

MONTGOMERY IDENTITIES FOR FRACTIONAL INTEGRALS AND RELATED FRACTIONAL INEQUALITIES

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Abstract: In the present work we develop some integral identities and inequalities for the fractional integral. We have obtained Montgomery identities for fractional integrals and a generalization for double fractional integrals. We also produced Ostrowski and Grüss inequalities for fractional integrals.



Montgomery Identities for
Fractional Integrals

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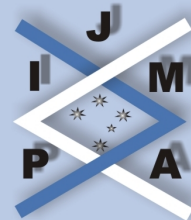
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1. Introduction

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [1]:

$$(1.1) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \int_a^b P_1(x, t) f'(t) dt,$$

where $P_1(x, t)$ is the Peano kernel

$$(1.2) \quad P_1(x, t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x, \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

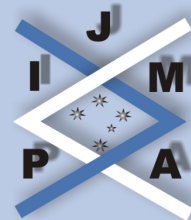
Suppose now that $w : [a, b] \rightarrow [0, \infty)$ is some probability density function, i.e. it is a positive integrable function satisfying $\int_a^b w(t) dt = 1$, and $W(t) = \int_a^x w(x) dx$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The following identity (given by Pečarić in [4]) is the weighted generalization of the Montgomery identity:

$$(1.3) \quad f(x) = \int_a^b w(t) f(t) dt + \int_a^b P_w(x, t) f'(t) dt,$$

where the weighted Peano kernel is

$$P_w(x, t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases}$$

In [2, 3], the authors obtained two identities which generalized (1.1) for functions of two variables. In fact, for a function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ such that the partial



derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ all exist and are continuous on $[a, b] \times [c, d]$, so for all $(x, y) \in [a, b] \times [c, d]$ we have:

$$(1.4) \quad (d-c)(b-a)f(x,y) = \int_c^d \int_a^b f(s,t) ds dt \\ + \int_c^d \int_a^b \frac{\partial f(s,t)}{\partial s} p(x,s) ds dt + \int_a^b \int_c^d \frac{\partial f(s,t)}{\partial t} q(y,t) dt ds \\ + \int_c^d \int_a^b \frac{\partial^2 f(s,t)}{\partial s \partial t} p(x,s)q(y,t) ds dt,$$

where

$$(1.5) \quad p(x,s) = \begin{cases} s-a, & a \leq s \leq x, \\ s-b, & x < s \leq b, \end{cases} \quad \text{and} \quad q(y,t) = \begin{cases} t-c, & c \leq t \leq y, \\ t-d, & y < t \leq d. \end{cases}$$

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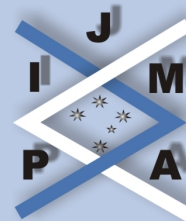
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2. Fractional Calculus

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 2.1. *The Riemann-Liouville integral operator of order $\alpha > 0$ with $a \geq 0$ is defined as*

$$(2.1) \quad \begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Properties of the operator can be found in [8]. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.



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3. Montgomery Identities for Fractional Integrals

Montgomery identities can be generalized in fractional integral forms, the main results of which are given in the following lemmas.

Lemma 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity for fractional integrals holds:*

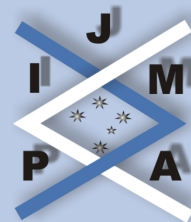
$$(3.1) \quad f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x, b)f(b)) + J_a^\alpha(P_2(x, b)f'(b)), \quad \alpha \geq 1,$$

where $P_2(x, t)$ is the fractional Peano kernel defined by:

$$(3.2) \quad P_2(x, t) = \begin{cases} \frac{t-a}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), & a \leq t \leq x, \\ \frac{t-b}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), & x < t \leq b. \end{cases}$$

