Journal of Inequalities in Pure and Applied Mathematics

HARDY TYPE INEQUALITIES FOR INTEGRAL TRANSFORMS ASSOCIATED WITH A SINGULAR SECOND ORDER DIFFERENTIAL OPERATOR

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ISSN (electronic): 1443-5756
211-04

volume 7, issue 1, article 38, 2006.

Received 04 November, 2004; accepted 17 January, 2006.

Communicated by: S.S. Dragomir

Abstract

We consider a singular second order differential operator Δ defined on $]0,\infty[$. We give nice estimates for the kernel which intervenes in the integral transform of the eigenfunction of Δ. Using these results, we establish Hardy type inequalities for Riemann-Liouville and Weyl transforms associated with the operator Δ.

2000 Mathematics Subject Classification: 44Xxx, 44A15.
Key words: Hardy type inequalities, Integral transforms, Differential operator.

Contents

1. Introduction

In this paper we consider the differential operator on $]0,\infty[$, defined by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx} + \rho^2,$$

where A is a real function defined on $[0,\infty[$, satisfying

$$A(x) = x^{2\alpha+1}B(x); \alpha > -\frac{1}{2}$$

and B is a positive, even C^∞ function on $\mathbb{R}$ such that $B(0) = 1$, and $\rho \geq 0$. We suppose that the function A satisfies the following assumptions

i) $A(x)$ is increasing, and $\lim_{+\infty} A(x) = +\infty$.

ii) $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{+\infty} \frac{A'(x)}{A(x)} = 2\rho$.

iii) there exists a constant $\delta > 0$, satisfying

$$\begin{cases} \frac{B'(x)}{B(x)} = 2\rho - \frac{2\alpha+1}{x} + e^{-\delta x}F(x), & \text{for } \rho > 0, \\ \frac{B'(x)}{B(x)} = e^{-\delta x}F(x), & \text{for } \rho = 0, \end{cases}$$

where F is C^∞ on $]0,\infty[$, bounded together with its derivatives on the interval $[x_0,\infty[$, $x_0 > 0$.

This operator plays an important role in harmonic analysis, for example, many special functions (orthogonal polynomials,...) are eigenfunctions of operators of the same type as Δ.

The Bessel and Jacobi operators defined respectively by

$$\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}; \quad \alpha > -\frac{1}{2}$$

and

$$\Delta_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\frac{d}{dx} + (\alpha+\beta+1)^2,$$
$$\alpha \geq \beta > -\frac{1}{2},$$

are of the type Δ, with

$$A(x) = x^{2\alpha+1}; \quad \rho = 0,$$

respectively

$$A(x) = \sinh^{2\alpha+1} x \cosh^{2\beta+1} x; \quad \rho = \alpha+\beta+1.$$

Also, the radial part of the Laplacian - Betrami operator on the Riemannian symmetric space, is of type Δ.

The operator Δ has been studied from many points of view ([1], [7], [13], [14], [15], [16]).In particular, K. Trimèche has proved in [15] that the differential equation

$$\Delta u(x) = -\lambda^2 u(x), \quad \lambda \in \mathbb{C}$$

has a unique solution on $[0, \infty[$, satisfying the conditions $u(0) = 1$, $u'(0) = 0$. We extend this solution on $\mathbb{R}$ by parity and we denote it by φ_λ. He has also proved that the eigenfunction φ_λ has the following Mehler integral representation

$$\varphi_\lambda(x) = \int_0^x k(x,t)\cos\lambda t dt,$$

where the kernel $k(x,t)$ is defined by

$$k(x,t) = 2h(x,t) + C_\alpha A^{-\frac{1}{2}}(x)x^{\frac{1}{2}-\alpha}(x^2 - t^2)^{\alpha-\frac{1}{2}}, \quad 0 < t < x$$

with

$$h(x,t) = \frac{1}{\Pi}\int_0^\infty \psi(x,\lambda)\cos(\lambda t)d\lambda,$$

$$C_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\Pi}\Gamma(\alpha+\frac{1}{2})},$$

and

$$\forall\, \lambda \in \mathbb{R},\ x \in \mathbb{R}; \quad \psi(x,\lambda) = \varphi_\lambda(x) - x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)j_\alpha(\lambda x),$$

where

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}$$

and J_α is the Bessel function of the first kind and order α ([8]).

The Riemann - Liouville and Weyl transforms associated with the operator Δ are respectively defined, for all non-negative measurable functions f by

$$\mathcal{R}(f)(x) = \int_0^x k(x,t)f(t)dt$$

and

$$\mathcal{W}(f)(t) = \int_t^\infty k(x,t) f(x) A(x) dx.$$

These operators have been studied on regular spaces of functions. In particular, in [15], the author has proved that the Riemann-Liouville transform $\mathcal{R}$ is an isomorphism from $\mathcal{E}^*(\mathbb{R})$ (the space of even infinitely differentiable functions on $\mathbb{R}$) onto itself, and that the Weyl transform $\mathcal{W}$ is an isomorphism from $\mathcal{D}_*(\mathbb{R})$ (the space of even infinitely differentiable functions on $\mathbb{R}$ with compact support) onto itself.

The Weyl transform has also been studied on Schwarz space $S_*(\mathbb{R})$ ([13]).

Our purpose in this work is to study the operators $\mathcal{R}$ and $\mathcal{W}$ on the spaces $L^p([0,\infty[, A(x)dx)$ consisting of measurable functions f on $[0,\infty[$ such that

$$||f||_{p,A} = \left(\int_0^\infty |f(x)|^p \, A(x) \, dx \right)^{\frac{1}{p}} < \infty; \quad 1 < p < \infty.$$

The main results of this paper are the following Hardy type inequalities

- For $\rho > 0$ and $p > \max(2, 2\alpha+2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0,\infty[, A(x)dx)$,

(1.1) $$||\mathcal{R}(f)||_{p,A} \le C_{p,\alpha} ||f||_{p,A}$$

and for all $g \in L^{p'}([0,\infty[, A(x)dx)$,

(1.2) $$\left\| \frac{1}{A(x)} \mathcal{W}(g) \right\|_{p',A} \le C_{p,\alpha} ||g||_{p',A},$$

where $p' = \frac{p}{p-1}$.

