journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 1, Article 30, 9 pp.

ABOUT A CLASS OF LINEAR POSITIVE OPERATORS OBTAINED BY CHOOSING THE NODES

OVIDIU T. POP AND MIRCEA D. FĂRCAŞ

NATIONAL COLLEGE "MIHAI EMINESCU"
5 MIHAI EMINESCU STREET
SATU MARE 440014, ROMANIA
ovidiutiberiu@yahoo.com

mirceafarcas2005@yahoo.com

Received 15 June, 2007; accepted 18 March, 2009
Communicated by S.S Dragomir

ABSTRACT. In this paper we consider the given linear positive operators $(L_m)_{m\geq 1}$ and with their help, we construct linear positive operators $(\mathcal{K}_m)_{m\geq 1}$. We study the convergence, the evaluation for the rate of convergence in terms of the first modulus of smoothness for the operators $(\mathcal{K}_m)_{m\geq 1}$.

Key words and phrases: Linear positive operators, convergence theorem, the first order modulus of smoothness, approximation theorem.

2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A35, 41A36.

1. INTRODUCTION

In this section, we recall some notions and operators which we will use in this article.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m\}$ (see [5] or [24]). For the following construction, see [15]. Define the natural number m_0 by

$$m_0 = \begin{cases} \max(1, -[\beta]), & \text{if} \quad \beta \in \mathbb{R} - \mathbb{Z}; \\ \max(1, 1-\beta), & \text{if} \quad \beta \in \mathbb{Z}, \end{cases} \tag{1.3}$$

where $[x]$, $\{x\}$ denote the integer and fractional parts respectively of a real number x.

201-07

For the real number β, we have that

$$m+\beta \geq \gamma_\beta \tag{1.4}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0+\beta = \begin{cases} \max\left(1+\beta, \{\beta\}\right), & \text{if} \quad \beta \in \mathbb{R}-\mathbb{Z}; \\ \max(1+\beta, 1), & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.5}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta; \\ 1+\frac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.6}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.7}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \dots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.3) – (1.6), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.8}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$. These operators are called Stancu operators, and were introduced and studied in 1969 by D.D. Stancu in the paper [23]. In [23], the domain of definition of Stancu's operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.9}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R}$, f is bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$, consider the operators $S_m : C_2\left([0,\infty)\right) \to C\left([0,\infty)\right)$ defined for any function $f \in C_2\left([0,\infty)\right)$ by

$$\left(S_m f\right)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.10}$$

for any $x \in [0,\infty)$, where

$$C_2\left([0,\infty)\right) = \left\{ f \in C\left([0,\infty)\right) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

The operators $\left(S_m\right)_{m\geq 1}$ are called Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [12].

They were intensively studied by J. Favard in 1944 in [8] and O. Szász in 1950 in [25].

For $m \in \mathbb{N}$, the operator $V_m : C_2([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C_2([0,\infty))$ by

$$(1.11) \qquad (V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right),$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [2].

W. Meyer-König and K. Zeller have introduced in [11] a sequence of linear and positive operators. After a slight adjustment, given by E.W. Cheney and A. Sharma in [6], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(1.12) \qquad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are called the Meyer-König and Zeller operators.

Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In [10], M. Ismail and C.P. May consider the operators $(R_m)_{m\geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(1.13) \qquad (R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right)$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$.

If $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$(1.14) \qquad \omega(f;\delta) = \sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}.$$

2. PRELIMINARIES

For the following construction and result see [16] and [18], where $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$. Let $I, J \subset [0,\infty)$ be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$. Let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, respectively J so that the sum

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$$

exists for any $f \in E(I)$, $x \in J$ and $m \in \mathbb{N}$. For any $x \in I$ consider the functions $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ and $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0,1,2\}$. In the following, we suppose that for any $x \in I$ we have $\psi_x \in E(I)$ and $e_i \in E(I)$, $i \in \{0,1,2\}$.

For $m \in \mathbb{N}$, let the given operator $L_m : E(I) \to F(J)$ defined by

$$(2.1) \qquad (L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$$

with the property that the convergence

$$\lim_{m\to\infty}(L_m f)(x)=f(x) \tag{2.2}$$

is uniform on any compact $K\subset I\cap J$, for any $f\in E(I)\cap C(I)$.

Remark 1. From (2.2), for the operators $(L_m)_{m\geq 1}$ we have that the following convergences

$$\lim_{m\to\infty}(L_m e_i)(x)=e_i(x), \tag{2.3}$$

$i\in\{0,1,2\}$ and

$$\lim_{m\to\infty}(L_m\psi_x^2)(x)=0 \tag{2.4}$$

are uniform on any compact $K\subset I\cap J$.

