

HARDY TYPE INEQUALITIES FOR INTEGRAL TRANSFORMS ASSOCIATED WITH A SINGULAR SECOND ORDER DIFFERENTIAL OPERATOR

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

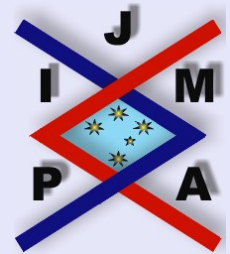
We consider a singular second order differential operator Δ defined on $]0, \infty[$. We give nice estimates for the kernel which intervenes in the integral transform of the eigenfunction of Δ . Using these results, we establish Hardy type inequalities for Riemann-Liouville and Weyl transforms associated with the operator Δ .

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Contents

1	Introduction	3
2	The Eigenfunctions of the Operator Δ	9
3	The Kernel h	14
4	Hardy Type Operators T_φ	23
5	The Riemann - Liouville and Weyl Transforms Associated with the Operator Δ	27
References		



Hardy Type Inequalities For Integral Transforms Associated With A Singular Second Order Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 2 of 40

1. Introduction

In this paper we consider the differential operator on $]0, \infty[$, defined by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} + \rho^2,$$

where A is a real function defined on $]0, \infty[$, satisfying

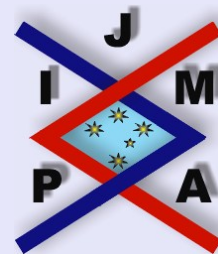
$$A(x) = x^{2\alpha+1}B(x); \alpha > -\frac{1}{2}$$

and B is a positive, even C^∞ function on \mathbb{R} such that $B(0) = 1$, and $\rho \geq 0$. We suppose that the function A satisfies the following assumptions

- i) $A(x)$ is increasing, and $\lim_{+\infty} A(x) = +\infty$.
- ii) $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{+\infty} \frac{A'(x)}{A(x)} = 2\rho$.
- iii) there exists a constant $\delta > 0$, satisfying

$$\begin{cases} \frac{B'(x)}{B(x)} = 2\rho - \frac{2\alpha+1}{x} + e^{-\delta x}F(x), & \text{for } \rho > 0, \\ \frac{B'(x)}{B(x)} = e^{-\delta x}F(x), & \text{for } \rho = 0, \end{cases}$$

where F is C^∞ on $]0, \infty[$, bounded together with its derivatives on the interval $[x_0, \infty[$, $x_0 > 0$.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 3 of 40

This operator plays an important role in harmonic analysis, for example, many special functions (orthogonal polynomials,...) are eigenfunctions of operators of the same type as Δ .

The Bessel and Jacobi operators defined respectively by

$$\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx}; \quad \alpha > -\frac{1}{2}$$

and

$$\Delta_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha + 1) \coth x + (2\beta + 1) \tanh x) \frac{d}{dx} + (\alpha + \beta + 1)^2,$$

$$\alpha \geq \beta > -\frac{1}{2},$$

are of the type Δ , with

$$A(x) = x^{2\alpha+1}; \quad \rho = 0,$$

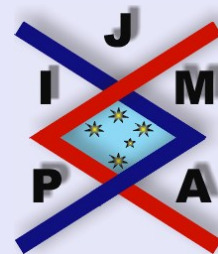
respectively

$$A(x) = \sinh^{2\alpha+1} x \cosh^{2\beta+1} x; \quad \rho = \alpha + \beta + 1.$$

Also, the radial part of the Laplacian - Btrami operator on the Riemannian symmetric space, is of type Δ .

The operator Δ has been studied from many points of view ([1], [7], [13], [14], [15], [16]). In particular, K. Trimèche has proved in [15] that the differential equation

$$\Delta u(x) = -\lambda^2 u(x), \quad \lambda \in \mathbb{C}$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 4 of 40

has a unique solution on $[0, \infty[$, satisfying the conditions $u(0) = 1, u'(0) = 0$. We extend this solution on \mathbb{R} by parity and we denote it by φ_λ . He has also proved that the eigenfunction φ_λ has the following Mehler integral representation

$$\varphi_\lambda(x) = \int_0^x k(x, t) \cos \lambda t dt,$$

where the kernel $k(x, t)$ is defined by

$$k(x, t) = 2h(x, t) + C_\alpha A^{-\frac{1}{2}}(x) x^{\frac{1}{2}-\alpha} (x^2 - t^2)^{\alpha-\frac{1}{2}}, \quad 0 < t < x$$

with

$$h(x, t) = \frac{1}{\Pi} \int_0^\infty \psi(x, \lambda) \cos(\lambda t) d\lambda,$$

$$C_\alpha = \frac{2\Gamma(\alpha + 1)}{\sqrt{\Pi}\Gamma(\alpha + \frac{1}{2})},$$

and

$$\forall \lambda \in \mathbb{R}, x \in \mathbb{R}; \quad \psi(x, \lambda) = \varphi_\lambda(x) - x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x) j_\alpha(\lambda x),$$

