MONTGOMERY IDENTITIES FOR FRACTIONAL INTEGRALS AND RELATED FRACTIONAL INEQUALITIES

G. ANASTASSIOU
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA
EMail: ganastss@memphis.edu

M. R. HOOSHMANDASL, A. GHASEMI AND F. MOFTAKHARZADE
Department of Mathematical Sciences
Yazd University, Yazd, Iran
EMail: hooshmandasl@yazduni.ac.ir esfahan.ghasemi@yahoo.com f_moftakhar@yahoo.com

Received: 06 August, 2009

Accepted: 29 November, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 26A33.

Key words: Montgomery identity; Fractional integral; Ostrowski inequality; Grüss inequality.

Abstract: In the present work we develop some integral identities and inequalities for the fractional integral. We have obtained Montgomery identities for fractional integrals and a generalization for double fractional integrals. We also produced Ostrowski and Grüss inequalities for fractional integrals.

vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then the following Montgomery identity holds [1]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P_1(x,t) f'(t)\,dt, \tag{1.1}$$

where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases} \tag{1.2}$$

Suppose now that $w : [a,b] \to [0,\infty)$ is some probability density function, i.e. it is a positive integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^x w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The following identity (given by Pečarić in [4]) is the weighted generalization of the Montgomery identity:

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt, \tag{1.3}$$

where the weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

In [2, 3], the authors obtained two identities which generalized (1.1) for functions of two variables. In fact, for a function $f : [a,b] \times [c,d] \to \mathbb{R}$ such that the partial

journal of inequalities in pure and applied mathematics
issn: 1443-5756

derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ all exist and are continuous on $[a,b]\times[c,d]$, so for all $(x,y)\in[a,b]\times[c,d]$ we have:

$$(1.4)\quad (d-c)(b-a)f(x,y)=\int_c^d\int_a^b f(s,t)\,ds\,dt \\ +\int_c^d\int_a^b \frac{\partial f(s,t)}{\partial s}p(x,s)\,ds\,dt+\int_a^b\int_c^d \frac{\partial f(s,t)}{\partial t}q(y,t)\,dt\,ds \\ +\int_c^d\int_a^b \frac{\partial^2 f(s,t)}{\partial s\partial t}p(x,s)q(y,t)\,ds\,dt,$$

where

$$(1.5)\quad p(x,s)=\begin{cases} s-a, & a\le s\le x,\\ s-b, & x<s\le b,\end{cases} \quad \text{and} \quad q(y,t)=\begin{cases} t-c, & c\le t\le y,\\ t-d, & y<t\le d.\end{cases}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Fractional Calculus

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 2.1. *The Riemann-Liouville integral operator of order* $\alpha > 0$ *with* $a \geq 0$ *is defined as*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{2.1}$$
$$J_a^0 f(x) = f(x).$$

Properties of the operator can be found in [8]. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Montgomery Identities for Fractional Integrals
G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi and F. Moftakharzade
vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Montgomery Identities for Fractional Integrals

Montgomery identities can be generalized in fractional integral forms, the main results of which are given in the following lemmas.

Lemma 3.1. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable on* $[a,b]$, *and* $f' : [a,b] \to \mathbb{R}$ *be integrable on* $[a,b]$, *then the following Montgomery identity for fractional integrals holds:*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) - J_a^{\alpha-1}(P_2(x,b) f(b)) + J_a^{\alpha}(P_2(x,b) f'(b)), \qquad \alpha \geq 1, \tag{3.1}$$

where $P_2(x,t)$ *is the fractional Peano kernel defined by:*

$$P_2(x,t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t \leq x, \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x < t \leq b. \end{cases} \tag{3.2}$$