Remark 2. From Remark 1 it results that for any compact $K\subset I\cap J$ the sequences $(u_m(K))_{m\geq 1}$, $(v_m(K))_{m\geq 1}$, $(w_m(K))_{m\geq 1}$ depending on K exist, so that the convergences

$$\lim_{m\to\infty}u_m(K)=\lim_{m\to\infty}v_m(K)=\lim_{m\to\infty}w_m(K)=0 \tag{2.5}$$

are uniform on K and

$$|(L_m e_0)(x)-1|\leq u_m(K), \tag{2.6}$$

$$|(L_m e_1)(x)-x|\leq v_m(K), \tag{2.7}$$

$$(L_m\psi_x^2)(x)\leq w_m(K), \tag{2.8}$$

for any $x\in K$ and any $m\in\mathbb{N}$.

In the following, for $m\in\mathbb{N}$ and $k\in\{0,1,\dots,p_m\}\cap\mathbb{N}_0$ we consider the nodes $y_{m,k}\in I$ so that

$$\alpha_m=\sup_{k\in\{0,1,\dots,p_m\}\cap\mathbb{N}_0}|x_{m,k}-y_{m,k}|<\infty \tag{2.9}$$

for any $m\in\mathbb{N}$ and

$$\lim_{m\to\infty}\alpha_m=0. \tag{2.10}$$

For $m\in\mathbb{N}$ and $k\in\{0,1,\dots,p_m\}\cap\mathbb{N}_0$ we note that $\alpha_{m,k}=x_{m,k}-y_{m,k}$.

Definition 2.1. For $m\in\mathbb{N}$, define the operator $\mathcal{K}_m:E(I)\to F(J)$ by

$$(\mathcal{K}_m f)(x)=\sum_{k=0}^{p_m}\varphi_{m,k}(x)f(y_{m,k}), \tag{2.11}$$

for any $x\in I$ and any $f\in E(I)$.

Remark 3. Similar ideas to the construction above can be found in the recent papers [9] and [13].

3. MAIN RESULTS

In this section, we study the operators defined by (2.11).

Theorem 3.1. *For any $f \in E(I) \cap C(I)$ we have that the convergence*

$$\lim_{m\to\infty} (\mathcal{K}_m f)(x) = f(x) \tag{3.1}$$

is uniform on any compact $K \subset I \cap J$.

Proof. For $x \in K$ and $m \in \mathbb{N}$ we have that

$$\begin{aligned}
(\mathcal{K}_m \psi_x^2)(x) &= (\mathcal{K}_m e_2)(x) - 2x(\mathcal{K}_m e_1)(x) + x^2(\mathcal{K}_m e_0)(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k} + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \alpha_{m,k})^2 \\
&\qquad - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \alpha_{m,k}) + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k}^2 - 2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k}\alpha_{m,k} \\
&\qquad + \sum_{k=0}^{p_m} \varphi_{m,k}(x)\alpha_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k} \\
&\qquad + 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x)\alpha_{m,k} + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \\
&\le (L_m \psi_x^2)(x) + 2\alpha_m (L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x).
\end{aligned}$$

Taking Remark 1 and Remark 2 into account, it results that (3.1) holds. □

Theorem 3.2. *If $f \in E(I \cap J) \cap C(I \cap J)$, then for any $x \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$, we have that*

$$\begin{aligned}
|(\mathcal{K}_m f)(x) - f(x)| &\le |f(x)|\,|(L_m e_0(x)) - 1| + ((L_m e_0)(x) + 1)\omega(f; \delta_{m,x}) \\
&\le M u_m(K) + (2 + u_m(K))\omega(f; \delta_m),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
\delta_{m,x} &= \sqrt{(L_m e_0)(x)[(L_m \psi_x^2)(x) + 2\alpha_m (L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x)]}, \\
\delta_m &= \sqrt{(1 + u_m(K))[w_m(K) + 2\alpha_m(b + v_m(K) + (\alpha_m^2 + 2b\alpha_m)(1 + u_m(K))]}
\end{aligned}$$

and

$$M = \sup\{|f(x)| : x \in K\}.$$

Proof. We apply the Shisha-Mond Theorem (see [22] or [24]) for the operator $\mathcal{K}_m$ and taking the inequality from the proof of the Theorem 3.1 into account verified by $(\mathcal{K}_m \psi_x^2)(x)$ and Remark 2, the inequality (3.2) follows. □

