journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 1, Article 2, 13 pp.

A SOBOLEV-TYPE INEQUALITY WITH APPLICATIONS

RAMÓN G. PLAZA

DEPARTAMENTO DE MATEMATICAS Y MECANICA
IIMAS-UNAM
APDO. POSTAL 20-726
C.P. 01000 MEXICO DF, MEXICO
plaza@mym.iimas.unam.mx

Received 02 August, 2006; accepted 09 February, 2007
Communicated by C. Bandle

ABSTRACT. In this note, a Sobolev-type inequality is proved. Applications to obtaining linear decay rates for perturbations of viscous shocks are also discussed.

Key words and phrases: Sobolev-type inequality, Linear decay rates, Viscous shocks.

2000 *Mathematics Subject Classification.* 26D10, 35L67.

1. THE INEQUALITY

The purpose of this contribution is to prove the following.

Theorem 1.1. *Let ψ be a real-valued smooth localized function with non-zero integral,*

$$\int_{\mathbb{R}} \psi(x)\,dx = M \neq 0, \tag{1.1}$$

satisfying

$$\int_{\mathbb{R}} |x^i \partial^j \psi(x)|\,dx \leq C, \qquad \text{for all } \; i,j \geq 0. \tag{1.2}$$

Then there exists a uniform constant $C_* > 0$ such that

$$\sup_x |u(x)| \leq C_* \, \|u\|_{L^2}^{1/2} \, \|u_x - \alpha\psi\|_{L^2}^{1/2}, \tag{1.3}$$

for all $u \in H^1(\mathbb{R})$ and all $\alpha \in \mathbb{R}$.

Clearly, this result is an extension of the classical Sobolev inequality

$$\|u\|_\infty^2 \leq 2\|u\|_{L^2}\|u_x\|_{L^2}.$$

Assuming ψ satisfies (1.1) and (1.2), inequality (1.3) is valid for any $u \in H^1(\mathbb{R})$ and all $\alpha \in \mathbb{R}$; here the constant $C_* > 0$ is independent of u and α, but depends on ψ. This result may be useful while studying the asymptotic behavior of solutions to evolution equations that decay to

This research was partially supported by the University of Leipzig and the Max Planck Society, Germany. This is gratefully acknowledged.
209-06

a manifold spanned by a certain function ψ (see Section 2 below). It is somewhat surprising that the result holds for all $\alpha \in \mathbb{R}$. The crucial fact is that the antiderivative of ψ cannot be in L^2, thanks to hypothesis (1.1). In this fashion we avoid the case $u_x \in \mathrm{span}\{\psi\}$.

We would like to establish (1.3) by extremal functions. Since the solution to the minimization problem associated with (1.3) may not exist, our approach will consist of studying a parametrized family of inequalities for which we can explicitly compute extremal functions for each parameter value.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists a constant $c_* > 0$ such that*

$$c_* \leq \rho^{-1}\|u\|_{L^2}^2 + \rho\|u_x - \alpha\psi\|_{L^2}^2, \tag{1.4}$$

for all $\rho > 0$, $\alpha \in \mathbb{R}$, and u in a dense subset of $H^1(\mathbb{R})$ with $u(0) = 1$. Moreover, c_* is also uniform under translation $\tilde{\psi}(\cdot) = \psi(\cdot + y)$, where $y \in \mathbb{R}$ (even though hypothesis (1.2) *is not uniform by translation).*

Proposition 1.3. *Theorem 1.2 implies Theorem 1.1.*

Proof. It suffices to show that

$$|u(0)| \leq C_*\|u\|_{L^2}^{1/2}\,\|u_x - \alpha\psi\|_{L^2}^{1/2}, \tag{1.5}$$

with uniform $C_* > 0$, also by translation. Indeed, we can always take, for any $y \in \mathbb{R}$, $\tilde{u}(x) := u(x + y)$, $\tilde{\psi}(x) := \psi(x + y)$, yielding

$$\begin{aligned}
|u(y)| = |\tilde{u}(0)| &\leq C_*\|\tilde{u}\|_{L^2}^{1/2}\,\|\tilde{u}_x - \alpha\tilde{\psi}\|_{L^2}^{1/2} \\
&= C_*\|u(\cdot + y)\|_{L^2}^{1/2}\,\|u_x(\cdot + y) - \alpha\psi(\cdot + y)\|_{L^2}^{1/2} \\
&= C_*\|u\|_{L^2}^{1/2}\|u_x - \alpha\psi\|_{L^2}^{1/2}, \qquad \forall y \in \mathbb{R},
\end{aligned}$$

by uniformity of C_* and by translation invariance of L^p norms. This shows (1.3).

Now assume Theorem 1.2 holds. If $u(0) = 0$ then (1.5) holds trivially. In the case $u(0) \neq 0$, consider $\tilde{u} = u/u(0)$, $\tilde{\alpha} = \alpha/u(0)$ and apply (1.4),

$$c_* u(0)^2 \leq \rho^{-1}\|u\|_{L^2}^2 + \rho\|u_x - \alpha\psi\|_{L^2}^2.$$

Minimizing over ρ yields $\rho = \|u\|_{L^2}/\|u_x - \alpha\psi\|_{L^2}$, so that

$$c_* u(0)^2 \leq 2\|u\|_{L^2}\|u_x - \alpha\psi\|_{L^2}.$$

This proves (1.5) with $C_* = \sqrt{2/c_*}$. □

Therefore, we are left to prove Theorem 1.2.