- For $\rho = 0$ and $p > 2\alpha + 2$ there exists a positive constant $C_{p,\alpha}$ such that (1.1) and (1.2) hold.

In ([5], [6]) we have obtained (1.1) and (1.2) in the cases

$$A(x) = x^{2\alpha+1}, \quad \alpha > -\frac{1}{2}$$

respectively

$$A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x); \qquad \alpha \geq \beta > -\frac{1}{2}.$$

This paper is arranged as follows. In the first section, we recall some properties of the eigenfunctions of the operator Δ. The second section deals with the study of the behavior of the kernel $h(x,t)$. In the third section, we introduce the following integral operator

$$T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t)\nu(t)dt$$

where

- φ is a measurable function defined on $]0,1[$,
- ν is a measurable non-negative function on $]0,\infty[$ locally integrable.

Then we give the criteria in terms of the function φ to obtain the following Hardy type inequalities for T_φ,

for all real numbers, $1 < p \leq q < \infty$, there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f and g we have

$$\left(\int_0^\infty (T_\varphi(f(x)))^q \, \mu(x)dx\right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \, \nu(x)dx\right)^{\frac{1}{p}}.$$

In the fourth section, we use the precedent results to establish the Hardy type inequalities (1.1) and (1.2) for the operators $\mathcal{R}$ and $\mathcal{W}$.

2. The Eigenfunctions of the Operator $\triangle$

As mentioned in the introduction, the equation

$$\Delta u(x) = -\lambda^2 u(x), \quad \lambda \in \mathbb{C} \tag{2.1}$$

has a unique solution on $[0,\infty[$, satisfying the conditions $u(0) = 1$, $u'(0) = 0$. We extend this solution on $\mathbb{R}$ by parity and we denote it φ_λ. Equation (2.1) possesses also two solutions $\phi_{\mp\lambda}$ linearly independent having the following behavior at infinity $\phi_{\mp\lambda}(x) \sim e^{(\mp\lambda-\rho)x}$. Then there exists a function c such that

$$\varphi_\lambda(x) = c(\lambda)\phi_\lambda(x) + c(-\lambda)\phi_{-\lambda}(x).$$

In the case of the Bessel operator Δ_α, the functions φ_λ, ϕ_λ and c are given respectively by

$$j_\alpha(\lambda x) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(\lambda x)}{(\lambda x)^\alpha}, \ \lambda x \neq 0, \tag{2.2}$$

$$k_\alpha(i\lambda x) = 2^\alpha \Gamma(\alpha+1)\frac{K_\alpha(i\lambda x)}{(i\lambda x)^\alpha}, \quad \lambda x \neq 0,$$

$$c(\lambda) = 2^\alpha \Gamma(\alpha+1) e^{-i(\alpha+\frac{1}{2})\frac{\Pi}{2}} \lambda^{-(\alpha+\frac{1}{2})}, \quad \lambda > 0,$$

where J_α and K_α are respectively the Bessel function of first kind and order α, and the MacDonald function of order α.

In the case of the Jacobi operator $\Delta_{\alpha,\beta}$, the functions $\varphi_\lambda, \phi_\lambda$ and c are respectively

$$\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), (\alpha+1), -\sinh^2(x)\right), \quad x \geq 0, \lambda \in \mathbb{C},$$

$$\phi_\lambda^{\alpha,\beta}(x) = (2\sinh x)^{(i\lambda-\rho)}\ {}_2F_1\left(\frac{1}{2}(\rho-2\alpha-i\lambda), \frac{1}{2}(\rho-i\lambda), 1-i\lambda, (\sinh x)^{-2}\right);$$
$$x>0,\ \lambda \in \mathbb{C}-(-i\mathbb{N})$$

and

$$c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(\rho-i\lambda)\right)\Gamma\left(\frac{1}{2}(\alpha-\beta+1+i\lambda)\right)}$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

From ([1], [2], [15], [16]) we have the following properties:

i) We have:

- For $\rho = 0$: $\quad \forall\ x \geq 0,\ \varphi_0(x) = 1,$
- For $\rho \geq 0$: there exists a constant $k > 0$ such that

$$\forall x \geq 0, \quad e^{-\rho x} \leq \varphi_0(x) \leq k(1+x)e^{-\rho x}. \tag{2.3}$$

ii) For $\lambda \in \mathbb{R}$ and $x \geq 0$ we have

$$|\varphi_\lambda(x)| \leq \varphi_0(x). \tag{2.4}$$

iii) For $\lambda \in \mathbb{C}$such that $|\Im\lambda| \leq \rho$ and $x \geq 0$ we have $|\varphi_\lambda(x)| \leq 1$.

iv) We have the integral representation of Mehler type,

$$\forall x > 0,\ \forall \lambda \in \mathbb{C}, \qquad \varphi_\lambda(x) = \int_0^x k(x,t)\cos(\lambda t)dt, \tag{2.5}$$

where $k(x,\cdot)$ is an even positive C^∞ function on $]-x,x[$ with support in $[-x,x]$.

v) For $\lambda \in \mathbb{R}$, we have $c(-\lambda) = \overline{c(\lambda)}$.

vi) The function $|c(\lambda)|^{-2}$ is continuous on $[0, +\infty[$ and there exist positive constants $k,\ k_1,\ k_2$ such that

- If $\rho \geq 0\ : \forall \lambda \in \mathbb{C},\ |\lambda| > k$
$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$
- If $\rho > 0\ : \forall \lambda \in \mathbb{C},\ |\lambda| \leq k$
$$k_1|\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^2,$$
- If $\rho = 0, \alpha > 0 : \forall \lambda \in \mathbb{C},\ |\lambda| \leq k$
$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}. \tag{2.6}$$

Now, let us put
$$v(x) = A^{\frac{1}{2}}(x)u(x).$$
The equation (2.1) becomes
$$v''(x) - (G(x) - \lambda^2)v(x) = 0,$$
where
$$G(x) = \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)}\right)' - \rho^2.$$
Let
$$\xi(x) = G(x) + \frac{\frac{1}{4} - \alpha^2}{x^2}.$$

Thus from hypothesis of the function A, we deduce the following results for the function ξ.

