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PARTIAL SUMS OF SOME MEROMORPHIC FUNCTIONS

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ABSTRACT. In the present paper we give some results concerning partial sums of certain meromorphic functions.We also consider the partial sums of certain integral operator.

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1. INTRODUCTION

Let Σ be the class consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are regular in the punctured disc $E = \{z : 0 < |z| < 1\}$ with a simple pole at the origin and residue 1 there.

Let $f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k$ be the nth partial sum of the series expansion for $f(z) \in \Sigma$. Let $\Sigma^*(A,B)$, $\Sigma_K(A,B)$, $\Sigma_c(A,B)$, $-1 \leq A < B \leq 1$ be the subclasses of functions in Σ satisfying

$$-\left\{\frac{zf'(z)}{f(z)}\right\} \prec \frac{1+Az}{1+Bz}, \qquad z \in \mathcal{U} = E \cup \{0\} \tag{1.2}$$

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$$-\left\{\frac{zf''(z)}{f'(z)}+1\right\} \prec \frac{1+Az}{1+Bz}, \qquad z\in\mathcal{U}. \tag{1.3}$$

$$-z^2f'(z) \prec \frac{1+Az}{1+Bz}, \qquad z\in\mathcal{U} \tag{1.4}$$

respectively [5]. The classes $\Sigma^*(2\alpha-1,1)$ and $\Sigma_K(2\alpha-1,1)$ are respectively the well known subclasses of Σ consisting of functions meromorphic starlike of order α and meromorphic convex of order α and meromorphically close to convex of order α denoted by $\Sigma^*(\alpha), \Sigma_K(\alpha)$ and $\Sigma_c(\alpha)$ respectively.

If $f(z) = \frac{1}{z}+\sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \frac{1}{z}+\sum_{k=1}^{\infty} b_k z^k$, then their Hadamard product (or convolution), denoted by $f(z) * g(z)$ is the function defined by the power series

$$f(z)*g(z) = \frac{1}{z}+\sum_{k=1}^{\infty} a_k b_k z^k.$$

In the present paper, we give sufficient conditions for $f(z)$ to be in $\Sigma^*(A,B), \Sigma_K(A,B)$ and further investigate the ratio of a function of the form (1.1) to its sequence of partial sums when the coefficients are sufficiently small to satisfy conditions

$$\sum_{k=1}^{\infty} k\{k(1+B)+(1+A)\}|a_k| \le B-A,$$

$$\sum_{k=1}^{\infty} \{k(1+B)+(1+A)\}|a_k| \le B-A.$$

More precisely, we will determine sharp lower bounds for $\Re\left\{\frac{f(z)}{f_n(z)}\right\}$, $\Re\left\{\frac{f_n(z)}{f(z)}\right\}$, $\Re\left\{\frac{f'(z)}{f_n'(z)}\right\}$ and $\Re\left\{\frac{f_n'(z)}{f'(z)}\right\}$. Further, we give a property for the partial sums of certain integral operators in connection with functions belonging to the class $\Sigma_c(A,B)$.

2. SOME PRELIMINARY RESULTS

Theorem 2.1. *Let* $f(z) = \frac{1}{z}+\sum_{k=1}^{\infty} a_k z^k, \quad z\in E$. *If*

$$\sum_{k=1}^{\infty} k\{k(1+B)+(1+A)\}|a_k| \le B-A, \quad \textit{then} \quad f(z)\in\Sigma_K(A,B). \tag{2.1}$$