Proof. In order to prove the Montgomery identity for fractional integrals in relation (3.1), by using the properties of fractional integrals and relation (3.2), we have

$$\begin{aligned} \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f'(b)) &= \int_a^b (b-t)^{\alpha-1} P_1(x,t) f'(t)\,dt \\ &= \int_a^x \frac{t-a}{b-a}(b-t)^{\alpha-1} f'(t)\,dt + \int_x^b \frac{t-b}{b-a}(b-t)^{\alpha-1} f'(t)\,dt \\ &= \int_a^x (b-t)^{\alpha-1} f'(t)\,dt - \frac{1}{b-a}\int_a^b (b-t)^{\alpha} f'(t)\,dt. \end{aligned} \tag{3.3}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Next, integrating by parts and using (3.3), we have

$$\begin{aligned}(3.4)\quad &\Gamma(\alpha)J_a^\alpha(P_1(x,b)f'(b)) \\ &= (b-x)^{\alpha-1}f(x) - \frac{\alpha}{b-a}\Gamma(\alpha)J_a^\alpha f(b) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f(t)dt \\ &= (b-x)^{\alpha-1}f(x) - \frac{1}{b-a}\Gamma(\alpha)J_a^\alpha f(b) + \Gamma(\alpha)J_a^{\alpha-1}(P_1(x,b)f(b)).\end{aligned}$$

Finally, from (3.4) for $\alpha \geq 1$, we obtain

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^\alpha(P_2(x,b)f'(b)),$$

and the proof is completed. □

Remark 1. Letting $\alpha = 1$, formula (3.1) reduces to the classic Montgomery identity (1.1).

Lemma 3.2. *Let* $w : [a,b] \to [0,\infty)$ *be a probability density function, i.e.* $\int_a^b w(t)\,dt = 1$, *and set* $W(t) = \int_a^t w(x)\,dx$ *for* $a \leq t \leq b$, $W(t) = 0$ *for* $t < a$ *and* $W(t) = 1$ *for* $t > b$, $\alpha \geq 1$. *Then the generalization of the weighted Montgomery identity for fractional integrals is in the following form:*

$$\begin{aligned}(3.5)\quad f(x) = (b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)f(b)) \\ - J_a^{\alpha-1}(Q_w(x,b)f(b)) + J_a^\alpha(Q_w(x,b)f'(b)),\end{aligned}$$

where the weighted fractional Peano kernel is

$$(3.6)\qquad Q_w(x,t) = \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha)W(t), & a \leq t \leq x, \\ (b-x)^{1-\alpha}\Gamma(\alpha)(W(t)-1), & x < t \leq b. \end{cases}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. From fractional calculus and relation (3.6), we have

$$J_a^\alpha(Q_w(x,b)f'(b)) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}Q_w(x,t)f'(t)dt = (b-x)^{1-\alpha}\left(\int_a^b (b-t)^{\alpha-1}W(t)f'(t)dt - \int_x^b (b-t)^{\alpha-1}f'(t)dt\right). \tag{3.7}$$

Using integration by parts in (3.7) and $W(a)=0, W(b)=1$, we have

$$\int_a^b (b-t)^{\alpha-1}W(t)f'(t)\,dt = -\Gamma(\alpha)J_a^\alpha(w(b)f(b)) + (\alpha-1)\int_a^b (b-t)^{\alpha-2}W(t)f(t)dt, \tag{3.8}$$

and

$$\int_x^b (b-t)^{\alpha-1}f'(t)dt = -(b-x)^{\alpha-1}f(x) + (\alpha-1)\int_x^b (b-t)^{\alpha-2}f(t)\,dt. \tag{3.9}$$

We apply (3.8) and (3.9) to (3.7), to get

$$J_a^\alpha(Q_w(x,b)f'(b)) = (b-x)^{1-\alpha}\left[-\Gamma(\alpha)J_a^\alpha(w(b)f(b)) - (\alpha-1)\int_x^b (b-t)^{\alpha-2}f(t)\,dt + (b-x)^{\alpha-1}f(x) + (\alpha-1)\int_a^b (b-t)^{\alpha-2}W(t)f(t)\,dt\right] \tag{3.10}$$