Proposition 2.1.

1. *The function* ξ *is continuous on* $]0,\infty[$.
2. *There exist* $\delta > 0$ *and* $a \in \mathbb{R}$ *such that the function* ξ *satisfies*

$$\xi(x) = \frac{a}{x^2} + \exp(-\delta x) F_1(x),$$

where F_1 *is* C^∞ *on* $]0,\infty[$, *bounded together with all its derivatives on the interval* $[x_0,\infty[$,$x_0 > 0$.

Proposition 2.2 ([15]). *Let*

$$\psi(x,\lambda) = \varphi_\lambda(x) - x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x) j_\alpha(\lambda x), \tag{2.7}$$

where j_α *is defined by (2.2).*

Then there exist positive constants C_1 *and* C_2 *such that*

$$\forall x > 0, \forall \lambda \in \mathbb{R}^*, \ |\psi(x,\lambda)| \le C_1 A^{\frac{-1}{2}}(x) \tilde{\xi}(x) \lambda^{-\alpha-\frac{3}{2}} \exp\left(C_2 \frac{\tilde{\xi}(x)}{\lambda}\right), \tag{2.8}$$

with

$$\tilde{\xi}(x) = \int_0^x |\xi(r)| dr.$$

The kernel $k(x,t)$ *given by the relation (2.5) can be written*

$$k(x,t) = 2h(x,t) + C_\alpha A^{\frac{-1}{2}}(x) x^{\frac{1}{2}-\alpha} (x^2 - t^2)^{\alpha-\frac{1}{2}}, \qquad 0 < t < x, \tag{2.9}$$

where

$$h(x,t) = \frac{1}{\Pi}\int_0^{\infty} \psi(x,t)\cos(\lambda t)d\lambda, \tag{2.10}$$

$$C_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\Pi}\Gamma(\alpha+\frac{1}{2})},$$

and $\psi(x,\lambda)$ is the function defined by the relation (2.7).

Since the Riemann-Liouville and Weyl transforms associated with the operator Δ are given by the kernel k, then, we need some properties of this function. But from the relation (2.9) it suffices to study the kernel h.

3. The Kernel h

In this section we will study the behaviour of the kernel h.

Lemma 3.1. *For any real $a > 0$ there exist positive constants $C_1(a)$,$C_2(a)$ such that for all $x \in [0, a]$,*

$$C_1(a)x^{2\alpha+1} \leq A(x) \leq C_2(a)x^{2\alpha+1}.$$

From Proposition 1, and [16], we deduce the following lemma.

Lemma 3.2. *There exist positive constants a_1, a_2, C_1 and C_2 such that for $|\lambda| > a_1$*

$$\varphi_\lambda(x) = \begin{cases} C(\alpha)x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)\left(j_\alpha(\lambda x) + O(\lambda x)\right) & \text{for} \quad |\lambda x| \leq a_2 \\ C(\alpha)\lambda^{-(\alpha+\frac{1}{2})}A^{-\frac{1}{2}}(x)\left(C_1 \exp -i\lambda x + C_2 \exp i\lambda x\right) & \\ \qquad \times \left(1 + O(\lambda^{-1}) + O((\lambda x)^{-1})\right) & \\ & \text{for} \quad |\lambda x| > a_2, \end{cases}$$

where

$$C(\alpha) = \Gamma(\alpha+1)A^{\frac{1}{2}}(1)\exp\left(-\frac{1}{2}\int_0^1 B(t)dt\right).$$

Theorem 3.3. *For any $a > 0$, there exists a positive constant $C_1(\alpha, a)$ such that*

$$\forall\, 0 < t < x \leq a; \quad |h(x,t)| \leq C_1(\alpha, a)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).$$

Proof. By (2.10) we have for $0 < t < x$,

$$\begin{aligned} |h(t,x)| &\leq \frac{1}{\Pi}\int_0^{\infty}|\psi(x,\lambda)|d\lambda \\ &= \frac{1}{\Pi}\int_0^{a_1}|\psi(x,\lambda)|d\lambda + \frac{1}{\Pi}\int_{a_1}^{\infty}|\psi(x,\lambda)|d\lambda \\ &= I_1(x) + I_2(x), \qquad (3.1) \end{aligned}$$

where a_1 is the constant given by Lemma 3.2.

We put

$$f_\lambda(x) = x^{\frac{1}{2}-\alpha}A^{\frac{1}{2}}(x)|\psi(x,\lambda)|, \qquad 0 < x < a, \quad \lambda \in \mathbb{R}.$$

From Proposition 2.2 the function

$$(x,\lambda) \longrightarrow f_\lambda(x)$$

is continuous on $[0,a]\times[0,a_1]$. Then

$$I_1(x) = \frac{1}{\Pi}\int_0^{a_1}|\psi(x,\lambda)|d\lambda \leq C_\alpha^1 x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x), \qquad (3.2)$$

where

$$C_\alpha^1 = \frac{a_1}{\Pi}\sup_{(x,\lambda)\in[0,a]\times[0,a_1]}|f_\lambda(x)|.$$

Let us study the second term

$$I_2(x) = \frac{1}{\Pi}\int_{a_1}^{\infty}|\psi(x,\lambda)|d\lambda.$$

i) Suppose $-\frac{1}{2} < \alpha \leq \frac{1}{2}$. From inequality (2.8) we get

$$I_2(x) \leq \frac{C_1}{\Pi} A^{-\frac{1}{2}}(x)\tilde{\xi}(x) \int_{a_1}^{\infty} \lambda^{-\alpha-\frac{3}{2}} \exp\left(C_2 \frac{\tilde{\xi}(x)}{|\lambda|}\right) d\lambda$$
$$\leq \tilde{C}_1 A^{-\frac{1}{2}}(x)\tilde{\xi}(x) \exp\left(C_2 \frac{\tilde{\xi}(x)}{a_1}\right) x^{\alpha-\frac{1}{2}}.$$

Since $\tilde{\xi}$ is bounded on $[0, \infty[$, we deduce that

$$I_2(x) \leq C_{2,\alpha} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x). \tag{3.3}$$

This completes the proof in the case $-\frac{1}{2} < \alpha \leq \frac{1}{2}$.
ii) Suppose now that $\alpha > \frac{1}{2}$.

