned}$$

which completes the proof of Theorem 3.3. □

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