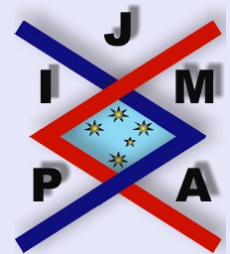
where

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha}$$

and J_α is the Bessel function of the first kind and order α ([8]).

The Riemann - Liouville and Weyl transforms associated with the operator Δ are respectively defined, for all non-negative measurable functions f by

$$\mathcal{R}(f)(x) = \int_0^x k(x, t) f(t) dt$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 5 of 40

and

$$\mathcal{W}(f)(t) = \int_t^\infty k(x, t) f(x) A(x) dx.$$

These operators have been studied on regular spaces of functions. In particular, in [15], the author has proved that the Riemann-Liouville transform \mathcal{R} is an isomorphism from $\mathcal{E}^*(\mathbb{R})$ (the space of even infinitely differentiable functions on \mathbb{R}) onto itself, and that the Weyl transform \mathcal{W} is an isomorphism from $\mathcal{D}_*(\mathbb{R})$ (the space of even infinitely differentiable functions on \mathbb{R} with compact support) onto itself.

The Weyl transform has also been studied on Schwarz space $S_*(\mathbb{R})$ ([13]).

Our purpose in this work is to study the operators \mathcal{R} and \mathcal{W} on the spaces $L^p([0, \infty[, A(x)dx)$ consisting of measurable functions f on $[0, \infty[$ such that

$$\|f\|_{p,A} = \left(\int_0^\infty |f(x)|^p A(x) dx \right)^{\frac{1}{p}} < \infty; \quad 1 < p < \infty.$$

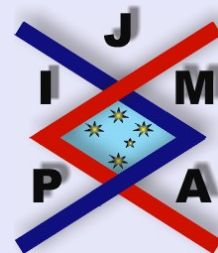
The main results of this paper are the following Hardy type inequalities

- For $\rho > 0$ and $p > \max(2, 2\alpha + 2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0, \infty[, A(x)dx)$,

$$(1.1) \quad \|\mathcal{R}(f)\|_{p,A} \leq C_{p,\alpha} \|f\|_{p,A}$$

and for all $g \in L^{p'}([0, \infty[, A(x)dx)$,

$$(1.2) \quad \left\| \frac{1}{A(x)} \mathcal{W}(g) \right\|_{p',A} \leq C_{p,\alpha} \|g\|_{p',A},$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents

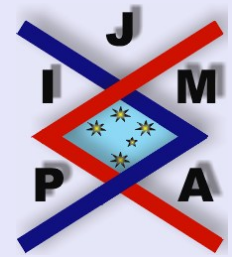


Go Back

Close

Quit

Page 6 of 40



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 7 of 40

where $p' = \frac{p}{p-1}$.

- For $\rho = 0$ and $p > 2\alpha + 2$ there exists a positive constant $C_{p,\alpha}$ such that (1.1) and (1.2) hold.

In ([5], [6]) we have obtained (1.1) and (1.2) in the cases

$$A(x) = x^{2\alpha+1}, \quad \alpha > -\frac{1}{2}$$

respectively

$$A(x) = \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x); \quad \alpha \geq \beta > -\frac{1}{2}.$$

This paper is arranged as follows. In the first section, we recall some properties of the eigenfunctions of the operator Δ . The second section deals with the study of the behavior of the kernel $h(x, t)$. In the third section, we introduce the following integral operator

$$T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t)\nu(t)dt$$

where

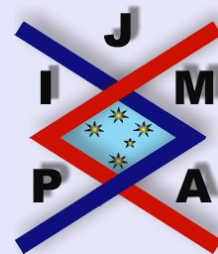
- φ is a measurable function defined on $]0, 1[$,
- ν is a measurable non-negative function on $]0, \infty[$ locally integrable.

Then we give the criteria in terms of the function φ to obtain the following Hardy type inequalities for T_φ ,

for all real numbers, $1 < p \leq q < \infty$, there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f and g we have

$$\left(\int_0^\infty (T_\varphi(f(x)))^q \mu(x) dx \right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x) dx \right)^{\frac{1}{p}}.$$

In the fourth section, we use the precedent results to establish the Hardy type inequalities (1.1) and (1.2) for the operators \mathcal{R} and \mathcal{W} .



