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ON THE ABSOLUTE CONVERGENCE OF SMALL GAPS FOURIER SERIES OF FUNCTIONS OF $\varphi\bigwedge BV$

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ABSTRACT. Let f be a 2π periodic function in $L^1[0,2\pi]$ and $\sum_{k=-\infty}^{\infty}\widehat{f}(n_k)e^{in_kx}$ be its Fourier series with 'small' gaps $n_{k+1}-n_k \geq q \geq 1$. Here we obtain a sufficiency condition for the convergence of the series $\sum_{k\in Z} \mid \widehat{f}(n_k) \mid^{\beta}$ (0< β ≤2) if f is of $\varphi\wedge BV$ locally. We also obtain beautiful interconnections between the types of lacunarity in Fourier series and the localness of the hypothesis to be satisfied by the generic function allows us to interpolate results concerning lacunary Fourier series and non-lacunary Fourier series.

Key words and phrases: Fourier series with small gaps, Absolute convergence of Fourier series and $\varphi\wedge$-bounded variation.

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1. INTRODUCTION

Let f be a 2π periodic function in $L^1[0,2\pi]$ and $\widehat{f}(n)$, $n\in\mathbb{Z}$, be its Fourier coefficients. The series

$$\sum_{k\in Z}\widehat{f}(n_k)e^{in_kx}, \tag{1.1}$$

wherein $\{n_k\}_1^{\infty}$ is a strictly increasing sequence of natural numbers and $n_{-k}=-n_k$, for all k, satisfies an inequality

$$(n_{k+1}-n_k)\geq q\geq 1 \quad \text{for all} \quad k=0,1,2,..., \tag{1.2}$$

is called the Fourier series of f with 'small' gaps.

Obviously, if $n_k=k$, for all k, (i.e. $n_{k+1}-n_k=q=1,$ for all k), then we get non-lacunary Fourier series and if $\{n_k\}$ is such that

$$(n_{k+1}-n_k)\to\infty \quad \text{as} \quad k\to\infty \tag{1.3}$$

then (1.1) is said to be the lacunary Fourier series.

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In 1982 M. Schramm and D. Waterman [3] have introduced the class $\varphi \wedge BV(I)$ of functions of $\varphi\wedge$-bounded variation over I and have studied sufficiency conditions for the absolute convergence of Fourier series of functions of $\wedge BV^{(p)}$ and $\varphi \wedge BV$.

Definition 1.1. Given a nonnegative convex function φ, defined on $[0,\infty)$ such that $\frac{\varphi(x)}{x} \to 0$ as $x \to 0$, for some constant $d \geq 2$, $\varphi(2x) \leq d\varphi(x)$ for all $x \geq 0$ and given a sequence of non-decreasing positive real numbers $\bigwedge = \{\lambda_m\}$ $(m = 1, 2, \ldots)$ such that $\sum_m \frac{1}{\lambda_m}$ diverges we say that $f \in \varphi \bigwedge BV$ (that is f is a function of $\varphi \bigwedge$-bounded variation over (I)) if

$$V_{\Lambda_\varphi}(f, I) = \sup_{\{I_m\}} \{V_{\Lambda_\varphi}(\{I_m\}, f, I)\} < \infty,$$

where

$$V_{\Lambda_\varphi}(\{I_m\}, f, I) = \left(\sum_m \frac{\varphi\left|f(b_m) - f(a_m)\right|}{\lambda_m} \right),$$

and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

Definition 1.2. For $p \geq 1$, the p-integral modulus of continuity $\omega^{(p)}(\delta, f, I)$ of f over I is defined as

$$\omega^{(p)}(\delta, f, I) = \sup_{0 \leq h \leq \delta} \left\|(T_h f - f)(x)\right\|_{p,I},$$

where $T_h f(x) = f(x + h)$ for all x and $\left\|(\cdot)\right\|_{p,I} = \left\|(\cdot)\chi_I\right\|_p$ in which χ_I is the characteristic function of I and $\left\|(\cdot)\right\|_p$ denotes the L^p-norm. $p = \infty$ gives the modulus of continuity $\omega(\delta, f, I)$.

By applying the Wiener-Ingham result [1, Vol. I, p. 222] for the finite trigonometric sums with 'small' gap (1.2) we have already studied the sufficiency conditions for the convergence of the series $\sum_{k \in Z} \left|\widehat{f}(n_k)\right|^\beta$ $(0 < \beta \leq 2)$ for the functions of $\bigwedge BV$ and $\wedge BV^{(p)}$ in terms of the modulus of continuity [6]. Here we obtain a sufficiency condition if function f is of $\varphi \bigwedge BV$. We prove the following theorem.