Proof. It suffices to show that

$$\left|\frac{\left(1+\frac{zf''(z)}{f(z)}\right)+1}{B\left(1+\frac{zf''(z)}{f'(z)}\right)+A}\right| < 1,$$

that is,

$$\left|\frac{zf''(z)+2f'(z)}{Bzf''(z)+(A+B)f'(z)}\right| < 1.$$

Consider

$$\left|\frac{zf''(z)+2f'(z)}{Bzf''(z)+(A+B)f'(z)}\right| \tag{2.2}$$

$$= \left|\frac{\sum_{k=1}^{\infty} k(k+1)a_k z^{k+1}}{(B-A)+B\sum_{k=1}^{\infty} k(k+1)a_k z^{k+1} - (2B-A)\sum_{k=1}^{\infty} k a_k z^{k+1}}\right|$$

$$\leq \frac{\Sigma k(k+1)|a_k|}{(B-A)-\sum_{k=1}^{\infty} k(kB+A)|a_k|}.$$

(2.2) is bounded by 1 if

$$\sum_{k=1}^{\infty} k(k+1)|a_k| \leq (B-A)\sum_{k=1}^{\infty} k(kB+A)|a_k|$$

which reduces to (2.1) □

Similarly we can prove the following theorem.

Theorem 2.2. *Let* $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \quad z \in E$. *If*

$$\sum_{k=1}^{\infty}\{k(1+B)+(1+A)\}|a_k| \leq B-A, \quad \textit{then} \quad f(z) \in \Sigma^*(A,B). \tag{2.3}$$

3. MAIN RESULTS

Theorem 3.1. *If* $f(z)$ *of the form (1.1) satisfies (2.3), then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{2(n+1+A)}{2n+2+A+B}, \quad z \in \mathcal{U}.$$

The result is sharp for every n, *with extremal function*

$$f(z) = \frac{1}{z} + \frac{B-A}{2n+2+A+B} z^{n+1}, \quad n \geq 0. \tag{3.1}$$

Proof. Consider

$$\frac{2n+2+A+B}{B-A}\left\{\frac{f(z)}{f_n(z)} - \frac{2n+A+B}{2n+2+A+B}\right\}$$

$$= \frac{1+\sum_{k=1}^{n} a_k z^{k+1} + \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}}$$

$$= \frac{1+w(z)}{1-w(z)},$$

where

$$w(z) = \frac{\frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{2+2\sum_{k=1}^{n} a_k z^{k+1} - \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty} a_k z^{k+1}}$$

and

$$|w(z)| \leq \frac{\frac{2n+2+A+B}{B-A}\sum_{k=1}^{\infty} |a_k|}{2-2\sum_{k=1}^{n} |a_k| - \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty} |a_k|}.$$

Now

$$|w(z)| \leq 1$$

if and only if

$$2\left(\frac{2n+2+A+B}{B-A}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 2-2\sum_{k=1}^{n}|a_k|,$$

which is equivalent to

$$\sum_{k=1}^{n}|a_k| + \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty}|a_k| \leq 1. \tag{3.2}$$

It suffices to show that the left hand side of (3.2) bounded above by

$$\sum_{k=1}^{\infty}\frac{2k+A+B}{(B-A)}|a_k|,$$

which is equivalent to

$$\sum_{k=1}^{n}\left(\frac{2(k+A)}{B-A}\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{2(k-n-1)}{B-A}\right)|a_k| \geq 0.$$

To see that the function $f(z)$ given by (3.1) gives the sharp result, we observe for

$$z = re^{\frac{\pi i}{n+2}}$$

that

$$\frac{f(z)}{f_n(z)} = 1+\frac{B-A}{2n+2+A+B}z^{n+2} \to 1-\frac{B-A}{2n+2+A+B} = \frac{2(n+1+A)}{2(n+1)+A+B}$$

when $r \to 1^-$.