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 8 of 40

2. The Eigenfunctions of the Operator Δ

As mentioned in the introduction, the equation

$$(2.1) \quad \Delta u(x) = -\lambda^2 u(x), \quad \lambda \in \mathbb{C}$$

has a unique solution on $[0, \infty[$, satisfying the conditions $u(0) = 1, u'(0) = 0$. We extend this solution on \mathbb{R} by parity and we denote it φ_λ . Equation (2.1) possesses also two solutions $\phi_{\mp\lambda}$ linearly independent having the following behavior at infinity $\phi_{\mp\lambda}(x) \sim e^{(\mp\lambda-\rho)x}$. Then there exists a function c such that

$$\varphi_\lambda(x) = c(\lambda)\phi_\lambda(x) + c(-\lambda)\phi_{-\lambda}(x).$$

In the case of the Bessel operator Δ_α , the functions φ_λ , ϕ_λ and c are given respectively by

$$(2.2) \quad j_\alpha(\lambda x) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(\lambda x)}{(\lambda x)^\alpha}, \quad \lambda x \neq 0,$$

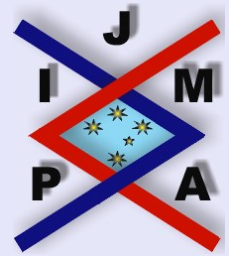
$$k_\alpha(i\lambda x) = 2^\alpha \Gamma(\alpha + 1) \frac{K_\alpha(i\lambda x)}{(i\lambda x)^\alpha}, \quad \lambda x \neq 0,$$

$$c(\lambda) = 2^\alpha \Gamma(\alpha + 1) e^{-i(\alpha + \frac{1}{2})\frac{\pi}{2}} \lambda^{-(\alpha + \frac{1}{2})}, \quad \lambda > 0,$$

where J_α and K_α are respectively the Bessel function of first kind and order α , and the MacDonald function of order α .

In the case of the Jacobi operator $\Delta_{\alpha,\beta}$, the functions φ_λ , ϕ_λ and c are respectively

$$\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1 \left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), (\alpha + 1), -\sinh^2(x) \right), \quad x \geq 0, \lambda \in \mathbb{C},$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 9 of 40

$$\phi_{\lambda}^{\alpha, \beta}(x) = (2 \sinh x)^{(i\lambda - \rho)} {}_2F_1 \left(\frac{1}{2}(\rho - 2\alpha - i\lambda), \frac{1}{2}(\rho - i\lambda), 1 - i\lambda, (\sinh x)^{-2} \right);$$

$$x > 0, \lambda \in \mathbb{C} - (-i\mathbb{N})$$

and

$$c(\lambda) = \frac{2^{\rho - i\lambda} \Gamma(\alpha + 1) \Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(\rho - i\lambda)\right) \Gamma\left(\frac{1}{2}(\alpha - \beta + 1 + i\lambda)\right)}$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

From ([1], [2], [15], [16]) we have the following properties:

i) We have:

- For $\rho = 0$: $\forall x \geq 0, \varphi_0(x) = 1$,
- For $\rho \geq 0$: there exists a constant $k > 0$ such that

$$(2.3) \quad \forall x \geq 0, \quad e^{-\rho x} \leq \varphi_0(x) \leq k(1 + x)e^{-\rho x}.$$

ii) For $\lambda \in \mathbb{R}$ and $x \geq 0$ we have

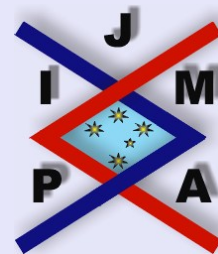
$$(2.4) \quad |\varphi_{\lambda}(x)| \leq \varphi_0(x).$$

iii) For $\lambda \in \mathbb{C}$ such that $|\Im \lambda| \leq \rho$ and $x \geq 0$ we have $|\varphi_{\lambda}(x)| \leq 1$.

iv) We have the integral representation of Mehler type,

$$(2.5) \quad \forall x > 0, \forall \lambda \in \mathbb{C}, \quad \varphi_{\lambda}(x) = \int_0^x k(x, t) \cos(\lambda t) dt,$$

where $k(x, \cdot)$ is an even positive C^{∞} function on $] -x, x[$ with support in $[-x, x]$.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents

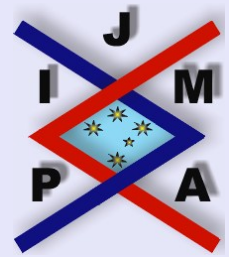


Go Back

Close

Quit

Page 10 of 40



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 11 of 40

v) For $\lambda \in \mathbb{R}$, we have $c(-\lambda) = \overline{c(\lambda)}$.

vi) The function $|c(\lambda)|^{-2}$ is continuous on $[0, +\infty[$ and there exist positive constants k, k_1, k_2 such that

- If $\rho \geq 0 : \forall \lambda \in \mathbb{C}, |\lambda| > k$

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

- If $\rho > 0 : \forall \lambda \in \mathbb{C}, |\lambda| \leq k$

$$k_1|\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^2,$$

- If $\rho = 0, \alpha > 0 : \forall \lambda \in \mathbb{C}, |\lambda| \leq k$

$$(2.6) \quad k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}.$$

Now, let us put

$$v(x) = A^{\frac{1}{2}}(x)u(x).$$

The equation (2.1) becomes

$$v''(x) - (G(x) - \lambda^2)v(x) = 0,$$

where

$$G(x) = \frac{1}{4} \left(\frac{A'(x)}{A(x)} \right)^2 + \frac{1}{2} \left(\frac{A'(x)}{A(x)} \right)' - \rho^2.$$

Let

$$\xi(x) = G(x) + \frac{\frac{1}{4} - \alpha^2}{x^2}.$$

Thus from hypothesis of the function A , we deduce the following results for the function ξ .

Proposition 2.1.

1. The function ξ is continuous on $]0, \infty[$.
2. There exist $\delta > 0$ and $a \in \mathbb{R}$ such that the function ξ satisfies

$$\xi(x) = \frac{a}{x^2} + \exp(-\delta x)F_1(x),$$

where F_1 is C^∞ on $]0, \infty[$, bounded together with all its derivatives on the interval $[x_0, \infty[$, $x_0 > 0$.

Proposition 2.2 ([15]). Let

$$(2.7) \quad \psi(x, \lambda) = \varphi_\lambda(x) - x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)j_\alpha(\lambda x),$$

where j_α is defined by (2.2).

Then there exist positive constants C_1 and C_2 such that

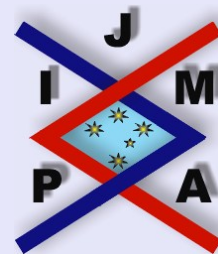
$$(2.8) \quad \forall x > 0, \forall \lambda \in \mathbb{R}^*, |\psi(x, \lambda)| \leq C_1 A^{-\frac{1}{2}}(x) \tilde{\xi}(x) \lambda^{-\alpha-\frac{3}{2}} \exp\left(C_2 \frac{\tilde{\xi}(x)}{\lambda}\right),$$

with

$$\tilde{\xi}(x) = \int_0^x |\xi(r)| dr.$$

The kernel $k(x, t)$ given by the relation (2.5) can be written

$$(2.9) \quad k(x, t) = 2h(x, t) + C_\alpha A^{-\frac{1}{2}}(x) x^{\frac{1}{2}-\alpha} (x^2 - t^2)^{\alpha-\frac{1}{2}}, \quad 0 < t < x,$$



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 12 of 40

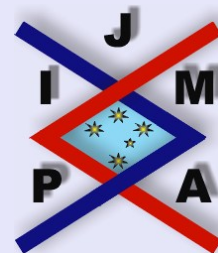
where

$$(2.10) \quad h(x, t) = \frac{1}{\Pi} \int_0^{\infty} \psi(x, t) \cos(\lambda t) d\lambda,$$

$$C_{\alpha} = \frac{2\Gamma(\alpha + 1)}{\sqrt{\Pi}\Gamma(\alpha + \frac{1}{2})},$$

and $\psi(x, \lambda)$ is the function defined by the relation (2.7).

Since the Riemann-Liouville and Weyl transforms associated with the operator Δ are given by the kernel k , then, we need some properties of this function. But from the relation (2.9) it suffices to study the kernel h .



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 13 of 40

3. The Kernel h

In this section we will study the behaviour of the kernel h .

Lemma 3.1. *For any real $a > 0$ there exist positive constants $C_1(a)$, $C_2(a)$ such that for all $x \in [0, a]$,*

$$C_1(a)x^{2\alpha+1} \leq A(x) \leq C_2(a)x^{2\alpha+1}.$$

From Proposition 1, and [16], we deduce the following lemma.