Proof. In order to prove the Montgomery identity for fractional integrals in relation (3.1), by using the properties of fractional integrals and relation (3.2), we have

$$(3.3) \quad \begin{aligned} & \Gamma(\alpha) J_a^\alpha(P_1(x, b)f'(b)) \\ &= \int_a^b (b-t)^{\alpha-1} P_1(x, t) f'(t) dt \\ &= \int_a^x \frac{t-a}{b-a} (b-t)^{\alpha-1} f'(t) dt + \int_x^b \frac{t-b}{b-a} (b-t)^{\alpha-1} f'(t) dt \\ &= \int_a^x (b-t)^{\alpha-1} f'(t) dt - \frac{1}{b-a} \int_a^b (b-t)^\alpha f'(t) dt. \end{aligned}$$



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Next, integrating by parts and using (3.3), we have

$$\begin{aligned}
 (3.4) \quad & \Gamma(\alpha) J_a^\alpha (P_1(x, b) f'(b)) \\
 &= (b-x)^{\alpha-1} f(x) - \frac{\alpha}{b-a} \Gamma(\alpha) J_a^\alpha f(b) + (\alpha-1) \int_a^x (b-t)^{\alpha-2} f(t) dt \\
 &= (b-x)^{\alpha-1} f(x) - \frac{1}{b-a} \Gamma(\alpha) J_a^\alpha f(b) + \Gamma(\alpha) J_a^{\alpha-1} (P_1(x, b) f(b)).
 \end{aligned}$$

Finally, from (3.4) for $\alpha \geq 1$, we obtain

$$f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1} (P_2(x, b) f(b)) + J_a^\alpha (P_2(x, b) f'(b)),$$

and the proof is completed. \square

Remark 1. Letting $\alpha = 1$, formula (3.1) reduces to the classic Montgomery identity (1.1).

Lemma 3.2. Let $w : [a, b] \rightarrow [0, \infty)$ be a probability density function, i.e. $\int_a^b w(t) dt = 1$, and set $W(t) = \int_a^t w(x) dx$ for $a \leq t \leq b$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$, $\alpha \geq 1$. Then the generalization of the weighted Montgomery identity for fractional integrals is in the following form:

$$\begin{aligned}
 (3.5) \quad f(x) &= (b-x)^{1-\alpha} \Gamma(\alpha) J_a^\alpha (w(b) f(b)) \\
 &\quad - J_a^{\alpha-1} (Q_w(x, b) f(b)) + J_a^\alpha (Q_w(x, b) f'(b)),
 \end{aligned}$$

where the weighted fractional Peano kernel is

$$(3.6) \quad Q_w(x, t) = \begin{cases} (b-x)^{1-\alpha} \Gamma(\alpha) W(t), & a \leq t \leq x, \\ (b-x)^{1-\alpha} \Gamma(\alpha) (W(t) - 1), & x < t \leq b. \end{cases}$$



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Proof. From fractional calculus and relation (3.6), we have

$$\begin{aligned}
 (3.7) \quad & J_a^\alpha(Q_w(x, b)f'(b)) \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} Q_w(x, t) f'(t) dt \\
 &= (b-x)^{1-\alpha} \left(\int_a^b (b-t)^{\alpha-1} W(t) f'(t) dt - \int_x^b (b-t)^{\alpha-1} f'(t) dt \right).
 \end{aligned}$$

Using integration by parts in (3.7) and $W(a) = 0, W(b) = 1$, we have

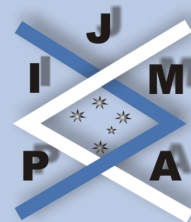
$$\begin{aligned}
 (3.8) \quad & \int_a^b (b-t)^{\alpha-1} W(t) f'(t) dt \\
 &= -\Gamma(\alpha) J_a^\alpha(w(b)f(b)) + (\alpha-1) \int_a^b (b-t)^{\alpha-2} W(t) f(t) dt,
 \end{aligned}$$

and

$$(3.9) \quad \int_x^b (b-t)^{\alpha-1} f'(t) dt = -(b-x)^{\alpha-1} f(x) + (\alpha-1) \int_x^b (b-t)^{\alpha-2} f(t) dt.$$