Therefore we complete the proof of Theorem 3.1. □

Corollary 3.2. *For $A = 2\alpha - 1, B = 1$, we get Theorem 2.1 in [3] which states as follows: If $f(z)$ of the form (1.1) satisfies condition*

$$\sum_{1}^{\infty}(k+\alpha)|a_k| \leq 1-\alpha,$$

then

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n+2\alpha}{n+1+\alpha}, \quad z \in \mathcal{U}.$$

The result is sharp for every n, with extremal function

$$f(z) = \frac{1}{z}+\frac{1-\alpha}{n+1+\alpha}z^{n+1}, \quad n \geq 0.$$

Theorem 3.3. *If $f(z)$ of the form (1.1) satisfies (2.1), then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{(n+2)(2n+A+B)}{(n+1)(2n+2+A+B)}, \quad z \in \mathcal{U}.$$

The result is sharp for every n, with extremal function

$$f(z) = \frac{1}{z}+\frac{B-A}{(n+1)(2n+2+A+B)}z^{n+1}, \quad n \geq 0. \tag{3.3}$$

Proof. Consider

$$\frac{(n+1)(2n+2+B+A)}{B-A}\left\{\frac{f(z)}{f_n(z)}-\frac{(n+2)(2n+A+B)}{(n+1)(2n+2+A+B)}\right\}$$
$$=\frac{1+\sum_{k=1}^{n}a_kz^{k+1}+\frac{(n+1)(2n+2+A+B)}{B-A}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{1+\sum_{k=1}^{n}a_kz^{k+1}}:=\frac{1+w(z)}{1-w(z)},$$

where

$$w(z)=\frac{\frac{(n+1)(2n+2+A+B)}{B-A}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{2+2\sum_{k=1}^{n}a_kz^{k+1}+\frac{(n+1)(2n+2+B+A)}{B-A}\sum_{k=n+1}^{\infty}a_kz^{k+1}}.$$

Now

$$|w(z)|\le\frac{\frac{(n+1)(2n+2+A+B)}{B-A}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\frac{(n+1)(2n+2+A+B)}{B-A}\sum_{k=n+1}^{\infty}|a_k|}\le 1$$

if

$$\sum_{k=1}^{n}|a_k|+\frac{(n+1)(2n+2+A+B)}{B-A}\sum_{k=n+1}^{\infty}|a_k|\le 1. \tag{3.4}$$

The left hand side of (3.4) is bounded above by

$$\sum_{k=1}^{\infty}\frac{k(2k+A+B)}{B-A}|a_k|$$

if

$$\frac{1}{B-A}\left\{\sum_{k=1}^{n}\left(k(2k+A+B)-(B-A)\right)|a_k|\right.$$
$$\left.+\sum_{k=n+1}^{\infty}\left(k(2k+A+B)-(n+1)(2n+2+A+B)\right)|a_k|\right\}\ge 0,$$

and the proof is completed. □

Corollary 3.4. *For $A=2\alpha-1, B=1$, we get Theorem 2.2 in [3] which reads:*
If $f(z)$ of the form (1.1) satisfies condition

$$\sum_{1}^{\infty}k(k+\alpha)|a_k|\le 1-\alpha,$$

then

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\}\ge\frac{(n+2)(n+\alpha)}{(n+1)(n+1+\alpha)},\quad z\in\mathcal{U}.$$

The result is sharp for every n, with extremal function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{(n+1)(n+1+\alpha)}z^{n+1},\quad n\ge 0.$$

We next determine bounds for $\Re\left\{\frac{f_n(z)}{f(z)}\right\}$.

Theorem 3.5.

(a) *If $f(z)$ of the form (1.1) satisfies the condition (2.3), then*

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\}\ge\frac{2n+2+A+B}{n+2},\quad z\in\mathcal{U}.$$

(b) *If $f(z)$ of the form (1.1) satisfies condition (2.1), then*

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{2(n+1)(2n+2+A+B)}{2(n+1)(n+2)-n(B-A)}, \quad z \in \mathcal{U}.$$

Equalities hold in (a) and (b) for the functions given by (3.1) and (3.3) respectively.

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted.