Lemma 3.2. *There exist positive constants a_1, a_2, C_1 and C_2 such that for $|\lambda| > a_1$*

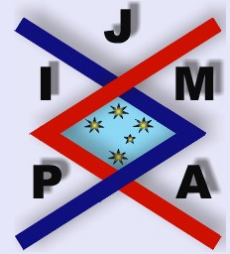
$$\varphi_\lambda(x) = \begin{cases} C(\alpha)x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x)(j_\alpha(\lambda x) + O(\lambda x)) & \text{for } |\lambda x| \leq a_2 \\ C(\alpha)\lambda^{-(\alpha+\frac{1}{2})}A^{-\frac{1}{2}}(x)(C_1 \exp -i\lambda x + C_2 \exp i\lambda x) \\ \quad \times (1 + O(\lambda^{-1}) + O((\lambda x)^{-1})) & \text{for } |\lambda x| > a_2, \end{cases}$$

where

$$C(\alpha) = \Gamma(\alpha + 1)A^{\frac{1}{2}}(1) \exp\left(-\frac{1}{2} \int_0^1 B(t)dt\right).$$

Theorem 3.3. *For any $a > 0$, there exists a positive constant $C_1(\alpha, a)$ such that*

$$\forall 0 < t < x \leq a; \quad |h(x, t)| \leq C_1(\alpha, a)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).$$



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 14 of 40

Proof. By (2.10) we have for $0 < t < x$,

$$\begin{aligned}
 |h(t, x)| &\leq \frac{1}{\Gamma} \int_0^\infty |\psi(x, \lambda)| d\lambda \\
 &= \frac{1}{\Gamma} \int_0^{a_1} |\psi(x, \lambda)| d\lambda + \frac{1}{\Gamma} \int_{a_1}^\infty |\psi(x, \lambda)| d\lambda \\
 (3.1) \quad &= I_1(x) + I_2(x),
 \end{aligned}$$

where a_1 is the constant given by Lemma 3.2.

We put

$$f_\lambda(x) = x^{\frac{1}{2}-\alpha} A^{\frac{1}{2}}(x) |\psi(x, \lambda)|, \quad 0 < x < a, \quad \lambda \in \mathbb{R}.$$

From Proposition 2.2 the function

$$(x, \lambda) \longrightarrow f_\lambda(x)$$

is continuous on $[0, a] \times [0, a_1]$. Then

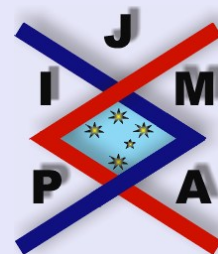
$$(3.2) \quad I_1(x) = \frac{1}{\Gamma} \int_0^{a_1} |\psi(x, \lambda)| d\lambda \leq C_\alpha^1 x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x),$$

where

$$C_\alpha^1 = \frac{a_1}{\Gamma} \sup_{(x, \lambda) \in [0, a] \times [0, a_1]} |f_\lambda(x)|.$$

Let us study the second term

$$I_2(x) = \frac{1}{\Gamma} \int_{a_1}^\infty |\psi(x, \lambda)| d\lambda.$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 15 of 40

i) Suppose $-\frac{1}{2} < \alpha \leq \frac{1}{2}$. From inequality (2.8) we get

$$\begin{aligned} I_2(x) &\leq \frac{C_1}{\Pi} A^{-\frac{1}{2}}(x) \tilde{\xi}(x) \int_{a_1}^{\infty} \lambda^{-\alpha-\frac{3}{2}} \exp\left(C_2 \frac{\tilde{\xi}(x)}{|\lambda|}\right) d\lambda \\ &\leq \tilde{C}_1 A^{-\frac{1}{2}}(x) \tilde{\xi}(x) \exp\left(C_2 \frac{\tilde{\xi}(x)}{a_1}\right) x^{\alpha-\frac{1}{2}}. \end{aligned}$$

Since $\tilde{\xi}$ is bounded on $[0, \infty[$, we deduce that

$$(3.3) \quad I_2(x) \leq C_{2,\alpha} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

This completes the proof in the case $-\frac{1}{2} < \alpha \leq \frac{1}{2}$.

ii) Suppose now that $\alpha > \frac{1}{2}$.

- Let a_1, a_2 be the constants given in Lemma 3.2. From this lemma we deduce that there exists a positive constant $C_1(\alpha)$ such that

$$(3.4) \quad \forall x > \frac{a_2}{a_1}, \lambda > a_1; |\varphi_\lambda(x)| \leq C_1(\alpha) A^{-\frac{1}{2}}(x) \lambda^{-(\alpha+\frac{1}{2})}.$$

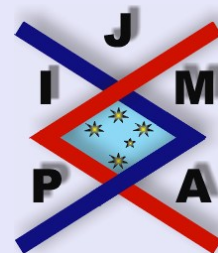
On the other hand, the function

$$s \longrightarrow s^{\alpha+\frac{1}{2}} j_\alpha(s)$$

is bounded on $[0, \infty[$.