Corollary 3.3. *If*

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1 \tag{3.3}$$

for any $x \in J$*, then for any* $f \in E(I \cap J) \cap C(I \cap J)$*, any* $x \in K = [a, b] \subset I \cap J$ *and any* $m \in \mathbb{N}$ *we have that*

$$|(\mathcal{K}_m f)(x) - f(x)| \le 2\omega(f; \delta_{m,x}) \le 2\omega(f; \delta'_m) \tag{3.4}$$

where $\delta'_m = \sqrt{w_m(K) + 2\alpha_m v_m(K) + \alpha_m^2 + 4b\alpha_m}$*.*

Proof. It results from Theorem 3.2, because $(L_m e_0)(x) = 1$, for any $m \in \mathbb{N}$ and $x \in J$, so $u_m(K) = 0$, for any $m \in \mathbb{N}$. □

Remark 4. From the conditions of Theorem 3.2 we have that

$$|(\mathcal{K}_m f)(x) - f(x)| \le M u_m(K) + (2 + u_m(K))\omega(f; \delta_m)$$

and because $\lim\limits_{m\to\infty} \delta_m = 0$, it results that the convergence $\lim\limits_{m\to\infty} (K_m f)(x) = f(x)$ is uniform on K.

In the following, by particularisation of the sequence $y_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ and applying Theorem 3.1 and Corollary 3.3, we can obtain a convergence and approximation theorem for the new operators. In Applications 1 – 2, let $p_m = m$, $\varphi_{m,k}(x) = p_{m,k}(x)$, where $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$ and $K = [0, 1]$.

Application 1. *If* $I = J = [0, 1]$*,* $E(I) = F(J) = C([0, 1])$*,* $x_{m,k} = \frac{k}{m}$*,* $m \in \mathbb{N}$*,* $k \in \{0, 1, \ldots, m\}$*, we obtain the Bernstein operators. We have that* $u_m([0, 1]) = 0$*,* $v_m([0, 1]) = 0$ *and* $w_m([0, 1]) = \frac{1}{4m}$*,* $m \in \mathbb{N}$*. We consider the nodes* $y_{m,k} = \frac{\sqrt{k(k+1)}}{m}$*,* $m \in \mathbb{N}$*,* $k \in \{0, 1, \ldots, m\}$*. Then it is verified immediately that* $\alpha_m = \frac{1}{m+\sqrt{m(m+1)}}$*,* $m \in \mathbb{N}$ *and* $\lim\limits_{m\to\infty} \alpha_m = 0$*. In this case, the operators* $(\mathcal{K}_m)_{m\ge1}$ *have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{\sqrt{k(k+1)}}{m}\right),$$

$f \in C([0, 1])$*,* $x \in [0, 1]$*,* $m \in \mathbb{N}$ *and* $\delta'_m < \sqrt{\frac{5}{4m} + \frac{2}{m+\sqrt{m(m+1)}}} < \frac{3}{2\sqrt{m}}$*,* $m \in \mathbb{N}$*.*

Application 2. *We study a particular case of the Stancu operators. Let* $\alpha = 10$ *and* $\beta = -\frac{1}{2}$*. We obtain* $I = [0, 22]$ *and for any* $f \in C([0, 22])$*,* $x \in [0, 1]$ *and* $m \in \mathbb{N}$

$$\left(P_m^{(10,-1/2)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{2k + 20}{2m - 1}\right).$$

We consider the nodes $y_{m,k} = \frac{(4k+40)m}{(2m-1)^2}$*. In this case, the operators* $(\mathcal{K}_m)_{m\ge1}$ *have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{m(4k + 40)}{(2m - 1)^2}\right),$$

where $f \in C([0, 22])$*,* $x \in [0, 1]$*,* $m \in \mathbb{N}$ *and* $\delta'_m < \frac{\sqrt{36m^3+2220m^2-399m+81}}{(2m-1)^2} < \frac{45}{\sqrt{2m-1}}$*,* $m \in \mathbb{N}$*.*