Consider

$$\frac{2(n+2)}{B-A}\left\{\frac{f_n(z)}{f(z)} - \frac{2n+2+A+B}{2(n+2)}\right\}$$
$$= \frac{1+\sum_{k=1}^{n} a_k z^{k+1} + \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}} := \frac{1+w(z)}{1-w(z)},$$

where

$$|w(z)| \leq \frac{\frac{n+2}{B-A}\sum_{k=n+1}^{\infty}|a_k|}{1-\sum_{k=1}^{n}|a_k| - \frac{n+A+B}{B-A}\sum_{k=n+1}^{\infty}|a_k|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=1}^{n}|a_k| + \frac{2n+2+A+B}{B-A}\sum_{k=n+1}^{\infty}|a_k| \leq 1. \tag{3.5}$$

Since the left hand side of (3.5) is bounded above by

$$\sum_{k=1}^{\infty}\frac{2k+A+B}{B-A}|a_k|,$$

the proof is completed. □

Corollary 3.6. *For $A = 2\alpha - 1, B = 1$, we get Theorem 2.3 in* [3] *which reads:*

(a) *If $f(z)$ of the form (1.1) satisfies condition*

$$\sum_{1}^{\infty}(k+\alpha)|a_k| \leq 1-\alpha,$$

then

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1+\alpha)}{(n+2)}, \quad z \in \mathcal{U}.$$

(b) *If $f(z)$ of the form (1.1) satisfies condition*

$$\sum_{1}^{\infty}k(k+\alpha)|a_k| \leq 1-\alpha,$$

then

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(n+1+\alpha)}{(n+1)(n+2)-n(1-\alpha)}, \quad z \in \mathcal{U}.$$

Equalities hold in (a) and (b) for the functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1+\alpha)}z^{n+1}, \quad n \geq 0,$$

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1)(n+1+\alpha)}z^{n+1}, \quad n \geq 0$$

respectively.

We turn to ratios involving derivatives. The proof of Theorem 3.7 is similar to that in Theorem 3.1 and (a) of Theorem 3.5 and so the details may be omitted.

Theorem 3.7. *If $f(z)$ of form (1.1) satisfies condition (2.3) with $A=-B$, then*

(a) $$\Re\left\{\frac{f'(z)}{f_n'(z)}\right\}\geq 0,\quad z\in\mathcal{U},$$

(b) $$\Re\left\{\frac{f_n'(z)}{f'(z)}\right\}\geq \frac{1}{2},\quad z\in\mathcal{U}.$$

In both the cases, the extremal function is given by (3.1) with $A=-B$.

Theorem 3.8. *If $f(z)$ of form (1.1) satisfies condition (2.1) then,*

(a) $$\Re\left\{\frac{f'(z)}{f_n'(z)}\right\}\geq \frac{2(n+A+B)}{2n+2+A+B},\quad z\in\mathcal{U},$$

(b) $$\Re\left\{\frac{f_n'(z)}{f'(z)}\right\}\geq \frac{2n+2+A+B}{2(n+2)},\quad z\in\mathcal{U}.$$

In both the cases, the extremal function is given by (3.3)

Proof. It is well known that $f(z)\in\Sigma_K(A,B)$ if and only if $-zf'(z)\in\Sigma^*(A,B)$. In particular, $f(z)$ satisfies condition (2.1) if and only if $-zf'(z)$ satisfies condition (2.3). Thus (a) is an immediate consequence of Theorem 3.1 and (b) follows directly from (a) of Theorem 3.5. □

For a function $f(z)\in\Sigma$, we define the integral operator $F(z)$ as follows

$$F(z)=\frac{1}{z^2}\int_0^z tf(t)dt=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{1}{k+2}a_kz^k,\quad z\in E.$$

The n^{th} partial sum $F_n(z)$ of the integral operator $F(z)$ is given by

$$F_n(z)=\frac{1}{z}+\sum_{k=1}^{n}\frac{1}{k+2}a_kz^k,\quad z\in E.$$

The following lemmas will be required for the proof of Theorem 3.11 below.

