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CAUCHY'S MEANS OF LEVINSON TYPE

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ABSTRACT. In this paper we introduce Levinson means of Cauchy's type. We show that these means are monotonic.

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1. INTRODUCTION AND PRELIMINARIES

Let $x_1, x_2, \dots, x_n$ and $p_1, p_2, \dots, p_n$ be real numbers such that $x_i \in [0, \frac{1}{2}]$, $p_i > 0$ with $P_n = \sum_{i=1}^{n} p_i$. Let G_n and A_n be the weighted geometric mean and arithmetic mean respectively defined by

$$G_n = \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}} \quad \text{and} \quad A_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i = \overline{x}.$$

In particular, consider the means

$$G'_n = \left(\prod_{i=1}^{n} (1-x_i)^{p_i}\right)^{\frac{1}{P_n}} \quad \text{and} \quad A'_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i (1-x_i).$$

The well known Levinson inequality is the following ([1, 2] see also [6, p. 71]).

Theorem 1.1. *Let f be a real valued 3-convex function on $[0, 2a]$. Then for $0 < x_i < a$, $p_i > 0$ we have*

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$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right). \tag{1.1}$$

In [4], the second author proved the following similar result.

Theorem 1.2. *Let f be a real valued 3-convex function on $[0, 2a]$ and x_i $(1 \leq i \leq n)$ n points on $[0, a]$. Then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a + x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (a + x_i)\right). \tag{1.2}$$

Lemma 1.3. *Let f be a log-convex function. If, $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\left[\frac{f(x_2)}{f(x_1)}\right]^{\frac{1}{x_2 - x_1}} \leq \left[\frac{f(y_2)}{f(y_1)}\right]^{\frac{1}{y_2 - y_1}}. \tag{1.3}$$

Lemma 1.4. *Let $f \in C^3(I)$ for some interval $I \subseteq \mathbb{R}$, such that f''' is bounded and $m = \min f'''$ and $M = \max f'''$. Consider the functions ϕ_1, ϕ_2 defined as,*

$$\phi_1(t) = \frac{M}{6}t^3 - f(t), \tag{1.4}$$

$$\phi_2(t) = f(t) - \frac{m}{6}t^3, \tag{1.5}$$

then ϕ_1 and ϕ_2 are 3-convex functions.

Lemma 1.5. *Let us define the function*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)(s-2)}, & s \neq 0,1,2; \\ \frac{1}{2}\log x, & s=0; \\ -x\log x, & s=1; \\ \frac{1}{2}x^2\log x, & s=2. \end{cases} \tag{1.6}$$

Then $\varphi_s'''(x) = x^{s-3}$, that is $\varphi_s(x)$ is 3-convex for $x > 0$.

Let $I \subseteq \mathbb{R}$ be an interval and let $\mathcal{F}$ be some appropriately chosen vector space of real valued functions defined on I. Let Ψ be a functional on $\mathcal{F}$ and let $A : \mathcal{F} \to \mathcal{R}$ be a linear operator, where $\mathcal{R}$ is the vector of all real valued functions defined on I.

Suppose that for each $f \in \mathcal{F}$, there is a $\xi \in I$ such that

$$\Psi(f) = A(f)(\xi). \tag{1.7}$$

J. Pećarić, I. Perić and H. Srivastava in [5] proved the following important result for ϕ and A defined above.

Theorem 1.6. *For every $f, g \in \mathcal{F}$, there is a $\xi \in I$ such that*

$$A(g)(\xi)\Psi(f) = A(f)(\xi)\Psi(g). \tag{1.8}$$

2. MAIN RESULTS

Theorem 2.1. *Let $f \in C^3(I)$. Then for $x_i > 0$ and $p_i > 0$ there exist $\xi \in I$ such that the following equality holds true,*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right) = \frac{f'''(\xi)}{6}\left[\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^3 + \frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)^3 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right)^3\right]. \tag{2.1}$$

Proof. Suppose that f''' is bounded, that is, $\min f''' = m$, $\max f''' = M$. By applying the Levinsen inequality (1.1) to the functions ϕ_1 and ϕ_2 defined in Lemma 1.4, we get the following inequalities,

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right) \le \frac{M}{6}\left[\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^3 + \frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)^3 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right)^3\right] \tag{2.2}$$

and

$$\frac{m}{6}\left[\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^3 + \frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)^3 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right)^3\right] \le f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right). \tag{2.3}$$

By combining both inequalities and using the fact that for $m \le \rho \le M$ there exist $\xi \in I$ such that $f'''(\xi) = \rho$, we get (2.1). Moreover, if f''' is (for example) bounded from above we have that (2.2) is valid and again (2.1) is valid.