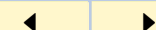
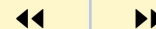
We apply (3.8) and (3.9) to (3.7), to get

$$\begin{aligned}
 (3.10) \quad & J_a^\alpha(Q_w(x, b)f'(b)) \\
 &= (b-x)^{1-\alpha} \left[-\Gamma(\alpha) J_a^\alpha(w(b)f(b)) - (\alpha-1) \int_x^b (b-t)^{\alpha-2} f(t) dt \right. \\
 &\quad \left. + (b-x)^{\alpha-1} f(x) + (\alpha-1) \int_a^b (b-t)^{\alpha-2} W(t) f(t) dt \right]
 \end{aligned}$$



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$$\begin{aligned}
 &= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha} J_a^\alpha(w(b)f(b)) + (b-x)^{1-\alpha}(\alpha-1) \\
 &\quad \times \left[\int_a^x (b-t)^{\alpha-2} W(t) f(t) dt + \int_x^b (b-t)^{\alpha-2} (W(t)-1) f(t) dt \right] \\
 &= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha} J_a^\alpha(w(b)f(b)) + J_a^{\alpha-1}(Q_w(x,b)f(b)).
 \end{aligned}$$

Finally, we have obtained that

$$\begin{aligned}
 (3.11) \quad f(x) &= (b-x)^{1-\alpha} \Gamma(\alpha) J_a^\alpha(w(b)f(b)) \\
 &\quad - J_a^{\alpha-1}(Q_w(x,b)f(b)) + J_a^\alpha(Q_w(x,b)f'(b)),
 \end{aligned}$$

proving the claim. □

Remark 2. Letting $\alpha = 1$, the weighted generalization of the Montgomery identity for fractional integrals in (3.5) reduces to the weighted generalization of the Montgomery identity for integrals in (1.3).

Lemma 3.3. Let a function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ have continuous partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ on $[a, b] \times [c, d]$, for all $(x, y) \in [a, b] \times [c, d]$ and $\alpha, \beta \geq 2$. Then the following two variables Montgomery identity for fractional integrals holds:

$$\begin{aligned}
 &(d-c)(b-a)f(x,y) \\
 &= (b-x)^{1-\alpha}(d-y)^{1-\beta} \Gamma(\alpha) \Gamma(\beta) \left[J_{a,c}^{\alpha,\beta} \left(q(y,d) \frac{\partial}{\partial t} f(b,d) \right) \right. \\
 &\quad \left. + J_{c,a}^{\beta,\alpha} \left(f(b,d) + p(x,b) \frac{\partial f(b,d)}{\partial s} + p(x,b) q(y,d) \frac{\partial^2 f(b,d)}{\partial s \partial t} \right) \right. \\
 &\quad \left. - J_{c,a}^{\beta,\alpha-1} \left(p(x,b) f(b,d) + p(x,b) q(y,d) \frac{\partial f(b,d)}{\partial t} \right) \right]
 \end{aligned}$$

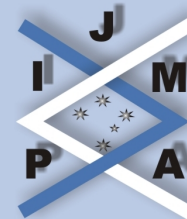
$$\begin{aligned}
 & - J_{c,a}^{\beta-1,\alpha} \left(q(y,d) f(b,d) + p(x,b) q(y,d) \frac{\partial f(b,d)}{\partial s} \right) \\
 & + J_{c,a}^{\beta-1,\alpha-1} \left(p(x,b) q(y,d) f(b,d) \right) \Big],
 \end{aligned}$$

where

$$J_{c,a}^{\beta,\alpha} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_c^y \int_a^x (x-s)^{\alpha-1} (y-t)^{\beta-1} f(s,t) ds dt.$$

Also, $p(x,s)$ and $q(y,t)$ are defined by (1.5).

Proof. Put into (1.4), instead of f , the function $g(x,y) = f(x,y)(b-x)^{\alpha-1}(d-y)^{\beta-1}$. □



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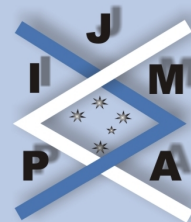
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4. An Ostrowski Type Fractional Inequality

In 1938, Ostrowski proved the following interesting integral inequality [5]:

$$(4.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{1}{(b-a)^2} \left(x - \frac{a+b}{2} \right)^2 \right] (b-a)M,$$

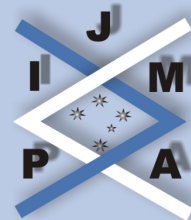
where $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function such that $|f'(x)| \leq M$, for every $x \in [a, b]$. Now we extend it to fractional integrals.