Montgomery Identities for Fractional Integrals
G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi and F. Moftakharzade
vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha} J_a^{\alpha}(w(b)f(b)) + (b-x)^{1-\alpha}(\alpha-1)$$
$$\times \left[\int_a^x (b-t)^{\alpha-2} W(t) f(t)\,dt + \int_x^b (b-t)^{\alpha-2}(W(t)-1) f(t)\,dt\right]$$
$$= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha} J_a^{\alpha}(w(b)f(b)) + J_a^{\alpha-1}(Q_w(x,b)f(b)).$$

Finally, we have obtained that

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha) J_a^{\alpha}(w(b)f(b)) - J_a^{\alpha-1}(Q_w(x,b)f(b)) + J_a^{\alpha}(Q_w(x,b)f'(b)), \tag{3.11}$$

proving the claim. □

Remark 2. Letting $\alpha = 1$, the weighted generalization of the Montgomery identity for fractional integrals in (3.5) reduces to the weighted generalization of the Montgomery identity for integrals in (1.3).

Lemma 3.3. *Let a function $f : [a,b]\times[c,d] \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ on $[a,b]\times[c,d]$, for all $(x,y) \in [a,b]\times[c,d]$ and $\alpha, \beta \geq 2$. Then the following two variables Montgomery identity for fractional integrals holds:*

$$(d-c)(b-a) f(x,y)$$
$$= (b-x)^{1-\alpha}(d-y)^{1-\beta}\Gamma(\alpha)\Gamma(\beta)\left[J_{a,c}^{\alpha,\beta}\left(q(y,d)\frac{\partial}{\partial t} f(b,d)\right)\right.$$
$$+ J_{c,a}^{\beta,\alpha}\left(f(b,d) + p(x,b)\frac{\partial f(b,d)}{\partial s} + p(x,b)\,q(y,d)\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right)$$
$$- J_{c,a}^{\beta,\alpha-1}\left(p(x,b)\,f(b,d) + p(x,b)\,q(y,d)\frac{\partial f(b,d)}{\partial t}\right)$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$- J_{c,a}^{\beta-1,\alpha}\left(q(y,d)\,f(b,d) + p(x,b)\,q(y,d)\,\frac{\partial f(b,d)}{\partial s}\right)$$
$$+ J_{c,a}^{\beta-1,\alpha-1}\Big(p(x,b)\,q(y,d)\,f(b,d)\Big)\Big],$$

where

$$J_{c,a}^{\beta,\alpha} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_c^y \int_a^x (x-s)^{\alpha-1}(y-t)^{\beta-1} f(s,t)\,ds\,dt.$$

Also, $p(x,s)$ *and* $q(y,t)$ *are defined by (1.5).*

Proof. Put into (1.4), instead of f, the function $g(x,y) = f(x,y)(b-x)^{\alpha-1}(d-y)^{\beta-1}$. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

4. An Ostrowski Type Fractional Inequality

In 1938, Ostrowski proved the following interesting integral inequality [5]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{1}{(b-a)^2} \left(x - \frac{a+b}{2} \right)^2 \right] (b-a)M, \tag{4.1}$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \leq M$, for every $x \in [a,b]$. Now we extend it to fractional integrals.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $|f'(x)| \leq M$, for every $x \in [a,b]$ and $\alpha \geq 1$. Then the following Ostrowski fractional inequality holds:*

$$\begin{aligned} &\left| f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) + J_a^{\alpha-1} P_2(x,b) f(b) \right| \\ &\qquad \leq \frac{M}{\alpha(\alpha+1)} \left[(b-x) \left(2\alpha \left(\frac{b-x}{b-a} \right) - \alpha - 1 \right) + (b-a)^{\alpha} (b-x)^{1-\alpha} \right]. \end{aligned} \tag{4.2}$$

Proof. From Lemma 3.1 we have

$$\begin{aligned} &\left| f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) + J_a^{\alpha-1} (P_2(x,b) f(b)) \right| \\ &\qquad = \left| J_a^{\alpha} (P_2(x,b) f'(b)) \right|. \end{aligned} \tag{4.3}$$