Then from equality (2.7), we have, for $x > \frac{a_2}{a_1}$

$$(3.5) \quad \frac{1}{\Pi} \int_{a_1}^{\infty} |\psi(x, \lambda)| d\lambda$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents

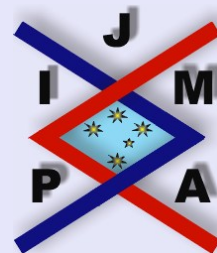


Go Back

Close

Quit

Page 16 of 40



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 17 of 40

$$\begin{aligned}
 &\leq \frac{1}{\Pi} \int_{a_1}^{\infty} |\varphi_{\lambda}(x)| d\lambda + \frac{1}{\Pi} x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_1}^{\infty} |j_{\alpha}(\lambda x)| d\lambda \\
 &\leq \frac{C_1(\alpha)}{\Pi} A^{-\frac{1}{2}}(x) \int_{a_1}^{\infty} \lambda^{-(\alpha+\frac{1}{2})} d\lambda + \frac{1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_2}^{\infty} |j_{\alpha}(u)| du \\
 &\leq \frac{C_1(\alpha)}{(\alpha-\frac{1}{2})\Pi} A^{-\frac{1}{2}}(x) \left(\frac{1}{a_1}\right)^{(\alpha-\frac{1}{2})} + \frac{1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_2}^{\infty} |j_{\alpha}(u)| du \\
 &\leq \frac{C_1(\alpha)}{(\alpha-\frac{1}{2})\Pi} A^{-\frac{1}{2}}(x) \left(\frac{x}{a_2}\right)^{(\alpha-\frac{1}{2})} + \frac{1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_2}^{\infty} |j_{\alpha}(u)| du \\
 &\leq C_2(\alpha) x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x),
 \end{aligned}$$

where

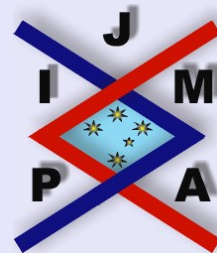
$$C_2(\alpha) = \frac{C_1(\alpha)}{(\alpha-\frac{1}{2})\Pi} (a_2)^{(-\alpha+\frac{1}{2})} + \frac{1}{\Pi} \int_{a_2}^{\infty} |j_{\alpha}(u)| du.$$

- $0 < x < \frac{a_2}{a_1}$. From Lemma 3.2 and the fact that

$$\forall x \in \mathbb{R}, \quad |j_{\alpha}(\lambda x)| \leq 1$$

we deduce that there exists a positive constant $M_1(\alpha)$ such that

$$\forall 0 < x < \frac{a_2}{a_1}, \quad 0 \leq \lambda \leq \frac{a_2}{x} \quad |\psi(x, \lambda)| \leq M_1(\alpha) x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x).$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 18 of 40

This involves

$$(3.6) \quad \begin{aligned} \frac{1}{\Pi} \int_{a_1}^{\frac{a_2}{x}} |\psi(x, \lambda)| d\lambda &\leq \frac{M_1(\alpha)}{\Pi} x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x) \left(\frac{a_2}{x} - a_1 \right) \\ &\leq \frac{a_2}{\Pi} M_1(\alpha) x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x). \end{aligned}$$

Moreover

$$(3.7) \quad \begin{aligned} \frac{1}{\Pi} \int_{\frac{a_2}{x}}^{\infty} |\psi(x, \lambda)| d\lambda &\leq \frac{C_1(\alpha)}{\Pi} A^{-\frac{1}{2}}(x) \int_{\frac{a_2}{x}}^{\infty} \lambda^{-(\alpha+\frac{1}{2})} d\lambda \\ &\quad + \frac{1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_2}^{\infty} |j_\alpha(u)| du \\ &\leq \frac{C_1(\alpha)}{(\alpha - \frac{1}{2}) \Pi} A^{-\frac{1}{2}}(x) \left(\frac{1}{a_2} \right)^{(\alpha-\frac{1}{2})} \\ &\quad + \frac{1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_{a_2}^{\infty} |j_\alpha(u)| du \\ &\leq C_2(\alpha) x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x). \end{aligned}$$