- Let a_1, a_2 be the constants given in Lemma 3.2. From this lemma we deduce that there exists a positive constant $C_1(\alpha)$ such that

$$\forall x > \frac{a_2}{a_1},\ \lambda > a_1;\ |\varphi_\lambda(x)| \leq C_1(\alpha) A^{-\frac{1}{2}}(x) \lambda^{-(\alpha+\frac{1}{2})}. \tag{3.4}$$

On the other hand, the function

$$s \longrightarrow s^{\alpha+\frac{1}{2}} j_\alpha(s)$$

is bounded on $[0, \infty[$.

Then from equality (2.7), we have, for $x > \frac{a_2}{a_1}$

$$\frac{1}{\Pi} \int_{a_1}^{\infty} |\psi(x, \lambda)| d\lambda \tag{3.5}$$

$$
\begin{aligned}
&\le \frac{1}{\Pi}\int_{a_1}^{\infty}|\varphi_\lambda(x)|d\lambda + \frac{1}{\Pi}x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_1}^{\infty}|j_\alpha(\lambda x)|d\lambda \\
&\le \frac{C_1(\alpha)}{\Pi}A^{-\frac{1}{2}}(x)\int_{a_1}^{\infty}\lambda^{-(\alpha+\frac{1}{2})}d\lambda + \frac{1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_2}^{\infty}|j_\alpha(u)|du \\
&\le \frac{C_1(\alpha)}{\left(\alpha-\frac{1}{2}\right)\Pi}A^{-\frac{1}{2}}(x)\left(\frac{1}{a_1}\right)^{(\alpha-\frac{1}{2})} + \frac{1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_2}^{\infty}|j_\alpha(u)|du \\
&\le \frac{C_1(\alpha)}{\left(\alpha-\frac{1}{2}\right)\Pi}A^{-\frac{1}{2}}(x)\left(\frac{x}{a_2}\right)^{(\alpha-\frac{1}{2})} + \frac{1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_2}^{\infty}|j_\alpha(u)|du \\
&\le C_2(\alpha)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x),
\end{aligned}
$$

where

$$C_2(\alpha) = \frac{C_1(\alpha)}{\left(\alpha-\frac{1}{2}\right)\Pi}(a_2)^{(-\alpha+\frac{1}{2})} + \frac{1}{\Pi}\int_{a_2}^{\infty}|j_\alpha(u)|du.$$

- $0 < x < \frac{a_2}{a_1}$. From Lemma 3.2 and the fact that

$$\forall x \in \mathbb{R}, \quad |j_\alpha(\lambda x)| \le 1$$

we deduce that there exists a positive constant $M_1(\alpha)$ such that

$$\forall\, 0 < x < \frac{a_2}{a_1},\ 0 \le \lambda \le \frac{a_2}{x} \qquad |\psi(x,\lambda)| \le M_1(\alpha)x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x).$$

This involves

$$\begin{aligned}\frac{1}{\Pi}\int_{a_1}^{\frac{a_2}{x}}|\psi(x,\lambda)|d\lambda &\leq \frac{M_1(\alpha)}{\Pi}x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)\left(\frac{a_2}{x}-a_1\right)\\ &\leq \frac{a_2}{\Pi}M_1(\alpha)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).\end{aligned} \tag{3.6}$$

Moreover

$$\begin{aligned}\frac{1}{\Pi}\int_{\frac{a_2}{x}}^{\infty}|\psi(x,\lambda)|d\lambda &\leq \frac{C_1(\alpha)}{\Pi}A^{-\frac{1}{2}}(x)\int_{\frac{a_2}{x}}^{\infty}\lambda^{-(\alpha+\frac{1}{2})}d\lambda\\ &\qquad + \frac{1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_2}^{\infty}|j_\alpha(u)|du\\ &\leq \frac{C_1(\alpha)}{\left(\alpha-\frac{1}{2}\right)\Pi}A^{-\frac{1}{2}}(x)\left(\frac{1}{a_2}\right)^{(\alpha-\frac{1}{2})}\\ &\qquad + \frac{1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_{a_2}^{\infty}|j_\alpha(u)|du\\ &\leq C_2(\alpha)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we deduce that

$$\forall\ 0<x<\frac{a_2}{a_1};\quad \frac{1}{\Pi}\int_{a_1}^{\infty}|\psi(x,\lambda)|d\lambda \leq M_2(\alpha)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x) \tag{3.8}$$

where

$$M_2(\alpha)=\frac{a_2}{\Pi}M_1(\alpha)+C_2(\alpha).$$

From (3.5), (3.8) it follows that

$$\forall\ 0 < x < a; \quad I_2(x) \le M_2(\alpha) x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

This completes the proof. □

In order to provide some estimates for the kernel h for later use, we need the following lemmas

Lemma 3.4.

i) For $\rho > 0$*, we have*

$$A(x) \sim e^{2\rho x}, \quad (x \longrightarrow +\infty)$$

ii) For $\rho = 0$*, we have*

$$A(x) \sim x^{2\alpha+1}, \ (x \longrightarrow +\infty).$$

This lemma can be deduced from hypothesis of the function A.

Lemma 3.5 ([2]). *For* $\rho = 0$ *and* $\alpha > \frac{1}{2}$ *there exist two positive constants* $D_1(\alpha)$ *and* $D_2(\alpha)$ *satisfying*

i)

$$|\varphi_\lambda(x)| \le D_1(\alpha) x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x), \quad x > 0,\ \lambda \ge 0.$$

ii)

$$|\varphi_\lambda(x)| \leq D_2(\alpha)|c(\lambda)|A^{-\frac{1}{2}}(x), \quad x > 1,\ \lambda x > 1$$

where

$$\lambda \longrightarrow c(\lambda)$$

is the spectral function given by (2.6).