Application 3. *If* $I = J = [0, \infty)$*,* $E(I) = C_2([0, \infty))$*,* $F(J) = C([0, \infty))$*,* $K = [0, b]$*,* $p_m = \infty$*,* $x_{m,k} = \frac{k}{m}$*,* $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$*,* $m \in \mathbb{N}$*,* $k \in \mathbb{N}_0$*, we obtain the Mirakjan-Favard-Szász operators and we have that* $u_m(K) = 0$*,* $v_m(K) = 0$ *and* $w_m(K) = \frac{b}{m}$*,* $m \in \mathbb{N}$*. We consider the nodes* $y_{m,k} = \frac{2k(k+1)}{m(2k+1)}$*,* $m \in \mathbb{N}$*,* $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{2m}$*,* $m \in \mathbb{N}$*. In this case, the operators* $(\mathcal{K}_m)_{m\ge1}$ *have the form*

$$(\mathcal{K}_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{2k(k + 1)}{m(2k + 1)}\right),$$

where $f \in C_2([0,\infty))$, $x \in [0,\infty)$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{3b}{m} + \frac{1}{4m^2}}$, $m \in \mathbb{N}$.

Application 4. *Let* $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $K = [0,b]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = (1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$. *In this case, we obtain the Baskakov operators and we have that* $u_m(K) = 0$, $v_m(K) = 0$ *and* $w_m(K) = \frac{b(1+b)}{2m}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\sqrt{4k^2+4k+2}}{2m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{m\sqrt{2}}$. *The operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = (1+x)^{-m}\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k f\left(\frac{\sqrt{4k^2+4k+2}}{2m}\right),$$

where $f \in C_2([0,\infty))$, $x \in [0,\infty)$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{b(b+1+2\sqrt{2})}{m} + \frac{1}{2m^2}}$, $m \in \mathbb{N}$.

Application 5. *If* $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $K = [0,b]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$,

$$\varphi_{m,k}(x) = \frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{\frac{-(k+m)x}{1+x}}, \qquad m \in \mathbb{N}, k \in \mathbb{N}_0,$$

we obtain the Ismail-May operators and we have that $u_m(K) = 0$, $v_m(K) = 0$ *and* $w_m(K) = \frac{b(1+b)^2}{m}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\sqrt[3]{k^2(k+1)}}{m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{3m}$. *In this case, the operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = e^{\frac{-mx}{1+x}}\sum_{k=0}^{\infty}\frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{\sqrt[3]{k^2(k+1)}}{m}\right),$$

where $f \in C([0,\infty))$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{b(7+6b+3b^2)}{3m} + \frac{1}{9m^2}}$, $m \in \mathbb{N}$.

Application 6. *We consider* $I = J = [0,\infty)$, $E(I) = F(J) = C_B([0,\infty))$, $K = [0,b]$, $p_m = m$, $x_{m,k} = \frac{k}{m+1-k}$, $\varphi_{m,k}(x) = \frac{1}{(1+x)^m}\binom{m}{k}x^k$, $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$. *In this case we obtain the Bleimann-Butzer-Hahn operators and we have that* $u_m(K) = 0$, $v_m(K) = b\left(\frac{b}{1+b}\right)^m$ *and* $w_m(K) = \frac{4b(1+b)^2}{m+2}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\beta_m k}{m+1-k}$, $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$, *where* $(\beta_m)_{m\geq 1}$ *is a sequence of positive real numbers such that* $\lim\limits_{m\to\infty} m(1-\beta_m) = 0$ *and we have* $\alpha_m = m|1-\beta_m|$, $m \in \mathbb{N}$. *The operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k f\left(\frac{\beta_m k}{m+1-k}\right),$$

where $x \in [0,\infty)$, $m \in \mathbb{N}$, $f \in C_B([0,\infty))$.

Application 7. *If* $I = J = [0,1]$, $E(I) = B([0,1])$, $F(J) = C([0,1])$, $K = [0,1]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m+k}$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *we obtain the Meyer-König and Zeller operators and we have that* $u_m([0,1]) = 0$, $v_m([0,1]) = 0$ *and* $w_m([0,1]) = \frac{1}{4(m+1)}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{k+\beta_m}{m+k+\beta_m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *where* $(\beta_m)_{m\geq 1}$ *is a sequence of positive real numbers so that* $\lim\limits_{m\to\infty}\frac{\beta_m}{m+\beta_m} = 0$. *Then it is verified immediately that* $\alpha_m = \frac{\beta_m}{m+\beta_m}$, $m \in \mathbb{N}$ *and the operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{\infty}\binom{m+k}{k}(1-x)^{m+1}x^k f\left(\frac{k+\beta_m}{m+k+\beta_m}\right),$$

where $f \in B([0,1])$, $x \in [0,1]$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{1}{4(m+1)} + \frac{\beta_m(4m+5\beta_m)}{(m+\beta_m)^2}}$, $m \in \mathbb{N}$.