From (3.6) and (3.7) we deduce that

$$(3.8) \quad \forall 0 < x < \frac{a_2}{a_1}; \quad \frac{1}{\Pi} \int_{a_1}^{\infty} |\psi(x, \lambda)| d\lambda \leq M_2(\alpha) x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x)$$

where

$$M_2(\alpha) = \frac{a_2}{\Pi} M_1(\alpha) + C_2(\alpha).$$

From (3.5), (3.8) it follows that

$$\forall 0 < x < a; \quad I_2(x) \leq M_2(\alpha)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).$$

This completes the proof. □

In order to provide some estimates for the kernel h for later use, we need the following lemmas

Lemma 3.4.

i) For $\rho > 0$, we have

$$A(x) \sim e^{2\rho x}, \quad (x \longrightarrow +\infty)$$

ii) For $\rho = 0$, we have

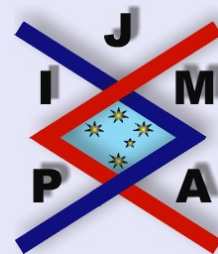
$$A(x) \sim x^{2\alpha+1}, \quad (x \longrightarrow +\infty).$$

This lemma can be deduced from hypothesis of the function A .

Lemma 3.5 ([2]). For $\rho = 0$ and $\alpha > \frac{1}{2}$ there exist two positive constants $D_1(\alpha)$ and $D_2(\alpha)$ satisfying

i)

$$|\varphi_\lambda(x)| \leq D_1(\alpha)x^{\alpha+\frac{1}{2}}A^{-\frac{1}{2}}(x), \quad x > 0, \lambda \geq 0.$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page	
Contents	
◀◀	▶▶
◀	▶
Go Back	
Close	
Quit	
Page 19 of 40	

ii)

$$|\varphi_\lambda(x)| \leq D_2(\alpha)|c(\lambda)|A^{-\frac{1}{2}}(x), \quad x > 1, \lambda x > 1$$

where

$$\lambda \longrightarrow c(\lambda)$$

is the spectral function given by (2.6).

Using previous results we will give the behavior of the function h for large values of the variable x

Theorem 3.6. For $\rho = 0$, $\alpha > \frac{1}{2}$, and $a > 0$ there exists a positive constant $C_{\alpha,a}$ such that

$$0 < t < x, \quad x > a, \quad |h(x, t)| \leq C_{\alpha,a}x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).$$

Proof. We have

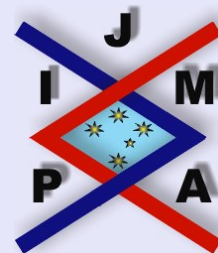
$$h(x, t) = \frac{1}{\Pi} \int_0^\infty |\psi(x, \lambda)| \cos(\lambda t) d\lambda,$$

then

$$\begin{aligned} (3.9) \quad |h(x, t)| &\leq \frac{1}{\Pi} \int_0^\infty |\psi(x, \lambda)| d\lambda \\ &= \frac{1}{\Pi} \int_0^1 |\psi(x, \lambda)| d\lambda + \frac{1}{\Pi} \int_1^\infty |\psi(x, \lambda)| d\lambda. \end{aligned}$$

From Proposition 2.2 and the fact that $\alpha > \frac{1}{2}$ we get

$$\frac{1}{\Pi} \int_1^\infty |\psi(x, \lambda)| d\lambda \leq \frac{C_1}{\Pi} A^{-\frac{1}{2}}(x) \tilde{\xi}(x) \exp\left(C_2(\tilde{\xi}(x))\right) \int_1^\infty \lambda^{-\alpha-\frac{3}{2}} d\lambda.$$



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 20 of 40

Since the function $\tilde{\xi}$ is bounded on $[0, \infty[$, we deduce that there exists $d_\alpha > 0$ verifying

$$(3.10) \quad \frac{1}{\Pi} \int_1^\infty |\psi(x, \lambda)| d\lambda \leq d_\alpha x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

On the other hand, we have

$$\frac{1}{\Pi} \int_0^1 |\psi(x, \lambda)| d\lambda \leq \frac{1}{\Pi} \int_0^1 |\varphi_\lambda(x)| d\lambda + \frac{1}{\Pi} x^{\alpha+\frac{1}{2}} A^{-\frac{1}{2}}(x) \int_0^1 |j_\alpha(\lambda x)| d\lambda.$$