1.1. **Proof of Theorem 1.2.** Without loss of generality assume that

$$\|\psi\|_{L^2} = 1. \tag{1.6}$$

Since $u \in H^1$, we may use the Fourier transform, and the constraint $u(0) = 1$ becomes

$$\int_{\mathbb{R}} \hat{u}(\xi)\, d\xi = 1, \tag{1.7}$$

up to a constant involving π. Note that the expression on the right of (1.4) defines a family of functionals parametrized by $\rho > 0$,

$$\mathcal{J}^\rho[u] := \rho^{-1}\int_{\mathbb{R}} |\hat{u}(\xi)|^2\, d\xi + \rho\int_{\mathbb{R}} |i\xi\hat{u}(\xi) - \alpha\hat{\psi}(\xi)|^2\, d\xi. \tag{1.8}$$

We shall see by direct computation that the minimizer u exists and is unique (given by a simple formula) for each ρ and α. Denote $\hat{u} = v + iw$, $\hat{\psi} = \eta + i\theta$ (real and imaginary parts). Then each functional (1.8) can be written as

$$\mathcal{J}^\rho[u] = \rho^{-1}\int_{\mathbb{R}} (v^2+w^2)\,d\xi + \rho\int_{\mathbb{R}} (\xi^2(v^2+w^2) + 2\alpha\xi(w\eta - v\theta) + \alpha^2(\eta^2+\theta^2))\,d\xi. \tag{1.9}$$

The constraint (1.7) splits into $\int v\,d\xi = 1$ and $\int w\,d\xi = 0$. Hence, we have the following minimization problem

$$\min_{u\in H^1(\mathbb{R})} \mathcal{J}^\rho[(v,w)]$$

subject to

$$\mathcal{I}_1[(v,w)] = \int_{\mathbb{R}} v\,d\xi - 1 = 0,$$
$$\mathcal{I}_2[(v,w)] = \int_{\mathbb{R}} w\,d\xi = 0,$$

for each $\rho > 0$ and $\alpha \in \mathbb{R}$. The Lagrange multiplier conditions

$$\tfrac{1}{2}D_{(h_1,0)}\mathcal{J}[(v,w)] = \mu D_{(h_1,0)}\mathcal{I}_1[(v,w)],$$
$$\tfrac{1}{2}D_{(0,h_2)}\mathcal{J}[(v,w)] = \nu D_{(0,h_2)}\mathcal{I}_2[(v,w)],$$

yield

$$\int_{\mathbb{R}} (\rho^{-1}v + \rho\xi^2 v - \rho\alpha\theta\xi)h_1\,d\xi = \mu\int_{\mathbb{R}} h_1\,d\xi,$$
$$\int_{\mathbb{R}} (\rho^{-1}w + \rho\xi^2 w + \rho\alpha\eta\xi)h_2\,d\xi = \nu\int_{\mathbb{R}} h_2\,d\xi,$$

for some $(\mu,\nu)\in\mathbb{R}^2$ and for all test functions (h_1,h_2). Therefore

$$\rho^{-1}v + \rho\xi^2 v - \rho\alpha\xi\theta = \mu,$$
$$\rho^{-1}w + \rho\xi^2 w + \rho\alpha\xi\eta = \nu.$$

Denote $\lambda = \mu + i\nu$. Multiply the second equation by i, and solve for v and w to obtain

$$\hat{u} = \frac{\rho\lambda - i\alpha\rho^2\xi\hat{\psi}(\xi)}{1+\rho^2\xi^2}. \tag{1.10}$$

Equation (1.10) is, in fact, the expression for the minimizer. Whence, we can compute the minimum value of $\mathcal{J}^\rho$ for each $\rho > 0$, in terms of λ and α. Substituting (1.10) one obtains (after some computations),

$$\rho^{-1}|\hat{u}|^2 + \rho|i\xi\hat{u} - \alpha\hat{\psi}|^2 = \frac{\rho(|\lambda|^2 + \alpha^2|\hat{\psi}|^2)}{1+\rho^2\xi^2}.$$

Hence we easily find that the minimum value of $\mathcal{J}^\rho$ is given by

$$\begin{aligned}\mathcal{J}^\rho_{\min} &= |\lambda|^2\int_{\mathbb{R}} \frac{\rho d\xi}{1+\rho^2\xi^2} + \alpha^2\rho\int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{1+\rho^2\xi^2}\,d\xi \\ &= \pi|\lambda|^2 + \alpha^2\Gamma(\rho), \end{aligned} \tag{1.11}$$

where

$$\Gamma(\rho) := \rho\int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{1+\rho^2\xi^2}\,d\xi. \tag{1.12}$$

Now we find the Lagrange multiplier λ in terms of α using the constraint (1.7), which implies

$$1 = \lambda \int_{\mathbb{R}} \frac{\rho d\xi}{1+\rho^2\xi^2} - \alpha\rho^2 \int_{\mathbb{R}} \frac{i\xi\hat{\psi}(\xi)}{1+\rho^2\xi^2}\, d\xi = \lambda\pi + \alpha\Theta(\rho),$$

where

$$\Theta(\rho) := -\rho^2 \int_{\mathbb{R}} \frac{i\xi\hat{\psi}(\xi)}{1+\rho^2\xi^2}\, d\xi. \tag{1.13}$$

Solving for λ we find,

$$\lambda = \frac{1}{\pi}(1 - \alpha\Theta(\rho)). \tag{1.14}$$

Observe that since ψ is real, then $\overline{\hat{\psi}(\xi)} = \hat{\psi}(-\xi)$ and therefore $\Theta(\rho) \in \mathbb{R}$ for all $\rho > 0$. This readily implies that $\lambda \in \mathbb{R}$ and, upon substitution in (1.11), that

$$\mathcal{J}^{\rho}_{\min} = \frac{1}{\pi}(1 - \alpha\Theta(\rho))^2 + \alpha^2\Gamma(\rho). \tag{1.15}$$

The latter expression is a real quadratic polynomial in $\alpha \in \mathbb{R}$. Minimizing over α we get

$$\alpha = \frac{\Theta(\rho)}{\pi\Gamma(\rho) + \Theta(\rho)^2} \in \mathbb{R}. \tag{1.16}$$

Thus, we can substitute (1.16) in (1.15), obtaining in this fashion the lower bound

$$\mathcal{J}^{\rho}_{\min} \geq \mathcal{I}(\rho) := \frac{\Gamma(\rho)}{\pi\Gamma(\rho) + \Theta(\rho)^2} > 0.$$

Remark 1.4. The choice (1.16) corresponds to taking $\alpha = \int i\xi\hat{u}\overline{\hat{\psi}}d\xi \in \mathbb{R}$, as the reader may easily verify using (1.10). Intuitively, the most we can do with α in (1.8) is to remove the $\hat{\psi}$-component of $\hat{u}$. In other words, if we minimize $\|u_x - \alpha\psi\|_{L^2}$ over α we obtain $\alpha = \int u_x\psi\, dx / \int \psi^2\, dx = \int i\xi\hat{u}\overline{\hat{\psi}}d\xi$ (recall $\|\psi\|_{L^2} = 1$). We can substitute its value in the expression of the minimizer to compute the lower bound $\mathcal{I}(\rho)$.