REFERENCES

[1] O. AGRATINI, Aproximare prin operatori liniari, *Presa Universitară Clujeană*, Cluj-Napoca, 2000 (Romanian).

[2] V.A. BASKAKOV, An example of a sequence of linear positive operators in the space of continuous functions, *Dokl. Acad. Nauk*, USSR, **113** (1957), 249–251.

[3] M. BECKER AND R.J. NESSEL, A global approximation theorem for Meyer-König and Zeller operators, *Math. Zeitschr.*, **160** (1978), 195–206.

[4] G. BLEIMANN, P.L. BUTZER AND L.A. HAHN, Bernstein-type operator approximating continuous functions on the semi-axis, *Indag. Math.*, **42** (1980), 255–262.

[5] S.N. BERNSTEIN, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, *Commun. Soc. Math. Kharkow* (**2**), **13** (1912-1913), 1–2.

[6] E.W. CHENEY AND A. SHARMA, Bernstein power series, *Canadian J. Math.*, **16**(2) (1964), 241–252.

[7] Z. DITZIAN AND V. TOTIK, Moduli of Smoothness, *Springer Verlag*, Berlin, 1987.

[8] J. FAVARD, Sur les multiplicateurs d'interpolation, *J. Math. Pures Appl.*, **23**(9) (1944), 219–247.

[9] M. FĂRCAŞ, An extension for the Bernstein-Stancu operators, *An. Univ. Oradea Fasc. Mat.*, Tom **XV** (2008), 23–27.

[10] M. ISMAIL AND C.P. MAY, On a family of approximation operators, *J. Math. Anal. Appl.*, **63** (1978), 446–462.

[11] W. MEYER-KÖNIG AND K. Zeller, Bernsteinsche Potenzreihen, *Studia Math.*, **19** (1960), 89–94.

[12] G.M. MIRAKJAN, Approximation of continuous functions with the aid of polynomials, *Dokl. Acad. Nauk SSSR*, **31** (1941), 201–205 (Russian).

[13] C. MORTICI AND I. OANCEA, A nonsmooth extension for the Bernstein-Stancu operators and an application, *Studia Univ. "Babeş-Bolyai", Mathematica*, **LI**(2) (2006), 69–81.

[14] M.W. MÜLLER, Die Folge der Gammaoperatoren, Dissertation, Stuttgart, 1967.

[15] O.T. POP, New properties of the Bernstein-Stancu operators, *An. Univ. Oradea Fasc. Mat.*, Tom **XI** (2004), 51–60.

[16] O.T. POP, The generalization of Voronovskaja's theorem for a class of linear and positive operators, *Rev. Anal. Num. Théor. Approx.*, **34**(1) (2005), 79–91.

[17] O.T. POP, About a class of linear and positive operators, *Carpathian J. Math.*, **21**(1-2) (2005), 99–108.

[18] O.T. POP, About some linear and positive operators defined by infinite sum, *Dem. Math.*, **XXXIX**(2) (2006), 377–388.

[19] O.T. POP, On operators of the type Bleimann, Butzer and Hahn, *Anal. Univ. Timişoara*, **XLIII**(1) (2005), 115–124.

[20] O.T. POP, The generalization of Voronovskaja's theorem for exponential operators, *Creative Math & Inf.*, **16** (2007), 54–62.

[21] O.T. POP, About a general property for a class of linear positive operators and applications, *Rev. Anal. Num. Théor. Approx.*, **34**(2) (2005), 175–180.

[22] O. SHISHA AND B. MOND, The degree of convergence of linear positive operators, *Proc. Nat. Acad. Sci. USA*, **60** (1968), 1196–1200.

[23] D.D. STANCU, Asupra unei generalizări a polinoamelor lui Bernstein, *Studia Univ. Babeş-Bolyai, Ser. Math.-Phys.*, **14** (1969), 31–45 (Romanian).

[24] D.D. STANCU, GH. COMAN, O. AGRATINI AND R. TRÎMBIŢAŞ, Analiză numerică şi teoria aproximării, I, *Presa Universitară Clujeană*, Cluj-Napoca, 2001 (Romanian).

[25] O. SZÁSZ, Generalization of. S.N. Bernstein's polynomials to the infinite interval, *J. Research, National Bureau of Standards*, **45** (1950), 239–245.

[26] A.F. TIMAN, *Theory of Approximation of Functions of Real Variable*, New York: Macmillan Co., 1963, MR22#8257.