Theorem 1.1. *Let $f \in L[-\pi, \pi]$ possess a Fourier series with 'small' gaps (1.2) and I be a subinterval of length $\delta_1 > \frac{2\pi}{q}$. If $f \in \varphi \bigwedge BV(I)$, $1 \leq p < 2r$, $1 \leq r < \infty$, and*

$$\sum_{k=1}^{\infty} \left(\left[\varphi^{-1} \left(\frac{\left(\omega^{((2-p)s+p)}\left(\frac{1}{n_k}, f, I\right)\right)^{2r-p}}{\sum_{j=1}^{n_k} \frac{1}{\lambda_j}} \right) \right]^{\frac{1}{r}} \Big/ k \right)^{\frac{\beta}{2}} < \infty,$$

where $\frac{1}{r} + \frac{1}{s} = 1$, then

$$\sum_{k \in Z} \left|\widehat{f}(n_k)\right|^\beta < \infty \qquad (0 < \beta \leq 2). \tag{1.4}$$

Theorem 1.1 with $\beta = 1$ is a 'small' gaps analogue of the Schramm and Waterman result [3, Theorem 2]. Observe that the interval I considered in the theorem for the gap condition (1.2) is of length $> \frac{2\pi}{q}$, so that when $n_k = k$, for all k, I is of length 2π. Hence for non-lacunary Fourier series (equality throughout in (1.2)) the theorem with $\beta = 1$ gives the Schramm and Waterman result [3, Theorem 2] as a particular case.

We need the following lemmas to prove the theorem.

Lemma 1.2 ([2, Lemma 2]). *Let f and I be as in Theorem 1.1. If $f \in L^2(I)$ then*

$$\sum_{k \in Z} \left|\widehat{f}(n_k)\right|^2 \leq A_\delta \left|I\right|^{-1} \parallel f \parallel_{2,I}^2, \tag{1.5}$$

where A_δ depends only on δ.

Lemma 1.3. *If $|n_k| > p$ then for $t \in \mathbb{N}$ one has*

$$\int_0^{\frac{\pi}{p}} \sin^{2t} |n_k|\, hdh \geq \frac{\pi}{2^{t+1}p}.$$

Proof. Obvious. □

Lemma 1.4 (Stechkin, refer to [5]). *If $u_n \geq 0$ for $n \in \mathbb{N}$, $u_n \neq 0$ and a function $F(u)$ is concave, increasing, and $F(0) = 0$, then*

$$\sum_1^\infty F(u_n) \leq 2 \sum_1^\infty F\left(\frac{u_n + u_{n+1} + \cdots}{n}\right).$$

Proof of Theorem 1.1. Let $I = \left[x_0 - \frac{\delta_1}{2}, x_0 + \frac{\delta_1}{2}\right]$ for some x_0 and δ_2 be such that $0 < \frac{2\pi}{q} < \delta_2 < \delta_1$. Put $\delta_3 = \delta_1 - \delta_2$ and $J = \left[x_0 - \frac{\delta_2}{2}, x_0 + \frac{\delta_2}{2}\right]$. Suppose integers T and j satisfy

$$|n_T| > \frac{4\pi}{\delta_3} \quad \text{and} \quad 0 \leq j \leq \frac{\delta_3 |n_T|}{4\pi}. \tag{1.6}$$

$f \in \varphi \wedge BV(I)$ implies

$$|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + C\varphi^{-1}(V_{\wedge_\varphi}(f, I)) \quad \text{for all } x \in I.$$

Since f is bounded over I, we have $f \in L^2(I)$, so that (1.5) holds and $f \in L^2[-\pi, \pi]$. If we put $f_j = (T_{2jh}f - T_{(2j-1)h}f)$ then $f_j \in L^2(I)$ and the Fourier series of f_j also possess gaps (1.2). Hence by Lemma 1.2 we get

$$\sum_{k \in Z} \left|\hat{f}(n_k)\right|^2 \sin^2\left(\frac{n_k h}{2}\right) = O\left(\|f_j\|_{2,J}^2\right) \tag{1.7}$$

because

$$\hat{f}_j(n_k) = 2i\hat{f}(n_k)e^{in_k(2j-\frac{1}{2}h)} \sin\left(\frac{n_k h}{2}\right).$$

Integrating both the sides of (1.7) over $(0, \frac{\pi}{n_T})$ with respect to h and using Lemma 1.3, we get

$$R_{n_T} = \sum_{|n_k| \geq n_T}^\infty \left|\hat{f}(n_k)\right|^2 = O(n_T) \int_0^{\frac{\pi}{n_T}} \left(\|f_j\|_{2,J}^2\right) dh. \tag{1.8}$$