We do not need to show that (1.10) is the actual minimizer. The variational formulation simply helps us to compute a lower bound for the functional in terms of ρ. Next, we study the behavior of $\Theta(\rho)$ and $\Gamma(\rho)$ for all $\rho > 0$. We are particularly interested in what happens for large ρ. In addition, we have to prove that the lower bound is uniform in $y \in \mathbb{R}$ if we substitute $\psi(\cdot)$ by $\psi(\cdot + y)$, a property that was required in the proof of Proposition 1.3.

Lemma 1.5. *There holds*

(i) $\Gamma(\rho) \in \mathbb{R}^+$ *for all* $\rho > 0$ *and it is invariant under translation* $\psi(\cdot) \to \psi(\cdot + y)$ *for any* $y \in \mathbb{R}$*,*
(ii) $C^{-1}\rho \leq \Gamma(\rho) \leq C\rho$ *for* $\rho \sim 0^+$*, and some* $C > 0$*,*
(iii) $\Gamma(\rho) \to \pi M^2$ *as* $\rho \to +\infty$*,*
(iv) $\Theta(\rho) \leq C\rho^2$ *for* $\rho \sim 0^+$*, and*
(v) $\Theta(\rho)$ *is uniformly bounded under translation* $\psi(\cdot) \to \psi(\cdot + y)$ *with* $y \in \mathbb{R}$*, as* $\rho \to +\infty$*.*

Proof. (i) is obvious, as $|\widehat{\psi(\cdot + y)}(\xi)| = |e^{i\xi y}\hat{\psi}(\xi)| = |\hat{\psi}(\xi)|$; also by (1.1), it is clear that $\Gamma(\rho) > 0$, for all $\rho > 0$.

(ii) follows directly from $\Gamma(\rho) \leq \rho \int |\hat{\psi}|^2 d\xi = \rho$ for all $\rho > 0$, because of (1.6), and from noticing that

$$\begin{aligned}\Gamma(\rho) &= \int_{\mathbb{R}} \frac{|\hat{\psi}(\zeta/\rho)|^2}{\zeta^2+1}\, d\zeta \\ &= \int_{|\zeta|\leq 1} + \int_{|\zeta|\geq 1} \\ &\geq \frac{1}{2}\int_{|\zeta|\leq 1} |\hat{\psi}(\zeta/\rho)|^2\, d\zeta = \frac{\rho}{2}\int_{|\xi|\leq 1/\rho} |\hat{\psi}(\xi)|^2\, d\xi.\end{aligned}$$

Since $\|\psi\|_{L^2} = 1$, we have for ρ sufficiently small,

$$\int_{|\xi|\leq 1/\rho} |\hat{\psi}(\xi)|^2\, d\xi \geq \frac{1}{2},$$

and thus $\Gamma(\rho) \geq \frac{1}{4}\rho = C^{-1}\rho$ for $\rho \sim 0^+$.

(iii) to prove (iii), notice that $|\hat{\psi}|$ is bounded, $\hat{\psi}(\zeta/\rho) \to \hat{\psi}(0)$ as $\rho \to +\infty$ pointwise, and $(\zeta^2+1)^{-1}$ is integrable; therefore we clearly have

$$\Gamma(\rho) = \int_{\mathbb{R}} \frac{|\hat{\psi}(\zeta/\rho)|^2}{\zeta^2+1}\, d\zeta \longrightarrow \int_{\mathbb{R}} \frac{|\hat{\psi}(0)|^2}{1+\zeta^2}\, d\zeta = \pi|\hat{\psi}(0)|^2 = \pi M^2 > 0,$$

as $\rho \to +\infty$.

(iv) follows directly from hypothesis (1.2), as

$$|\Theta(\rho)| \leq \rho^2 \int_{\mathbb{R}} \frac{|\xi\hat{\psi}(\xi)|}{1+\rho^2\xi^2}\, d\xi \leq \rho^2 \int_{\mathbb{R}} |\xi\hat{\psi}(\xi)|\, d\xi \leq C\rho^2.$$

Note that this estimate is valid also by translation, even though $\psi(\cdot + y)$ may not satisfy (1.2).

(v) in order to prove (v), we first assume that ψ itself satisfies (1.1) and (1.2). Split the integral into two parts,

$$\Theta(\rho) = -\int_{|\xi|\leq 1} \frac{i\xi\hat{\psi}(\xi)}{\xi^2+1/\rho^2}\, d\xi - \int_{|\xi|\geq 1} \frac{i\xi\hat{\psi}(\xi)}{\xi^2+1/\rho^2}\, d\xi := I_1 + I_2.$$

I_2 is clearly bounded as $\rho \to +\infty$ by hypothesis (1.2),

$$|I_2| \leq \int_{|\xi|\geq 1} \frac{|\xi\hat{\psi}(\xi)|}{\xi^2+1/\rho^2}\, d\xi \leq \int_{\mathbb{R}} |\xi\hat{\psi}(\xi)|\, d\xi \leq C.$$

Denote

$$\phi(\xi) := \begin{cases} \frac{1}{\xi}(\hat{\psi}(\xi) - \hat{\psi}(0)) & \text{for } \xi \neq 0, \\ \frac{d\hat{\psi}}{d\xi}(0) & \text{for } \xi = 0. \end{cases}$$