Of course (2.1) is obvious if f''' is not bounded from above and below. □

Theorem 2.2. *Let $f, g \in C^3(I)$. Then for $x_i > 0$ and $p_i > 0,\ i = 1, \dots, n$ there exist $\xi \in I$ such that the following equality holds true,*

$$\frac{f'''(\xi)}{g'''(\xi)} = \frac{f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i f(2a-x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i (2a-x_i)\right)}{g\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i g(2a-x_i) - g\left(\frac{1}{P_n}\sum_{i=1}^n p_i (2a-x_i)\right)}. \tag{2.4}$$

Proof. Consider the linear functionals Ψ and A as in (1.7) for $\mathcal{F} = C^3(I)$ and $\mathcal{R}$ the vector space of real valued functions such that $\Psi(k) = A(k)(\xi)$ for some function k. Let A be defined as:

$$A(f)(\xi) = \frac{f'''(\xi)}{6}\left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^n p_i x_i^3 + \frac{1}{P_n}\sum_{i=1}^n p_i (2a-x_i)^3 - \left(\frac{1}{P_n}\sum_{i=1}^n p_i (2a-x_i)\right)^3\right]. \tag{2.5}$$

Also, consider the linear combination $k = c_1 f - c_2 g$, where $f, g \in C^3(I)$ and c_1, c_2 are defined by

$$\begin{aligned} c_1 &= \Psi(g) \\ &= g\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) \\ &\qquad + \frac{1}{P_n}\sum_{i=1}^n p_i g(2a-x_i) - g\left(\frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i)\right), \end{aligned} \tag{2.6}$$

$$\begin{aligned} c_2 &= \Psi(f) \\ &= f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \\ &\qquad + \frac{1}{P_n}\sum_{i=1}^n p_i f(2a-x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i)\right). \end{aligned} \tag{2.7}$$

Obviously, we have $\Psi(k) = 0$. This implies that (as in Theorem 1.6):

$$\Psi(g)A(f)(\xi) = \Psi(f)A(g)(\xi). \tag{2.8}$$

Now since $\Psi(g) \neq 0$ and $A(g)(\xi) \neq 0$ we have from the last equation

$$\frac{\Psi(f)}{\Psi(g)} = \frac{A(f)(\xi)}{A(g)(\xi)}. \tag{2.9}$$

After putting in the values we get (2.4). □

Corollary 2.3. *Let $\frac{f'''}{g'''}$ be invertible then (2.4) suggests new means. That is,*

$$\xi = \left(\frac{f'''}{g'''}\right)^{-1}\left[\frac{f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i f(2a-x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i)\right)}{g\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i g(2a-x_i) - g\left(\frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i)\right)}\right] \tag{2.10}$$

is a new mean.

Definition 2.1. Define the function

$$\xi_s = \frac{1}{P_n}\sum_{i=1}^n p_i\Big(\varphi_s(2a-x_i) - \varphi_s(x_i)\Big) - \varphi_s(2a-\overline{x}) + \varphi_s(\overline{x}), \tag{2.11}$$

when $s \neq 0,1,2$. $s = 0,1,2$ are limiting cases defined by

$$\xi_0 = \frac{1}{2}\ln\left(\frac{G_n^a A_n}{G_n A_n^a}\right),$$

where

$$G_n^a = \left[\prod_{i=1}^n (2a-x_i)^{p_i}\right]^{\frac{1}{P_n}} \quad \text{and} \quad A_n^a = \frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i),$$

$$\xi_1 = \frac{1}{P_n}\sum_{i=1}^n p_i\Big(x_i\ln x_i - (2a-x_i)\ln(2a-x_i)\Big) + (2a-\overline{x})\ln(2a-\overline{x}) - \overline{x}\ln\overline{x},$$

$$\xi_2 = \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i\Big((2a-x_i)^2\ln(2a-x_i) - x_i^2\ln x_i\Big) - (2a-\overline{x})^2\ln(2a-\overline{x}) + \overline{x}^2\ln\overline{x}\right].$$