Montgomery Identities for Fractional Integrals
G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi and F. Moftakharzade
vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Therefore, from (4.3) and (2.1) and $|f'(x)| \leq M$, we have

$$(4.4)\quad \frac{1}{\Gamma(\alpha)}\left|\int_a^b (b-t)^{\alpha-1} P_2(x,t) f'(t)\,dt\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\left|P_2(x,t)\right|\left|f'(t)\right|dt$$
$$\leq \frac{M}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\left|P_2(x,t)\right|dt$$
$$\leq M\frac{(b-x)^{1-\alpha}}{b-a}\left(\int_a^x (b-t)^{\alpha-1}(t-a)\,dt + \int_x^b (b-t)^{\alpha}\,dt\right)$$
$$= \frac{M}{\alpha(\alpha+1)}\left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right)-\alpha-1\right)+(b-a)^{\alpha}(b-x)^{1-\alpha}\right].$$

This proves inequality (4.2). □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

5. A Grüss Type Fractional Inequality

In 1935, Grüss proved one of the most celebrated integral inequalities [6], which can be stated as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right| \le \frac{1}{4}(M-m)(N-n), \tag{5.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the conditions

$$m \le f(x) \le M, \qquad n \le g(x) \le N,$$

for all $x \in [a,b]$, where m, M, n, N are given real constants.

A great deal of attention has been given to the above inequality and many papers dealing with various generalizations, extensions, and variants have appeared in the literature [7].

Proposition 5.1. *Given that $f(x)$ and $g(x)$ are two integrable functions for all $x \in [a,b]$, and satisfy the conditions*

$$m \le (b-x)^{\alpha-1} f(x) \le M, \qquad n \le (b-x)^{\alpha-1} g(x) \le N,$$

where $\alpha > 1/2$, and m, M, n, N are real constants, the following Grüss fractional inequality holds:

$$\left|\frac{\Gamma(2\alpha-1)}{(b-a)\Gamma^2(\alpha)} J_a^{2\alpha-1}(fg)(b) - \frac{1}{(b-a)^2} J_a^{\alpha} f(b) J_a^{\alpha} g(b)\right| \le \frac{1}{4\,\Gamma^2(\alpha)}(M-m)(N-n). \tag{5.2}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. If substitute $h(x) = (b-x)^{\alpha-1} f(x)$ and $k(x) = (b-x)^{\alpha-1} g(x)$ in (5.1), we will obtain (5.2). □

In [10] some related fractional inequalities are given.

Montgomery Identities for Fractional Integrals
G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi and F. Moftakharzade
vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[2] N.S. BARNETT AND S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. Math.*, **27**(1) (2001), 1–10.

[3] S.S. DRAGOMIR, P. CERONE, N.S. BARNETT AND J. ROUMELIOTIS, An inequlity of the Ostrowski type for double integrals and applications for cubature formulae, *Tamsui Oxf. J. Math. Sci.*, **16**(1) (2000), 1–16.

[4] J.E. PEČARIĆ, On the Čebyšev inequality, *Bul. Şti. Tehn. Inst. Politehn. "Traian Vuia" Timişoara*, **25**(39)(1) (1980), 5–9.

[5] A. OSTROWSKI, Über die absolutabweichung einer differentiebaren funktion von ihren integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[6] G. GRÜSS, Über das maximum des absoluten betrages von $[1/(b-a)]\int_a^b f(x)g(x)\,dx - [1/(b-a)^2]\int_a^b f(x)\,dx \int_a^b g(x)\,dx$, *Math. Z.*, **39** (1935), 215–226.

[7] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[8] S. MILLER AND B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 199, p. 2.

[9] J. PEČARIĆ AND A. VUKELIĆ, Montgomery identities for function of two variables, *J. Math. Anal. Appl.*, **332** (2007), 617–630.

vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

[10] G. ANASTASSIOU, *Fractional Differentiation Inequalities*, Springer, N.Y., Berlin, 2009.

vol. 10, iss. 4, art. 97, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756