ϕ is continuous. Then, I_1 can be further decomposed into

$$I_1 = -\hat{\psi}(0)\int_{|\xi|\leq 1} \frac{i\xi\, d\xi}{\xi^2+1/\rho^2}\, d\xi - \int_{|\xi|\leq 1} \frac{i\xi^2\phi(\xi)}{\xi^2+1/\rho^2}\, d\xi.$$

The first integral is clearly zero for all $\rho > 0$, and the second is clearly bounded as

$$\int_{|\xi|\leq 1} \frac{\xi^2|\phi(\xi)|}{\xi^2+1/\rho^2}\, d\xi \leq \int_{|\xi|\leq 1} |\phi(\xi)|\, d\xi \leq C.$$

Therefore, $\Theta(\rho)$ is bounded as $\rho \to +\infty$.

Now, let us suppose that $\psi(\cdot) = \psi_0(\cdot + y)$ for some fixed $y \in \mathbb{R}, y \neq 0$, where ψ_0 satisfies (1.1) and (1.2). Then clearly $\hat{\psi}(\xi) = e^{i\xi y}\hat{\psi}_0(\xi)$ and

$$\Theta(\rho) = -\int_{\mathbb{R}} \frac{i\xi e^{i\xi y}\hat{\psi}_0(\xi)}{\xi^2 + 1/\rho^2}\, d\xi.$$

Assume that $y > 0$ (the case $y < 0$ is analogous); then consider the function

$$g(z) = \frac{ize^{izy}\hat{\psi}_0(z)}{z^2 + 1/\rho^2},$$

for z in $\operatorname{Im} z > 0$, and take the upper contour

$$\mathcal{C} = [-R, R] \cup \{z = Re^{i\theta}; \theta \in [0, \pi]\},$$

for some $R > 0$ large. Then $g(z)$ is analytic inside $\mathcal{C}$ except at the simple pole $z = i/\rho$. (When $y < 0$ one takes the lower contour that encloses the pole at $z = -i/\rho$.) By complex integration of g along $\mathcal{C}$ in the counterclockwise direction, and by the residue theorem, one gets

$$\int_{\mathcal{C}} g(z)\, dz = 2\pi i \operatorname{Res}_{z=i/\rho} g(z) = -\pi e^{-y/\rho}\hat{\psi}_0(i/\rho).$$

Therefore it is easy to see that the value $\Theta(\rho)$ is uniformly bounded in $y \in \mathbb{R}$ as

$$|\Theta(\rho)| \leq \pi|\hat{\psi}_0(i/\rho)| \to \pi|M| > 0$$

when $\rho \to +\infty$. This completes the proof of the lemma.

Remark 1.6. If we consider the solution u^ρ to

$$-u_{xx} + \frac{1}{\rho^2}u = \psi_x, \tag{1.17}$$

then, after taking Fourier transform, one finds

$$\hat{u}^\rho(\xi) = \frac{i\xi\hat{\psi}(\xi)}{\xi^2 + 1/\rho^2},$$

so that $u^\rho(0) = \int \hat{u}^\rho d\xi = -\Theta(\rho)$. The claim that $u^\rho(0)$ is bounded as $\rho \to +\infty$ is plausible because in the limit (formally) we have $-u^\rho_{xx} = \psi_x$ or $u^\rho_x = -\psi$. Since ψ is integrable, u^ρ should be bounded. The bound $\Theta(\rho) \sim e^{-|y|/\rho}$ represents the (slow) exponential decay of the Green's function solution to (1.17).

In Lemma 1.5, we have shown that $\Theta(\rho)$ and $\Gamma(\rho)$ are uniformly bounded for ρ large and in $y \in \mathbb{R}$. The same applies to $\mathcal{I}(\rho)$. For ρ near 0, since both tend to zero as $\rho \to 0^+$, by L'Hôpital's rule we get

$$\lim_{\rho\to 0^+} \mathcal{I}(\rho) = \lim_{\rho\to 0^+} \frac{\frac{d\Gamma}{d\rho}}{\pi\frac{d\Gamma}{d\rho} + 2\Theta\frac{d\Theta}{d\rho}} = \pi^{-1} > 0,$$

because (ii) implies $(d\Gamma/d\rho)_{|\rho=0^+} \geq C^{-1} > 0$, and $d\Theta/d\rho$ is bounded as $\rho \to 0^+$ by (iv).

Therefore, the constant $\mathcal{I}(\rho)$ is uniformly bounded from above and below for all $\rho > 0$, in particular for $\rho \to +\infty$. This implies the uniform boundedness from below of $\mathcal{J}^\rho_{\min}$ and of $\mathcal{J}^\rho[u]$ for all u in the constrained class of functions considered in Theorem 1.2. Furthermore, the lower bound is uniform by translation as well. This completes the proof. □

Remark 1.7. The corresponding Fourier L^1 estimate

$$\|\hat{u}\|_{L^1} \leq C\|\hat{u}\|_{L^2}^{1/2} \|i\xi\hat{u} - \alpha\hat{\psi}\|_{L^2}^{1/2},$$

(from which the result can be directly deduced), does not hold. Here it is a counterexample: let ψ be a nonnegative function with compact support and let Ψ be its antiderivative. Set

$$u(x) := \Psi(x) - \Psi(x/L),$$

where $L > 0$ is large. Then there is $R > 0$ such that u vanishes outside $|x| \leq RL$. Henceforth $\|u\|_{L^2} \leq CL$ for some $C > 0$. Moreover, we also have $u_x - \psi = \psi(x)/L$, and consequently $\|u_x - \psi\|_{L^2} \leq C/L$. This implies that the product $\|u\|_{L^2}\|u_x - \psi\|_{L^2}$ remains uniformly bounded in L. Now, the Fourier transform of u is

$$\hat{u}(\xi) = \hat{\Psi}(\xi) - L\hat{\Psi}(L\xi) = \frac{i}{\xi}\left(\hat{\psi}(L\xi) - \hat{\psi}(\xi)\right).$$