Using previous results we will give the behavior of the function h for large values of the variable x

Theorem 3.6. *For* $\rho = 0$*,* $\alpha > \frac{1}{2}$*, and* $a > 0$ *there exists a positive constant* $C_{\alpha,a}$ *such that*

$$0 < t < x, \quad x > a, \quad |h(x,t)| \leq C_{\alpha,a} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

Proof. We have

$$h(x,t) = \frac{1}{\Pi}\int_0^\infty |\psi(x,\lambda)| \cos(\lambda t) d\lambda,$$

then

$$\begin{aligned} |h(x,t)| &\leq \frac{1}{\Pi}\int_0^\infty |\psi(x,\lambda)| d\lambda \\ &= \frac{1}{\Pi}\int_0^1 |\psi(x,\lambda)| d\lambda + \frac{1}{\Pi}\int_1^\infty |\psi(x,\lambda)| d\lambda. \end{aligned} \tag{3.9}$$

From Proposition 2.2 and the fact that $\alpha > \frac{1}{2}$ we get

$$\frac{1}{\Pi}\int_1^\infty |\psi(x,\lambda)| d\lambda \leq \frac{C_1}{\Pi} A^{-\frac{1}{2}}(x)\tilde{\xi}(x) \exp\left(C_2(\tilde{\xi}(x)\right) \int_1^\infty \lambda^{-\alpha-\frac{3}{2}} d\lambda.$$

Since the function $\tilde{\xi}$ is bounded on $[0,\infty[$, we deduce that there exists $d_\alpha > 0$ verifying

$$\frac{1}{\Pi}\int_1^\infty |\psi(x,\lambda)|d\lambda \le d_\alpha x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x). \tag{3.10}$$

On the other hand, we have

$$\frac{1}{\Pi}\int_0^1 |\psi(x,\lambda)|d\lambda \le \frac{1}{\Pi}\int_0^1 |\varphi_\lambda(x)|d\lambda + \frac{1}{\Pi}x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)\int_0^1 |j_\alpha(\lambda x)|d\lambda.$$

However,

$$\frac{1}{\Pi}\int_0^1 |\varphi_\lambda(x)|d\lambda = \frac{1}{\Pi}\int_0^{\frac{1}{x}} |\varphi_\lambda(x)|d\lambda + \frac{1}{\Pi}\int_{\frac{1}{x}}^1 |\varphi_\lambda(x)|d\lambda$$

from Lemma 3.5 i) we have

$$\frac{1}{\Pi}\int_0^{\frac{1}{x}} |\varphi_\lambda(x)|d\lambda \le \frac{C_1}{\Pi}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x). \tag{3.11}$$

Furthermore from Lemma 3.5 ii) and the relation (2.6) it follows that there exists $d_2(\alpha) > 0$ such that

$$\begin{aligned}\frac{1}{\Pi}\int_{\frac{1}{x}}^1 |\varphi_\lambda(x)|d\lambda &\le \frac{d_2(\alpha)}{\Pi}A^{-\frac{1}{2}}(x)\int_{\frac{1}{x}}^1 \lambda^{-(\alpha+\frac{1}{2})}d\lambda \\ &\le \frac{d_2(\alpha)}{\Pi}A^{-\frac{1}{2}}(x)\int_{\frac{1}{x}}^\infty \lambda^{-(\alpha+\frac{1}{2})}d\lambda \\ &\le \frac{d_2(\alpha)}{\Pi(\alpha-\frac{1}{2})}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x). \end{aligned} \tag{3.12}$$

The theorem follows from the relations (3.9), (3.10), (3.11) and (3.12). ☐

Theorem 3.7. *For* $\rho > 0$ *and* $a > 1$ *there exists a positive constant* $C_{\alpha,a}$ *such that*

$$\forall\, 0 < t < x;\ x \geq a; \quad |h(x,t)| \leq C_2(\alpha,a)x^{\gamma}A^{-\frac{1}{2}}(x),$$

where $\gamma = \max\left(1, \alpha + \frac{1}{2}\right)$.

Proof. This theorem can be obtained in the same manner as Theorem 3.6, using the properties (2.3) and (2.4). ☐

4. Hardy Type Operators T_φ

In this section, we will define a class of integral operators and we recall some of their properties which we use in the next section to obtain the main results of this paper.

Let

$$\varphi :]0,1[\quad \longrightarrow \quad]0,\infty[$$

be a measurable function, then we associate the integral operator T_φ defined for all non-negative measurable functions f by

$$\forall x > 0; \qquad T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t)\nu(t)dt$$

where

- ν is a measurable non negative function on $]0,\infty[$ such that

$$\forall a > 0, \qquad \int_0^a \nu(t)dt < \infty \tag{4.1}$$

  and

- μ is a non-negative function on $]0,\infty[$ satisfying

$$\forall\ 0 < a < b, \qquad \int_a^b \mu(t)dt < \infty. \tag{4.2}$$

These operators have been studied by many authors. In particular, in [5], see also ([6], [10], [11]), we have proved the following results.

Theorem 4.1. *Let p, q be two real numbers such that*

$$1 < p \leq q < \infty.$$

Let ν and μ be two measurable non-negative functions on $]0,\infty[$, satisfying (4.1) and (4.2). Lastly, suppose that the function

$$\varphi :]0,1[\quad \longrightarrow \quad]0,\infty[$$

is continuous non increasing and satisfies

$$\forall x, y \in]0,1[, \quad \varphi(xy) \leq D(\varphi(x) + \varphi(y))$$

where D is a positive constant. Then the following assertions are equivalent

1. *There exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f:*

$$\left(\int_0^\infty (T_\varphi(f)(x))^q \mu(x)dx\right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x)dx\right)^{\frac{1}{p}}.$$

2. *The functions*

$$F(r) = \left(\int_r^\infty \mu(x)dx\right)^{\frac{1}{q}} \left(\int_0^r \left(\varphi\left(\frac{x}{r}\right)\right)^{p'} \nu(x)dx\right)^{\frac{1}{p'}}$$

and

$$G(r) = \left(\int_r^\infty \left(\varphi\left(\frac{r}{x}\right)\right)^{q} \mu(x)dx\right)^{\frac{1}{q}} \left(\int_0^r \nu(x)dx\right)^{\frac{1}{p'}}$$

are bounded on $]0,\infty[$, where $p' = \frac{p}{p-1}$.