Since $2 = \frac{(2-p)s+p}{s} + \frac{p}{r}$, by using Hölder's inequality, we get from (1.8)

$$\begin{aligned} B &= \int_J |f_j(x)|^2\, dx \\ &\leq \left(\int_J |f_j(x)|^{(2-p)s+p}\, dx\right)^{\frac{1}{s}} \left(\int_J |f_j(x)|^p\, dx\right)^{\frac{1}{r}} \\ &\leq \Omega_{h,J}^{1/r} \left(\int_J |f_j(x)|^p\, dx\right)^{\frac{1}{r}}, \end{aligned}$$

where $\Omega_{h,J} = (\omega^{(2-p)s+p}(h, f, J))^{2r-p}$. Thus

$$B^r \leq \Omega_{h,J} \int_J |f_j(x)|^p\, dx. \tag{1.9}$$

Since f is bounded over I, there exists some positive constant $M \geq \frac{1}{2}$ such that $|f(x)| \leq M$ for all $x \in I$. Dividing f by the positive constant M alters $\omega_p(h, f, J)$ by the same constant M

and $\varphi(2\,|f(x)|) \leq d\varphi(|f(x)|)$ for all x, we may assume that $|f(x)| \leq 1$ for all $x \in I$. Hence from (1.9) we get

$$B^r \leq \Omega_{h,J} \int_J |f_j(x)|\, dx.$$

Since $\varphi(2x) \leq d\varphi(x)$, we have $\varphi(ax) \leq d^{\log_2 a}\varphi(x)$, so

$$\begin{aligned}
\varphi\left(\frac{B^r}{\delta_2}\right) &\leq d^{\log_2 C\Omega_{h,J}}\varphi\left(\frac{\int_J |f_j(x)|\, dx}{\delta_2}\right) \\
&= C\Omega_{h,J}^{\log_2 d}\varphi\left(\frac{\int_J |f_j(x)|\, dx}{\delta_2}\right) \\
&= C\Omega_{h,J}^{\log_2 d-1}\Omega_{h,J}\varphi\left(\frac{\int_J |f_j(x)|\, dx}{\int_J 1dx}\right) \\
&\leq C\Omega_{h,J}\left(\frac{\int_J \varphi\,|f_j(x)|\, dx}{\delta_2}\right) \qquad \text{(by Jensen's inequality for integrals)} \\
&= C\Omega_{h,J}\left(\int_J \varphi\,|f_j(x)|\, dx\right).
\end{aligned}$$

Multiplying both the sides of the equation by $\frac{1}{\lambda_j}$ and then taking the summation over $j = 1$ to n_T $(T \in \mathbb{N})$ we get

$$\varphi\left(\frac{B^r}{\delta_2}\right) \leq C\left(\frac{\Omega_{h,J}}{\sum_{j=1}^{n_T}\left(\frac{1}{\lambda_j}\right)}\right)\left(\int_J \left(\sum_{j=1}^{n_T}\frac{\varphi\,|f_j(x)|}{\lambda_j}\right) dx\right). \tag{1.10}$$

Observe that for x in J, h in $(0, \frac{\pi}{n_T})$ and for each j of the summation the points $x + 2jh$ and $x + (2j-1)h$ lie in I; moreover $f \in \varphi \wedge BV(I)$ implies

$$\sum_{j=1}^{n_T}\frac{\varphi\,|f_j(x)|}{\lambda_j} = O(1).$$

Therefore, it follows from (1.10) that

$$B = O\left(\left[\varphi^{-1}\left(\frac{\Omega_{1/n_T,I}}{\sum_{j=1}^{n_T}\left(\frac{1}{\lambda_j}\right)}\right)\right]^{\frac{1}{r}}\right).$$

Substituting back the value of B in the equation (1.8), we get

$$R_{n_T} = \sum_{|n_k|\geq n_T}^{\infty}\left|\hat{f}(n_k)\right|^2 = O\left(\left[\varphi^{-1}\left(\frac{\Omega_{1/n_T,I}}{\sum_{j=1}^{n_T}\left(\frac{1}{\lambda_j}\right)}\right)\right]^{\frac{1}{r}}\right).$$

Thus

$$R_{n_T} = O\left(\left[\varphi^{-1}\left(\frac{\left(\omega^{(2-p)s+p}\left(\frac{1}{n_T}, f, I\right)\right)^{2r-p}}{\sum_{j=1}^{n_T}\left(\frac{1}{\lambda_j}\right)}\right)\right]^{\frac{1}{r}}\right).$$

Finally, Lemma 1.4 with $u_k = \left|\hat{f}(n_k)\right|^2$ $(k \in \mathbb{Z})$ and $F(u) = u^{\beta/2}$ gives

$$\begin{aligned}
\sum_{|k|=1}^{\infty} \left|\hat{f}(n_k)\right|^{\beta} &= 2\sum_{k=1}^{\infty} F\left(\left|\hat{f}(n_k)\right|^2\right) \\
&\leq 4\sum_{k=1}^{\infty} F\left(\frac{R_{n_k}}{k}\right) \\
&= 4\sum_{k=1}^{\infty} \left(\frac{R_{n_k}}{k}\right)^{\frac{\beta}{2}} \\
&= O(1)\left(\sum_{k=1}^{\infty}\left[\left[\varphi^{-1}\left(\frac{(\omega^{(2-p)s+p}(\frac{1}{n_k}, f, I))^{2r-p}}{\sum_{j=1}^{n_k}\frac{1}{\lambda_j}}\right)\right]^{\frac{1}{r}} \Big/ k\right]^{\frac{\beta}{2}}\right).
\end{aligned}$$

This proves the theorem. □

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