Then we define the new means $M_{s,t}$ as:

Definition 2.2. Let us denote:

$$M_{s,t} = \left(\frac{\xi_s}{\xi_t}\right)^{\frac{1}{s-t}} \tag{2.12}$$

for $s \neq t \neq 0,1,2$. We define these limiting cases as

$$M_{s,s} = \exp\left[\frac{\eta}{\frac{1}{P_n}\sum_{i=1}^n p_i((2a-x_i)^s - x_i^s) - (2a-\bar{x})^s + \bar{x}^s} - \frac{3s^2-6s+2}{s(s-1)(s-2)}\right],$$

where

$$\eta = \frac{1}{P_n}\sum_{i=1}^n p_i((2a-x_i)^s\log(2a-x_i) - x_i^s\log x_i) - (2a-\bar{x})^s\log(2a-\bar{x}) + \bar{x}^s\log\bar{x}$$

for $s \neq 0,1,2$

$$M_{0,0} = \exp\left[\frac{2\left(\frac{1}{P_n}\sum_{i=1}^n p_i[(\log(2a-x_i))^2 - (\log x_i)^2]\right)}{4\xi_0} - \frac{(\log(2a-\bar{x}))^2 - (\log\bar{x})^2 6\xi_0}{4\xi_0}\right],$$

$$M_{1,1} = \exp\left[\frac{\frac{1}{P_n}\sum_{i=1}^n p_i\left[(2a-x_i)(\log(2a-x_i))^2 - x_i(\log x_i)^2\right]}{2\xi_1} - \frac{(2a-\bar{x})(\log(2a-\bar{x}))^2 - \bar{x}(\log\bar{x})^2}{2\xi_1}\right],$$

$$M_{2,2} = \exp\left[\frac{\frac{1}{P_n}\sum_{i=1}^n p_i\left[(2a-x_i)^2(\log(2a-x_i))^2 - x_i^2(\log x_i)^2\right]}{3\xi_2} - \frac{(2a-\bar{x})^2(\log(2a-\bar{x}))^2 - \bar{x}^2(\log\bar{x})^2}{3\xi_2} - 1\right].$$

In our next result we prove that this new mean is monotonic.

Theorem 2.4. *Let $r \le s, t \le u, r \ne t, s \ne u$, then the following inequality is valid:*

$$M_{r,t} \le M_{s,u}. \tag{2.13}$$

Proof. Since ξ_s is log convex as proved in [3, Theorem 2.2], then applying Lemma 1.4 for $r \le s, t \le u, r \ne t, s \ne u$ we get our required result. □

3. RELATED RESULTS

Theorem 3.1. *Let $f \in C^3(I)$. For $x_i > 0$ and $p_i > 0,\ i = 1, \dots, n$ there exist $\xi \in I$ such that the following equality holds true,*

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) &- \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i f(a+x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i(a+x_i)\right) \\ &= \frac{f'''(\xi)}{6}\left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^n p_i x_i^3 + \frac{1}{P_n}\sum_{i=1}^n p_i(a+x_i)^3 - \left(\frac{1}{P_n}\sum_{i=1}^n p_i(a+x_i)\right)^3\right]. \end{aligned} \tag{3.1}$$

Proof. Similar to proof of Theorem 2.1. □

Theorem 3.2. *Let $f, g \in C^3(I)$. Then for $x_i > 0$ and $p_i > 0,\ i = 1, \dots, n$ there exist $\xi \in I$ such that the following equality holds true,*

$$\frac{f'''(\xi)}{g'''(\xi)} = \frac{f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i f(a+x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i(a+x_i)\right)}{g\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) + \frac{1}{P_n}\sum_{i=1}^n p_i g(a+x_i) - g\left(\frac{1}{P_n}\sum_{i=1}^n p_i(a+x_i)\right)} \tag{3.2}$$

Proof. Similar to proof of Theorem 2.2. □

Corollary 3.3. *Let $\frac{f'''}{g'''}$ be invertible. Then (3.2) suggests new means. That is,*

$$\xi=\left(\frac{f'''}{g'''}\right)^{-1}\left[\frac{f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)+\frac{1}{P_n}\sum_{i=1}^{n}p_if(a+x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i(a+x_i)\right)}{g\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_ig(x_i)+\frac{1}{P_n}\sum_{i=1}^{n}p_ig(a+x_i)-g\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i(a+x_i)\right)}\right] \tag{3.3}$$

is a new mean.