Theorem 4.2. *Let p and q be two real numbers such that*

$$1 < p \leq q < \infty$$

and μ, ν two measurable non-negative functions on $]0,\infty[$, satisfying the hypothesis of Theorem 4.1.

Let

$$\varphi :]0,1[\quad \longrightarrow \quad]0,\infty[$$

be a measurable non-decreasing function.

If there exists $\beta \in [0,1]$ such that the function

$$r \longrightarrow \left(\int_r^\infty \left(\varphi\left(\frac{r}{x}\right)\right)^{\beta q} \mu(x)dx\right)^{\frac{1}{q}} \left(\int_0^r \left(\varphi\left(\frac{x}{r}\right)\right)^{p'(1-\beta)} \nu(x)dx\right)^{\frac{1}{p'}}$$

is bounded on $]0,\infty[$, then there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f, we have

$$\left(\int_0^\infty \left(T_\varphi\left(f(x)\right)\right)^q \mu(x)dx\right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x)dx\right)^{\frac{1}{p}}$$

where $p' = \frac{p}{p-1}$.

The last result that we need is:

Corollary 4.3. *With the hypothesis of Theorem 4.1 and $\varphi = 1$, the following assertions are equivalent:*

1. *there exists a positive constant* $C_{p,q}$ *such that for all non-negative measurable functions* f *we have*

$$\left(\int_0^\infty (\mathcal{H}(f)(x))^q \mu(x)dx\right)^{\frac{1}{q}} \leq C_{p,q}\left(\int_0^\infty (f(x))^p \nu(x)dx\right)^{\frac{1}{p}},$$

2. *The function*

$$I(r) = \left(\int_r^\infty \mu(x)dx\right)^{\frac{1}{q}} \left(\int_0^r \nu(x)dx\right)^{\frac{1}{p'}}$$

   is bounded on $]0,\infty[$,

where $\mathcal{H}$ *is the Hardy operator defined by*

$$\forall x > 0, \quad \mathcal{H}(f)(x) = \int_0^x f(t)\nu(t)dt.$$

5. The Riemann - Liouville and Weyl Transforms Associated with the Operator $\triangle$

This section deals with the proof of the Hardy type inequalities (1.1) and (1.2) mentioned in the introduction.

We denote by

- $L^p\left([0,\infty[, A(x)dx\right);\ 1 < p < \infty$, the space of measurable functions on $[0,\infty[$, satisfying
$$||f||_{p,A} = \left(\int_0^\infty (f(x))^p A(x)dx\right)^{\frac{1}{p}} < \infty.$$
- $\mathcal{R}_0$ the operator defined for all non-negative measurable functions f by
$$\forall x > 0, \quad \mathcal{R}_0(f)(x) = \int_0^x h(x,t)f(t)dt,$$
where h is the kernel studied in the third section.
- $\mathcal{R}_1$ the operator defined for all non-negative measurable functions f by
$$\forall x > 0, \quad \mathcal{R}_1(f)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\Pi}\Gamma\left(\alpha+\frac{1}{2}\right)} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_0^x (x^2-t^2)^{\alpha-\frac{1}{2}} f(t)dt.$$

Definition 5.1.

1. *The Riemann-Liouville transform associated with the operator Δ is defined for all non-negative measurable functions f on $]0,\infty[$ by*

$$\mathcal{R}(f)(x) = \int_0^x k(x,t)f(t)dt.$$

2. *The Weyl transform associated with operator Δ is defined for all non-negative measurable functions f by*

$$\mathcal{W}(f)(t) = \int_t^\infty k(x,t)f(x)A(x)dx$$

where k is the kernel given by the relation (2.5).

Proposition 5.1.

1. *For $\rho > 0$, $\alpha > -\frac{1}{2}$ and $p > \max(2, 2\alpha+2)$ there exists a positive constant $C_1(\alpha,p)$ such that for all $f \in L^p([0,\infty[, A(x)dx)$,*

$$||\mathcal{R}_0(f)||_{p,A} \leq C_1(\alpha,p)||f||_{p,A}.$$

2. *For $\rho = 0$, $\alpha > \frac{1}{2}$ and $p > 2\alpha + 2$, there exists a positive constant $C_2(\alpha,p)$ such that for all $f \in L^p([0,\infty[, A(x)dx)$*

$$||\mathcal{R}_0(f)||_{p,A} \leq C_2(\alpha,p)||f||_{p,A}.$$

Proof. 1. Suppose that $\rho > 0$ and $p > \max(2, 2\alpha+2)$. Let

$$\nu(x) = A^{1-p'}(x)$$

and

$$\mu(x) = C_1(\alpha,a)x^{p(\alpha-\frac{1}{2})}A^{1-\frac{p}{2}}(x)1_{]0,a]}(x) + C_2(\alpha,a)x^{p\gamma}A^{1-\frac{p}{2}}(x)1_{[a,\infty[}(x),$$

with $a > 1$, $C_1(\alpha,a), C_2(\alpha,a)$ and γ are the constants given in Theorem 3.3 and Theorem 3.7.