Since $\hat{\psi}$ has compact support, it vanishes outside $|\xi| \leq \tilde{R}$, for some $\tilde{R} > 0$. Now, $|\hat{\psi}(0)| = M > 0$ implies that $|\hat{\psi}(\xi)| > 0$ near $\xi = 0$, and we can choose L sufficiently large such that $|\hat{\psi}(\xi)| \geq c_0$ for $\tilde{R}/L \leq |\xi| \leq \delta_*/2$, where $\delta_* = \sup\{\delta > 0\,;\, |\hat{\psi}(\xi)| > 0 \text{ for } 0 \leq |\xi| < \delta\}$, and c_0 is independent of L. Therefore

$$|\hat{u}(\xi)| = \frac{|\hat{\psi}(\xi)|}{|\xi|} \geq \frac{c_0}{|\xi|},$$

for all $\tilde{R}/L \leq |\xi| \leq \delta_*/2$, and the L^1 norm of $\hat{u}$ behaves like

$$\|\hat{u}\|_{L^1} \geq c_0 \int_{\tilde{R}/L \leq |\xi| \leq \delta_*/2} \frac{d\xi}{|\xi|} \;\sim\; c_0 \ln L \;\to +\infty,$$

as $L \to +\infty$.

2. APPLICATIONS TO VISCOUS SHOCK WAVES

To illustrate an application of uniform inequality (1.3), consider a scalar conservation law with second order viscosity,

$$u_t + f(u)_x = u_{xx}, \tag{2.1}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, f is smooth, and $f'' \geq a > 0$ (convex mode). Assume the triple (u_-, u_+, s) (with $u_+ < u_-$) is a classical shock front [5] satisfying the Rankine-Hugoniot jump condition[1] $-s[u] + [f(u)] = 0$, and Lax entropy condition $f'(u_+) < s < f'(u_-)$. A shock profile [1] is a traveling wave solution to (2.1) of form $u(x, t) = \bar{u}(x - st)$, where $\bar{u}$ satisfies $\bar{u}'' = f(\bar{u})_x - s\bar{u}'$, with $' = d/dz$, $z = x - st$, and $\bar{u} \to u_\pm$ as $z \to \pm\infty$. Without loss of generality we can assume $s = 0$ by normalizing f (see e. g. [3]), so that $f(u_\pm) = 0$, $f'(u_+) < 0 < f'(u_-)$ and the profile equation becomes

$$\bar{u}_x = f(\bar{u}). \tag{2.2}$$

Such a profile solution exists, and under the assumptions, it is both monotone $\bar{u}_x < 0$ and exponentially decaying up to two derivatives

$$|\partial_x^j(\bar{u}(x) - u_\pm)| \lesssim e^{-c|x|},$$

for all $0 \leq j \leq 2$ and some constant $c > 0$ (see [7, 8, 1] and the references therein)[2].

We will show that the following consequence of Theorem 1.1 is useful to obtain decay rates for solutions to the linearized equations for the perturbed problem.

[1] Here $[g]$ denotes the jump $g(u_+) - g(u_-)$ for any g.

[2] In the sequel "$\lesssim$" means "$\leq$" modulo a harmless positive constant.

Lemma 2.1. *Let $\bar{u}$ be the shock profile solution to* (2.2). *Then*

$$\|u\|_{L^\infty}^2 \lesssim \|u\|_{L^2}\|u_x - \alpha\bar{u}_x\|_{L^2}, \tag{2.3}$$

for all $u \in H^1(\mathbb{R})$ and all $\alpha \in \mathbb{R}$.

Proof. Follows immediately from Theorem 1.1 with $\psi = \bar{u}_x$, which satisfies hypotheses (1.1) and (1.2), as $\bar{u}_x$ is exponentially decaying and has non-zero integral $[u] \neq 0$. □

Consider a solution to (2.1) of the form $u+\bar{u}$, u being a perturbation; linearizing the resulting equation around the profile we obtain

$$u_t = Lu := u_{xx} - (f'(\bar{u})u)_x, \tag{2.4}$$

where L is a densely defined linear operator in, say, L^2. In [4], Goodman introduced the *flux transform* $\mathcal{F} : W^{1,p} \to L^p$, where $\mathcal{F}u := u_{xx} - f'(\bar{u})u_x$ as a way to cure the negative sign of $f''(\bar{u})\bar{u}_x < 0$. That is, if u solves (2.4) then clearly its flux variable $v := \mathcal{F}u$ satisfies the "integrated" equation [2],

$$u_t = \mathcal{L}u := u_{xx} - f'(\bar{u})u_x, \tag{2.5}$$

which leads to better energy estimates. Another feature of the flux transform formulation is the following inequality (see [4] for details, or [6] – Chapter 4, Proposition 4.6 – for the proof).

Lemma 2.2 (Poincaré-type inequality)**.** *There exists a constant $C > 0$ such that for all $1 \leq p \leq +\infty$ and $u \in L^p$,*

$$\|u - \delta\bar{u}_x\|_{L^p} \leq C\|\mathcal{F}u\|_{L^p}, \tag{2.6}$$

where δ is given by

$$\delta = \frac{1}{Z}\int_{\mathbb{R}} u\bar{u}_x \, dx, \tag{2.7}$$

and $Z = \int_{\mathbb{R}} \bar{u}_x^2 \, dx > 0$ is a constant.

Here we illustrate an application of the uniform estimate (2.3) to obtain sharp decay rates for solutions to the linearized perturbation equation, using the flux formulation due to Goodman.