Lemma 3.9. *For* $0\leq\theta\leq\pi,\quad \frac{1}{2}+\sum_{k=1}^{m}\frac{\cos(k\theta)}{k+1}\geq 0$

Lemma 3.10. *Let P be analytic in $\mathcal{U}$ with $P(0)=1$ and $\Re\{P(z)\}>\frac{1}{2}$ in $\mathcal{U}$. For any function Q analytic in $\mathcal{U}$ the function $P*Q$ takes values in the convex hull of the image on $\mathcal{U}$ under Q.*

Lemma 3.9 is due to Rogosinski and Szego [4] and Lemma 3.10 is a well known result ([2] and [6]) that can be derived from the Herglotz representation for P. Finally we derive

Theorem 3.11. *If $f(z)\in\Sigma_c(A,B)$, then $F_n(z)\in\Sigma_c(A,B)$.*

Proof. Let $f(z)$ be the form (1.1) and belong to the class $\Sigma_c(A,B)$.

We have,

(3.6) $$\Re\left\{1-\frac{1}{B-A}\sum_{k=1}^{\infty}ka_kz^{k+1}\right\}>\frac{1}{2},\quad z\in\mathcal{U}.$$

Applying the convolution properties of power series to $F_n'(z)$ we may write

$$-z^2F_n'(z) = 1 - \sum_{k=1}^{n} \frac{k}{k+2} a_k z^{k+1} \tag{3.7}$$
$$= \left(1 - \frac{1}{B-A}\sum_{k=1}^{\infty} k a_k z^{k+1}\right) * \left(1 + (B-A)\sum_{k=n+1}^{\infty} \frac{1}{k+1} z^k\right).$$

Putting $z = re^{i\theta}, \quad 0 \le r < 1, \quad 0 \le |\theta| \le \pi$, and making use of the minimum principle for harmonic functions along with Lemma 3.9, we obtain

$$\Re\left\{1 + (B-A)\sum_{k=1}^{n+1} \frac{1}{k+1} z^k\right\} = 1 + (B-A)\sum_{k=1}^{n+1} \frac{r^k \cos(k\theta)}{k+1} \tag{3.8}$$
$$> 1 + (B-A)\sum_{k=1}^{n+1} \frac{\cos k\theta}{k+1}$$
$$\ge \left\{1 - \left(\frac{B-A}{2}\right)\right\}.$$

In view of (3.6), (3.7), (3.8) and Lemma 3.10 we deduce that

$$-\Re\{z^2F_n'(z)\} > \left\{1 - \left(\frac{B-A}{2}\right)\right\}, \quad 0 \le A + B < 2, \quad z \in \mathcal{U},$$

which completes the proof of Theorem 3.11 □

Corollary 3.12. *For* $A = 2\alpha - 1, B = 1$, *we obtain Theorem* 2.8 *in* [3] *which reads: If* $f(z) \in \Sigma_c(\alpha)$, *then* $F_n(z) \in \Sigma_c(\alpha)$.

REFERENCES

[1] S. RUSCHWEYH, *Convolutions in Geometric Function Theory*, Les Presses de l'Universite de Montreal, 1982.

[2] A.W. GOODMAN, *Univalent Functions*, Vol. I, Mariner Publ. Co., Tampa, Fl., (1983).

[3] NAK EUN CHO AND S. OWA, On partial sums of certain meromorphic functions, *J. Inequal. in Pure and Appl. Math.*, **5**(2) (2004), Art. 30. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=377`].

[4] W. ROGOSINSKI AND G. SZEGO, Uber die abschimlte von potenzreihen die in ernein kreise be schranket bleiben, *Math. Z.*, **28** (1928), 73–94.

[5] S. LATHA, Some studies in the theory of univalent and multivalent functions, PhD Thesis, (1994).

[6] R. SINGH AND S. SINGH, Convolution properties of a class of starlike fuctions, *Proc. Amer. Math. Soc.*, **106** (1989), 145–152.