Then

$$\nu(x) \leq m_1(\alpha,p)x^{(2\alpha+1)(1-p')}$$

and

$$\mu(x) \leq m_2(\alpha,p)x^{2\alpha+1-p}.$$

These inequalities imply that

$$\forall b > 0; \qquad \int_0^b \nu(x)dx < \infty,$$

$$\forall\ 0 < b_1 < b_2; \qquad \int_{b_1}^{b_2} \mu(x)dx < \infty$$

and

$$\begin{aligned} I(r) &= \left(\int_r^\infty \mu(x)dx\right)^{\frac{1}{p}} \left(\int_0^r \nu(x)dx\right)^{\frac{1}{p'}} \\ &\leq \left(m_2(\alpha,p)\int_r^\infty x^{2\alpha+1-p}dx\right)^{\frac{1}{p}} \left(m_1(\alpha,p)\int_0^r x^{(2\alpha+1)(1-p')}dx\right)^{\frac{1}{p'}} \end{aligned}$$

$$\leq \frac{(m_2(\alpha,p))^{\frac{1}{p}}(m_1(\alpha,p))^{\frac{1}{p'}}}{(p-2\alpha-2)^{\frac{1}{p}}((2\alpha+1)(1-p')+1)^{\frac{1}{p'}}}$$
$$= \frac{(m_2(\alpha,p))^{\frac{1}{p}} \times ((p-1)m_1(\alpha,p))^{\frac{1}{p'}}}{p-2\alpha-2}.$$

From Corollary 4.3, there exists a positive constant $C_{p,\alpha}$ such that for all non-negative measurable functions g we have

$$\left(\int_0^\infty (\mathcal{H}(g)(x))^p \mu(x)dx\right)^{\frac{1}{p}} \leq C_{p,\alpha}\left(\int_0^\infty (g(x))^p \nu(x)dx\right)^{\frac{1}{p}}, \tag{5.1}$$

with

$$\mathcal{H}(g)(x) = \int_0^x g(t)\nu(t)dt.$$

Now let us put

$$T(f)(x) = \left(\frac{\mu(x)}{A(x)}\right)^{\frac{1}{p}} \int_0^x f(t)dt,$$

then we have

$$\mathcal{H}(g)(x) = \left(\frac{\mu(x)}{A(x)}\right)^{-\frac{1}{p}} T(f)(x),$$

where

$$g(x) = f(x)A^{p'-1}(x).$$

From inequality (5.1), we deduce that for all non-negative measurable functions f, we have

$$\left(\int_0^\infty (T(f)(x))^p A(x)dx\right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty (f(x))^p A(x)dx\right)^{\frac{1}{p}}. \tag{5.2}$$

On the other hand from Theorems 3.3 and 3.7 we deduce that the function

$$\mathcal{R}_0(f)(x) = \int_0^x h(x,t)f(t)dt$$

is well defined and we have

$$|\mathcal{R}_0(f)(x)| \leq T(|f|)(x). \tag{5.3}$$

Thus, the relations (5.2) and (5.3) imply that

$$\left(\int_0^\infty |\mathcal{R}_0(f)(x)|^p A(x)dx\right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty |f(x)|^p A(x)dx\right)^{\frac{1}{p}},$$

which proves 1).

2. Suppose that $\rho = 0$ and $\alpha > \frac{1}{2}$. From Theorems 3.3 and 3.6 we have

$$\forall\, 0 < t < x; \quad |h(t,x)| \leq C x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

Therefore if we take

$$\mu(x) = x^{(\alpha-\frac{1}{2})p} A^{1-\frac{p}{2}}(x)$$

and

$$\nu(x) = A^{1-p'}(x),$$

we obtain the result in the same manner as 1).

□

Proposition 5.2. *Suppose that* $-\frac{1}{2} < \alpha \leq \frac{1}{2}$, $\rho = 0$ *and that there exists a positive constant* a *such*

$$\forall\, 0 < t < x, \quad x > a, \quad h(x,t) = 0.$$

Then for all $p > 2\alpha + 2$*, we can find a positive constant* $C_{\alpha,a}$ *satisfying*

$$\forall f \in L^p([0,\infty[, A(x)dx); \quad ||\mathcal{R}_0(f)||_{p,A} \leq C_{\alpha,a}||f||_{p,A}.$$

Proof. The hypothesis and Theorem 3.3 imply that there exists a positive constant a such that

$$\forall\, 0 < t < x; \quad |h(t,x)| \leq C(\alpha,a)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\mathbf{1}_{]0,a]}(x).$$

Therefore, if we take

$$\mu(x) = C(\alpha,a)x^{p(\alpha-\frac{1}{2})}A^{1-\frac{p}{2}}(x)1_{]0,a]}(x)$$

and

$$\nu(x) = A^{1-p'}(x)$$

then, we obtain the result using a similar procedure to that in Proposition 1, 2). □

Now, let us study the operator $\mathcal{R}_1$ defined for all measurable non-negative functions f by

$$\mathcal{R}_1(f)(x) = C_\alpha x^{\frac{1}{2}-\alpha} A^{-\frac{1}{2}}(x) \int_0^x (x^2 - t^2)^{\alpha-\frac{1}{2}} f(t)dt,$$

where

$$C_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\Pi}\Gamma\left(\alpha+\frac{1}{2}\right)}.$$

Proposition 5.3.

1. *For $\alpha > -\frac{1}{2}$, $\rho > 0$ and $p > \max(2, 2\alpha+2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0,+\infty[, A(x)dx)$, we have*

$$||\mathcal{R}_1(f)||_{p,A} \leq C_{p,\alpha}||f||_{p,A}.$$

2. *For $\alpha > -\frac{1}{2}$, $\rho = 0$ and $p > 2\alpha+2$ there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0,+\infty[, A(x)dx)$, we have*

$$||\mathcal{R}_1(f)||_{p,A} \leq C_{p,\alpha}||f||_{p,A}.$$

Proof. Let T_φ the Hardy type operator defined for all non-negative measurable functions f by

$$T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t)\nu(t)dt,$$

where

$$\varphi(x) = (1-x^2)^{\alpha-\frac{1}{2}}$$

and

$$\nu(x) = A^{1-p'}(x).$$

Then for all non-negative measurable functions f, we have

$$\mathcal{R}_1(f)(x) = C_\alpha x^{-\frac{1}{2}+\alpha} A^{-\frac{1}{2}}(x) T_\varphi(g)(x), \tag{5.4}$$