Proposition 2.3 (Goodman [4])**.** *For all global solutions to $u_t = Lu$, with suitable initial conditions, there holds*

$$\|u(t) - \delta(t)\bar{u}_x\|_{L^\infty} \lesssim t^{-1/2}\|u(0)\|_{W^{1,1}}, \tag{2.8}$$

where $\delta(t)$ is given by (2.7)*.*

Remark 2.4. This is a linear stability result with a sharp decay rate (the power $t^{-1/2}$ is that of the heat equation, and therefore, optimal). Notice also that $\delta(t)$ depends on t, corresponding (at least at this linear level) to an instantaneous projection onto the manifold spanned by $\bar{u}$. The need of a uniform inequality for all $\delta \in \mathbb{R}$ such as (2.3) is thus clear. For a very comprehensive discussion on (nonlinear) "wave tracking" and stronger results, see Zumbrun [9].

Remark 2.5. The formal adjoint of the integrated operator is given by

$$\mathcal{L}^* u := u_{xx} + (f'(\bar{u})u)_x.$$

Note that if v and w are solutions to $v_t = \mathcal{L}v$ and $w_t = -\mathcal{L}^* w$, respectively, then

$$\frac{d}{dt}\int_{\mathbb{R}} v(t)w(t)\, dx = \int_{\mathbb{R}} (w\mathcal{L}v - v\mathcal{L}^* w)\, dx = 0,$$

and hence

$$\int_{\mathbb{R}} v(t)w(t)\, dx = \int_{\mathbb{R}} v(0)w(0)\, dx, \quad \text{for all } t \geq 0.$$

In the sequel, we will gloss over many details, such as global existence of the solutions to the linear equations, or the correct assumptions for initial conditions in suitable spaces (which are standard and can be found elsewhere [2, 9]), and concentrate on filling out the details of the proof of Proposition 2.3 sketched in [4].

2.1. **Energy Estimates.** We start with the basic energy estimate.

Lemma 2.6. *Let v be a solution to either $v_t = \mathcal{L}v$ or $v_t = \mathcal{L}^*v$. Then for all $t \geq s \geq 0$ we have the basic energy estimate*

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 \leq -\|v_x(t)\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|v(t)^2\,dx < 0, \tag{2.9}$$

and,

$$\|v(t)\|_{L^2}^2 \leq \|v(s)\|_{L^2}^2, \tag{2.10}$$

$$\int_s^t \|v_x(\tau)\|_{L^2}^2\,d\tau \leq \frac{1}{2}\|v(s)\|_{L^2}^2, \tag{2.11}$$

$$\int_s^t \int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|v(\tau)^2\,dxd\tau \leq \|v(s)\|_{L^2}^2. \tag{2.12}$$

Proof. Follows by standard arguments. Multiply $v_t = \mathcal{L}v$ by v and integrate by parts once to get (2.9). Likewise, multiply $v_t = \mathcal{L}^*v$ by v and integrate by parts twice to arrive at the same estimate. The negative sign in (2.9) is a consequence of compressivity of the wave $f''(\bar{u})\bar{u}_x < 0$. Estimates (2.10) – (2.12) follow directly from (2.9). $\square$

Next, we establish decay rates for v_t and w, and solutions to $v_t = \mathcal{L}v$ and $w_t = \mathcal{L}^*w$.

Lemma 2.7. *Let v be a solution to $v_t = \mathcal{L}v$. Then the following decay rate holds*

$$\|v_t(t)\|_{L^2} \lesssim t^{-1/2}\|v(0)\|_{L^2}. \tag{2.13}$$

Proof. First observe that $v_t = \mathcal{F}v_x$, and therefore $v_{tt} = (\mathcal{F}v_x)_t = \mathcal{F}v_{tx} = \mathcal{L}v_t$, that is, v_t solves the integrated equation as well, and hence, the estimates (2.9) – (2.12) hold for v_t also. In particular, the L^2 norm of v_t is decreasing. To show (2.13) it suffices to prove

$$\|v_t(t)\|_{L^2}^2 \lesssim \|v_x(s)\|_{L^2}^2, \tag{2.14}$$

for all $t > s+1, s \geq 0$. Integrate (2.14) in $s \in [0, t-1]$ and use (2.11) to obtain

$$\|v_t(t)\|_{L^2}^2 \lesssim (t-1)^{-1}\int_0^t \|v_x(s)\|_{L^2}^2\,ds \lesssim (t-1)^{-1}\|v(0)\|_{L^2}^2 \lesssim t^{-1}\|v(0)\|_{L^2}^2,$$

for all $t \geq 2$, yielding (2.13). To show (2.14) differentiate $v_t = \mathcal{L}u$ with respect to x, multiply by v_x and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|v_x(t)\|_{L^2}^2 = -\|v_{xx}(t)\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f'(\bar{u})_x v_x^2\,dx \leq M\|v_x(t)\|_{L^2}^2, \tag{2.15}$$

where $M := \sup|f'(\bar{u})_x|$. By Gronwall's inequality

$$\|v_x(T+t)\|_{L^2}^2 \leq e^{Mt}\|v_x(T)\|_{L^2}^2, \tag{2.16}$$

for all $t, T \geq 0$. Integrating (2.15) in $t \in [s, T]$,

$$\|v_x(T)\|_{L^2}^2 \leq \|v_x(s)\|_{L^2}^2 - \int_s^T \|v_{xx}(\tau)\|_{L^2}^2\,d\tau - \frac{1}{2}\int_s^T\int_{\mathbb{R}} f'(\bar{u})_x v_x(\tau)^2\,dxd\tau. \tag{2.17}$$

Estimate the last integral using (2.16), to obtain

$$\left|\int_s^T\int_{\mathbb{R}} f'(\bar{u})_x v_x^2\,dxd\tau\right| \le M\int_0^{T-s} e^{M\tau}\|v_x(\tau)\|_{L^2}^2\,d\tau \le e^{M(T-s)}\|v_x(s)\|_{L^2}^2.$$

Upon substitution in (2.17),

$$\int_s^T \|v_{xx}(\tau)\|_{L^2}^2\,d\tau \le \tfrac{1}{2}(1+e^{M(T-s)})\|v_x(s)\|_{L^2}^2.$$