Theorem 4.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$ and $|f'(x)| \leq M$, for every $x \in [a, b]$ and $\alpha \geq 1$. Then the following Ostrowski fractional inequality holds:*

$$(4.2) \quad \left| f(x) - \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1} P_2(x, b) f(b) \right| \\ \leq \frac{M}{\alpha(\alpha+1)} \left[(b-x) \left(2\alpha \left(\frac{b-x}{b-a} \right) - \alpha - 1 \right) + (b-a)^\alpha (b-x)^{1-\alpha} \right].$$

Proof. From Lemma 3.1 we have

$$(4.3) \quad \left| f(x) - \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1} (P_2(x, b) f(b)) \right| \\ = \left| J_a^\alpha (P_2(x, b) f'(b)) \right|.$$



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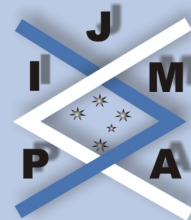
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Therefore, from (4.3) and (2.1) and $|f'(x)| \leq M$, we have

$$\begin{aligned} (4.4) \quad & \frac{1}{\Gamma(\alpha)} \left| \int_a^b (b-t)^{\alpha-1} P_2(x,t) f'(t) dt \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |P_2(x,t)| |f'(t)| dt \\ & \leq \frac{M}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |P_2(x,t)| dt \\ & \leq M \frac{(b-x)^{1-\alpha}}{b-a} \left(\int_a^x (b-t)^{\alpha-1} (t-a) dt + \int_x^b (b-t)^\alpha dt \right) \\ & = \frac{M}{\alpha(\alpha+1)} \left[(b-x) \left(2\alpha \left(\frac{b-x}{b-a} \right) - \alpha - 1 \right) + (b-a)^\alpha (b-x)^{1-\alpha} \right]. \end{aligned}$$

This proves inequality (4.2). □



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5. A Grüss Type Fractional Inequality

In 1935, Grüss proved one of the most celebrated integral inequalities [6], which can be stated as follows

$$(5.1) \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx \right| \leq \frac{1}{4}(M-m)(N-n),$$

provided that f and g are two integrable functions on $[a, b]$ and satisfy the conditions

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

for all $x \in [a, b]$, where m, M, n, N are given real constants.

A great deal of attention has been given to the above inequality and many papers dealing with various generalizations, extensions, and variants have appeared in the literature [7].

Proposition 5.1. *Given that $f(x)$ and $g(x)$ are two integrable functions for all $x \in [a, b]$, and satisfy the conditions*

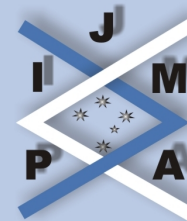
$$m \leq (b-x)^{\alpha-1} f(x) \leq M, \quad n \leq (b-x)^{\alpha-1} g(x) \leq N,$$

where $\alpha > 1/2$, and m, M, n, N are real constants, the following Grüss fractional inequality holds:

$$(5.2) \quad \left| \frac{\Gamma(2\alpha-1)}{(b-a)\Gamma^2(\alpha)} J_a^{2\alpha-1}(fg)(b) - \frac{1}{(b-a)^2} J_a^\alpha f(b) J_a^\alpha g(b) \right| \leq \frac{1}{4\Gamma^2(\alpha)}(M-m)(N-n).$$

Proof. If substitute $h(x) = (b-x)^{\alpha-1} f(x)$ and $k(x) = (b-x)^{\alpha-1} g(x)$ in (5.1), we will obtain (5.2). \square

In [10] some related fractional inequalities are given.



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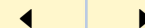
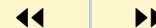
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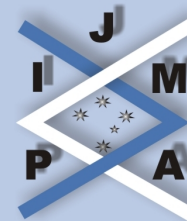
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