where

$$g(x) = f(x) A^{p'-1}(x).$$

Let

$$\mu(x) = x^{p(\alpha-\frac{1}{2})} A^{1-\frac{p}{2}}(x),$$

then, according to the hypothesis satisfied by the function A, it follows that there exist positive constants C_1, C_2 such that for all $\alpha > -\frac{1}{2}$ and $\rho > 0$ we have

$$\forall x > 0; \quad 0 \leq \mu(x) \leq C_1 x^{2\alpha+1-p} \tag{5.5}$$

$$\forall x > 0; \quad 0 \leq \nu(x) \leq C_2 x^{(2\alpha+1)(1-p')}. \tag{5.6}$$

Thus from the relations (5.5) and (5.6) we deduce that for $\alpha \geq \frac{1}{2}$, $\rho > 0$ and $p > 2\alpha + 2$, we have

- the function φ is continuous and non-increasing on $]0, 1[$.
- the functions φ, ν and μ satisfy the hypothesis of Theorem 4.1.

- the functions

$$F(r) = \left(\int_r^{\infty} \mu(x)dx\right)^{\frac{1}{p}} \left(\int_0^r \left(\varphi\left(\frac{x}{r}\right)\right)^{p'} \nu(x)dx\right)^{\frac{1}{p'}}$$

and

$$G(r) = \left(\int_r^{\infty} \left(\varphi\left(\frac{r}{x}\right)\right)^{p} \mu(t)dt\right)^{\frac{1}{p}} \left(\int_0^r \nu(t)dt\right)^{\frac{1}{p'}}$$

are bounded on $[0, \infty[$.

Hence from Theorem 4.1, there exists $C_{p,\alpha} > 0$ such that for all measurable non-negative functions f we have

$$\left(\int_0^{\infty} (T_{\varphi}(f(x)))^p \mu(x)dx\right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^{\infty} (f(x))^p \nu(x)dx\right)^{\frac{1}{p}}.$$

This inequality together with the relation (5.4) lead to

$$\left(\int_0^{\infty} (\mathcal{R}_1(f(x)))^p A(x)dx\right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^{\infty} (f(x))^p A(x)dx\right)^{\frac{1}{p}}$$

which proves the Proposition 1, 1) in the case $\alpha \geq \frac{1}{2}$.

For $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $p > 2$ we have

- the function φ is continuous and non-decreasing on $]0, 1[$.

- if we pick

$$\beta \in \left]\max\left(0, \frac{1-p(\frac{1}{2}+\alpha)}{p(\frac{1}{2}-\alpha)}\right), \min\left(1, \frac{1}{p(\frac{1}{2}-\alpha)}\right)\right[$$

and using inequalities (5.5) and (5.6), we deduce that the function

$$H(r) = \left(\int_r^{\infty}\left(\varphi\left(\frac{r}{x}\right)\right)^{\beta p}\mu(x)dx\right)^{\frac{1}{p}}\left(\int_0^{r}\left(\varphi\left(\frac{x}{r}\right)\right)^{(1-\beta)p'}\nu(x)dx\right)^{\frac{1}{p'}}$$

is bounded on $]0,\infty[$.

Consequently, the result follows from Theorem 4.2 and relation (5.4).
2) can be obtained in the same fashion as 1). □

Now we will give the main results of this paper.

Theorem 5.4.

1. *For* $\alpha > -\frac{1}{2}$, $\rho > 0$ *and* $p > \max(2, 2\alpha+2)$, *there exists a positive constant* $C_{p,\alpha}$ *such that for all* $f \in L^p([0,\infty[, A(x)dx)$,

$$||\mathcal{R}(f)||_{p,A} \leq C_{p,\alpha}||f||_{p,A}.$$

2. *For* $\alpha > -\frac{1}{2}$, $\rho > 0$ *and* $p > \max(2, 2\alpha+2)$, *there exists a positive constant* $C_{p,\alpha}$ *such that for all* $g \in L^{p'}([0,\infty[, A(x)dx)$,

$$\left\|\frac{1}{A(x)}\mathcal{W}(g)\right\|_{p',A} \leq C_{p,\alpha}||g||_{p',A}$$

where $p' = \frac{p}{p-1}$.

Proof. 1) follows from Proposition 1, 1) and Proposition 1, 1), and the fact that

$$\mathcal{R}(f) = \mathcal{R}_0(f) + \mathcal{R}_1(f).$$

2) follows from 1) and the relations

$$||g||_{p',A} = \max_{||f||_{p,A}\leq 1} \int_0^\infty f(x)g(x)A(x)dx, \tag{5.7}$$

for all measurable non-negative functions f and g

$$\int_0^\infty \mathcal{R}(f)(x)g(x)A(x)dx = \int_0^\infty \mathcal{W}(g)(x)f(x)dx. \tag{5.8}$$

□

Theorem 5.5.

1. *For* $\alpha > \frac{1}{2}$, $\rho = 0$ *and* $p > 2\alpha + 2$ *there exists a positive constant* $C_{p,\alpha}$ *such that for all* $f \in L^p([0,\infty[, A(x)dx)$

$$||\mathcal{R}(f)||_{p,A} \leq C_{p,\alpha}||f||_{p,A}.$$

2. *For* $\alpha > \frac{1}{2}$, $\rho = 0$ *and* $p > 2\alpha + 2$ *there exists a positive constant* $C_{p,\alpha}$ *such that for all* $g \in L^{p'}([0,\infty[, A(x)dx)$

$$\left\|\frac{1}{A(x)}\mathcal{W}(g)\right\|_{p',A} \leq C_{p,\alpha}||g||_{p',A}$$

where $p' = \frac{p}{p-1}$.

3. *For* $-\frac{1}{2} < \alpha \leq \frac{1}{2}$, $\rho = 0$, $p > 2\alpha + 2$ *and under the hypothesis of Proposition 5.2, the previous results hold.*

Proof. This theorem is obtained by using Propositions 1, 2), 5.2 and 1, 2). □

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