Likewise, from (2.16) it is easy to show that

$$\int_s^T\int_{\mathbb{R}} |f'(\bar{u})|v_x(\tau)^2\,dxd\tau \le \frac{m}{M}e^{M(T-s)}\|v_x(s)\|_{L^2}^2,$$

where $m := \sup|f'(\bar{u})|$. Denoting $\mu(t) := \max\left\{\frac{1}{2}(1+e^{Mt}), \frac{m}{N}e^{Mt}\right\}$, we see that both

$$\int_s^T \|v_{xx}(\tau)\|_{L^2}^2 d\tau, \quad \text{and} \quad \int_s^T\int_{\mathbb{R}} |f'(\bar{u})|v_x(\tau)^2\,dxd\tau,$$

are bounded by $\mu(T-s)\|v_x(s)\|_{L^2}^2$. Since the L^2 norm of v_t is decreasing, integrating inequality (2.10) for v_t we obtain

$$\begin{aligned}(T-s)\|v_t(T)\|_{L^2}^2 &\le \int_s^T \|v_t(\tau)\|_{L^2}^2\,d\tau \\ &= \int_s^T \|(Lv)(\tau)\|_{L^2}^2\,d\tau \\ &\lesssim \int_s^T \|v_{xx}(\tau)\|_{L^2}^2\,d\tau + \int_s^T\int_{\mathbb{R}} |f'(\bar{u})|v_x(\tau)^2\,dxd\tau \\ &\lesssim \mu(T-s)\|v_x(s)\|_{L^2}^2.\end{aligned}$$

Choose $T-s\equiv 1$ to finally arrive at

$$\|v_t(t)\|_{L^2}^2 \le \|v_t(1+s)\|_{L^2}^2 \lesssim \mu(1)\|v_x(s)\|_{L^2}^2,$$

for all $t > 1+s$, establishing (2.14). This proves the lemma. □

Lemma 2.8. *Let w be a solution to $w_t = \mathcal{L}^*w$. Then the following decay rate holds*

$$\|w(t)\|_{L^\infty} \lesssim t^{-1/4}\|w(0)\|_{L^2}. \tag{2.18}$$

Proof. Recall that (2.9) – (2.12) hold for w. In particular, by convexity $f'' \ge a > 0$ and (2.12), we have

$$\int_0^t\int_{\mathbb{R}} |\bar{u}_x|w(\tau)^2\,dxd\tau \le a^{-1}\|w(0)\|_{L^2}^2, \tag{2.19}$$

for all $t\ge 0$. Differentiate $w_t = \mathcal{L}^*w$ with respect to x, multiply by w_x and integrate by parts to obtain, for all $t \ge 0$,

$$\frac{1}{2}\frac{d}{dt}\|w_x(t)\|_{L^2}^2 = -\|w_{xx}(t)\|_{L^2}^2 - \frac{3}{2}\int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|w_x(t)^2\,dx - \frac{1}{2}\int_{\mathbb{R}} f'(\bar{u})_{xxx}w(t)^2\,dx.$$

The first two terms on the right hand side have the right sign for decay. We must control the term $-\int f'(\bar{u})_{xxx}w^2\,dx$. For that purpose, use the equation for w and the profile equation to compute

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |\bar{u}_x|w(t)^2\,dx = -\int_{\mathbb{R}} |\bar{u}_x|w_x(t)^2\,dx - \int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|^2w(t)^2\,dx.$$

This provides the cancellation we need, as the decreasing L^2 norm we seek will be that of w_x plus a multiple of $|\bar{u}_x|^{1/2}w$. First note by the smoothness of f and convexity that there exists $A > 0$ such that

$$|f'(\bar{u})_{xxx}| \leq A|\bar{u}_x|^2.$$

This implies

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|w_x(t)\|_{L^2}^2 + \frac{1}{2}Aa^{-1}\int_{\mathbb{R}}|\bar{u}_x|w(t)^2\,dx\right) \\
&= -\|w_{xx}(t)\|_{L^2}^2 - \frac{3}{2}\int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|w_x(t)^2\,dx \;-\frac{1}{2}\int_{\mathbb{R}} f'(\bar{u})_{xxx}w(t)^2\,dx + \\
&\qquad - Aa^{-1}\int_{\mathbb{R}}|\bar{u}_x|w_x(t)^2\,dx - Aa^{-1}\int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|^2 w(t)^2\,dx \\
&\leq J(t) - \frac{A}{2}\int_{\mathbb{R}}|\bar{u}_x|^2 w(t)^2\,dx,
\end{aligned}$$

where

$$J(t) := -\|w_{xx}(t)\|_{L^2}^2 - \frac{3}{2}\int_{\mathbb{R}} f''(\bar{u})|\bar{u}_x|w_x(t)^2\,dx - Aa^{-1}\int_{\mathbb{R}}|\bar{u}_x|w_x(t)^2\,dx \;\leq 0,$$

for all $t \geq 0$. Denoting $\bar{A} = Aa^{-1}$ and defining

$$R(t) := \|w_x(t)\|_{L^2}^2 + \bar{A}\int_{\mathbb{R}}|\bar{u}_x|w(t)^2\,dx,$$

we have thus shown that $R(t)$ is the decaying norm we were looking for, as $dR/dt \leq 0$. Integrating $R(t) \leq R(\tau)$ according to custom with respect to $\tau \in [0,t]$, for fixed $t \geq 0$, and using (2.11) and (2.19), one can estimate

$$tR(t) \leq \int_0^t R(\tau)\,d\tau \leq \tfrac{1}{2}\|w(0)\|_{L^2}^2 + \bar{A}a^{-1}\|w(0)\|_{L^2}^2 \lesssim \|w(0)\|_{L^2}^2.$$