Definition 3.1. Define the function

$$\bar{\xi}_s=\frac{1}{P_n}\sum_{i=1}^{n}p_i\Big(\varphi_s(a+x_i)-\varphi_s(x_i)\Big)-\varphi_s(a+\overline{x})+\varphi_s(\overline{x}), \tag{3.4}$$

when $s\neq 0,1,2$. $s=0,1,2$ are limiting cases defined by

$$\bar{\xi_0}=\frac{1}{2}\ln\left(\frac{\bar{G^a}_nA_n}{G_n\bar{A^a}_n}\right),$$

where

$$\bar{G^a}_n=\left(\prod_{i=1}^{n}(a+x_i)^{p_i}\right)^{\frac{1}{P_n}},\quad\text{and}\quad\bar{A^a}_n=\frac{1}{P_n}\sum_{i=1}^{n}p_i(a+x_i),$$

$$\bar{\xi_1}=\frac{1}{P_n}\sum_{i=1}^{n}p_i\Big(x_i\ln x_i-(a+x_i)\ln(2a-x_i)\Big)+(a+\overline{x})\ln(a+\overline{x})-\overline{x}\ln\overline{x},$$

$$\bar{\xi_2}=\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i\left((a+x_i)^2\ln(a+x_i)-x_i^2\ln x_i\right)-(a+\overline{x})^2\ln(a+\overline{x})+\overline{x}^2\ln\overline{x}\right].$$

We now define new means $\overline{M_{s,t}}$ as:

Definition 3.2. Let us denote:

$$\overline{M_{s,t}}=\left(\frac{\bar{\xi_s}}{\bar{\xi_t}}\right)^{\frac{1}{s-t}} \tag{3.5}$$

for $s\neq t\neq 0,1,2$. We define these limiting cases as

$$\overline{M_{s,s}}=\exp\left(\frac{\bar{\eta}}{\frac{1}{P_n}\sum_{i=1}^{n}p_i((a+x_i)^s-x_i^s)-(a+\bar{x})^s+\bar{x}^s}-\frac{3s^2-6s+2}{s(s-1)(s-2)}\right),$$

where

$$\bar{\eta}=\frac{1}{P_n}\sum_{i=1}^{n}p_i((a+x_i)^s\log(a+x_i)-x_i^s\log x_i)-(a+\bar{x})^s\log(a+\bar{x})+\bar{x}^s\log\bar{x}$$

for $s\neq 0,1,2$

$$\overline{M_{0,0}}=\exp\left[\frac{2\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i[(\log(a+x_i))^2-(\log x_i)^2]\right)}{4\bar{\xi_0}}-\frac{(\log(a+\bar{x}))^2-(\log\bar{x})^2 6\bar{\xi_0}}{4\bar{\xi_0}}\right],$$

$$\overline{M_{1,1}} = \exp\left[\frac{\frac{1}{P_n}\sum_{i=1}^{n} p_i((a+x_i)(\log(a+x_i))^2 - x_i(\log x_i)^2)}{2\bar{\xi_1}} - \frac{(a+\bar{x})(\log(a+\bar{x}))^2 - \bar{x}(\log\bar{x})^2}{2\bar{\xi_1}}\right],$$

$$\overline{M_{2,2}} = \exp\left[\frac{\frac{1}{P_n}\sum_{i=1}^{n} p_i((a+x_i)^2(\log(a+x_i))^2 - x_i^2(\log x_i)^2)}{3\bar{\xi_2}} - \frac{(a+\bar{x})^2(\log(a+\bar{x}))^2 - \bar{x}^2(\log\bar{x})^2}{3\bar{\xi_2}} - 1\right].$$

In our next result we prove that this new mean is monotonic.

Theorem 3.4. *Let $r \le s$, $t \le u$, $r \ne t$, $s \ne u$, then the following inequality is valid:*

$$\overline{M_{r,t}} \le \overline{M_{s,u}}. \tag{3.6}$$

Proof. Since $\bar{\xi_s}$ is log convex as proved in [3, Theorem 2.5] ($\bar{\xi_s} = \rho_s$), then applying Lemma 1.3 for $r \le s, t \le u, r \ne t, s \ne u$ we get our required result. □

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