Therefore,

$$\|w_x(t)\|_{L^2} \lesssim t^{-1/2}\|w(0)\|_{L^2},$$

for all $t > 0$ large. By the classical Sobolev inequality and (2.10) we obtain

$$\|w(t)\|_{L^\infty} \lesssim \|w_x(t)\|_{L^2}^{1/2}\|w(t)\|_{L^2}^{1/2} \lesssim t^{-1/4}\|w(0)\|_{L^2},$$

as claimed. □

2.2. **Proof of Proposition 2.3.** If u solves $u_t = Lu$, then its flux transform $v = \mathcal{F}u$ is a solution to $v_t = \mathcal{L}v$. Apply the uniform Sobolev-type inequality (2.3) to v, substituting[1] α by

$$\tilde{\delta}(t) = \frac{1}{Z}\int_{\mathbb{R}} v_x(t)\bar{u}_x\,dx,$$

(with $Z = \int_{\mathbb{R}}|\bar{u}_x|^2\,dx$), and the Poincaré-type inequality (2.6) (with $p = 2$), to obtain

$$\begin{aligned}
\|v(t)\|_{L^\infty}^2 &\lesssim \|v(t)\|_{L^2}\|v_x - \tilde{\delta}(t)\bar{u}_x\|_{L^2} \\
&\lesssim \|v(t)\|_{L^2}\|(\mathcal{F}v_x)(t)\|_{L^2} = \|v(t)\|_{L^2}\|v_t(t)\|_{L^2}.
\end{aligned}$$

Then, using the estimate (2.13), we arrive at

$$\|v(t)\|_{L^\infty}^2 \lesssim (t-s)^{-1/2}\|v(s)\|_{L^2}^2, \tag{2.20}$$

[1] Here the uniformity of inequality (2.3) in $\alpha \in \mathbb{R}$ plays a crucial role.

for all $t \geq s+2$. For fixed $T>0$ define the linear functional $\mathcal{A} : L^2 \to \mathbb{R}$ as

$$\mathcal{A}g := \int_{\mathbb{R}} v(T) g \, dx,$$

for all $g \in L^2$, with norm

$$\|\mathcal{A}\| = \sup_{\|g\|_{L^2}=1} \left| \int_{\mathbb{R}} v(T) g \, dx \right|.$$

For every $g \in L^2$ with $\|g\|_{L^2} = 1$, we can always solve the equation $w_t = -\mathcal{L}^* w = -w_{xx} - (f'(\bar{u})w)_x$ on $t \in [0,T]$ "backwards" in time, with $w(T) = g$. Thus, by Remark 2.5

$$\left| \int_{\mathbb{R}} v(T) g \, dx \right| = \left| \int_{\mathbb{R}} v(T) w(T) \, dx \right| = \left| \int_{\mathbb{R}} v(0) w(0) \, dx \right| \leq \|v(0)\|_{L^1} \|w(0)\|_{L^\infty},$$

for all $T>0$. Making the change of variables $\tilde{w}(x,t) = w(x, T-t)$ we readily see that $\tilde{w}$ satisfies $\tilde{w}_t = \mathcal{L}^* \tilde{w}$ with $\tilde{w}(0) = g$, and we can use estimate (2.18), yielding

$$\|w(0)\|_{L^\infty} = \|\tilde{w}(T)\|_{L^\infty} \lesssim T^{-1/4} \|g\|_{L^2}.$$

Thus,

$$\|v(T)\|_{L^2} = \sup_{\|g\|_{L^2}=1} \left| \int_{\mathbb{R}} v(T) g \, dx \right| \leq \|v(0)\|_{L^1} \|w(0)\|_{L^\infty} \lesssim T^{-1/4} \|v(0)\|_{L^1},$$

for all $T>0$. Choose $s = t/2$ in (2.20), and apply the last estimate with $T = t/2$, to get

$$\|v(t)\|_{L^\infty} \lesssim (t/2)^{-1/4} \|v(t/2)\|_{L^2} \lesssim t^{-1/2} \|v(0)\|_{L^1}, \tag{2.21}$$

which corresponds to the optimal decay rate for solutions to the integrated equation.

To prove the decay rate (2.8) for the original solution to the unintegrated equation $u_t = Lu$, apply the Poincaré-type inequality again (now with $p=\infty$) together with (2.21),

$$\|u(t) - \delta(t)\bar{u}_x\|_{L^\infty} \lesssim \|v(t)\|_{L^\infty} \lesssim t^{-1/2} \|v(0)\|_{L^1} \lesssim t^{-1/2} \|u(0)\|_{W^{1,1}}.$$

This completes the proof. $\square$

ACKNOWLEDGEMENTS

The motivation to prove inequality (1.3) originated during my research on viscous shock waves towards my doctoral dissertation [6], written under the direction of Prof. Jonathan Goodman. I thank him for many illuminating discussions, useful observations, and his encouragement. I am also grateful to Prof. Stefan Müller for suggesting the counterexample in Remark 1.7.

REFERENCES

[1] I.M. GELFAND, Some problems in the theory of quasi-linear equations, *Amer. Math. Soc. Transl.*, **29**(2) (1963), 295–381.

[2] J. GOODMAN, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Rational Mech. Anal.*, **95** (1986), 325–344.

[3] J. GOODMAN, Stability of viscous scalar shock fronts in several dimensions, *Trans. Amer. Math. Soc.*, **311**(2) (1989), 683–695.

[4] J. GOODMAN, Remarks on the stability of viscous shock waves, in *Viscous Profiles and Numerical Methods for Shock Waves*, M. Shearer, ed., SIAM, Philadelphia, PA, 1991, 66–72.

[5] P.D. LAX, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.*, **10** (1957), 537–566.

[6] R.G. PLAZA, *On the Stability of Shock Profiles*, PhD thesis, New York University, 2003.

[7] D. SERRE, *Systems of Conservation Laws 1: Hyperbolicity, entropies, shock waves*, Cambridge University Press, 1999.

[8] J. SMOLLER, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, Second ed., 1994.

[9] K. ZUMBRUN, Refined wave-tracking and nonlinear stability of viscous Lax shocks, *Methods Appl. Anal.*, **7**(4) (2000), 747–768.