However,

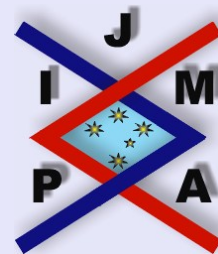
$$\frac{1}{\Pi} \int_0^1 |\varphi_\lambda(x)| d\lambda = \frac{1}{\Pi} \int_0^{\frac{1}{x}} |\varphi_\lambda(x)| d\lambda + \frac{1}{\Pi} \int_{\frac{1}{x}}^1 |\varphi_\lambda(x)| d\lambda$$

from Lemma 3.5 i) we have

$$(3.11) \quad \frac{1}{\Pi} \int_0^{\frac{1}{x}} |\varphi_\lambda(x)| d\lambda \leq \frac{C_1}{\Pi} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x).$$

Furthermore from Lemma 3.5 ii) and the relation (2.6) it follows that there exists $d_2(\alpha) > 0$ such that

$$(3.12) \quad \begin{aligned} \frac{1}{\Pi} \int_{\frac{1}{x}}^1 |\varphi_\lambda(x)| d\lambda &\leq \frac{d_2(\alpha)}{\Pi} A^{-\frac{1}{2}}(x) \int_{\frac{1}{x}}^1 \lambda^{-(\alpha+\frac{1}{2})} d\lambda \\ &\leq \frac{d_2(\alpha)}{\Pi} A^{-\frac{1}{2}}(x) \int_{\frac{1}{x}}^\infty \lambda^{-(\alpha+\frac{1}{2})} d\lambda \\ &\leq \frac{d_2(\alpha)}{\Pi(\alpha-\frac{1}{2})} x^{\alpha-\frac{1}{2}} A^{-\frac{1}{2}}(x). \end{aligned}$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 21 of 40

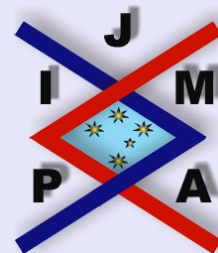
The theorem follows from the relations (3.9), (3.10), (3.11) and (3.12). \square

Theorem 3.7. For $\rho > 0$ and $a > 1$ there exists a positive constant $C_{\alpha,a}$ such that

$$\forall 0 < t < x; x \geq a; \quad |h(x, t)| \leq C_2(\alpha, a)x^\gamma A^{-\frac{1}{2}}(x),$$

where $\gamma = \max\left(1, \alpha + \frac{1}{2}\right)$.

Proof. This theorem can be obtained in the same manner as Theorem 3.6, using the properties (2.3) and (2.4). \square



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 22 of 40

4. Hardy Type Operators T_φ

In this section, we will define a class of integral operators and we recall some of their properties which we use in the next section to obtain the main results of this paper.

Let

$$\varphi :]0, 1[\longrightarrow]0, \infty[$$

be a measurable function, then we associate the integral operator T_φ defined for all non-negative measurable functions f by

$$\forall x > 0; \quad T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t) \nu(t) dt$$

where

- ν is a measurable non negative function on $]0, \infty[$ such that

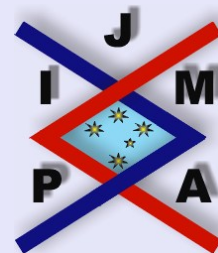
$$(4.1) \quad \forall a > 0, \quad \int_0^a \nu(t) dt < \infty$$

and

- μ is a non-negative function on $]0, \infty[$ satisfying

$$(4.2) \quad \forall 0 < a < b, \quad \int_a^b \mu(t) dt < \infty.$$

These operators have been studied by many authors. In particular, in [5], see also ([6], [10], [11]), we have proved the following results.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 23 of 40

Theorem 4.1. Let p, q be two real numbers such that

$$1 < p \leq q < \infty.$$

Let ν and μ be two measurable non-negative functions on $]0, \infty[$, satisfying (4.1) and (4.2). Lastly, suppose that the function

$$\varphi :]0, 1[\longrightarrow]0, \infty[$$

is continuous non increasing and satisfies

$$\forall x, y \in]0, 1[, \quad \varphi(xy) \leq D(\varphi(x) + \varphi(y))$$

where D is a positive constant. Then the following assertions are equivalent

1. There exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f :

$$\left(\int_0^\infty (T_\varphi(f)(x))^q \mu(x) dx \right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x) dx \right)^{\frac{1}{p}}.$$

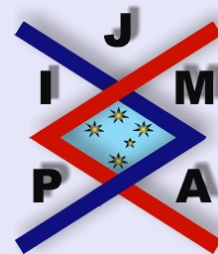
2. The functions

$$F(r) = \left(\int_r^\infty \mu(x) dx \right)^{\frac{1}{q}} \left(\int_0^r \left(\varphi \left(\frac{x}{r} \right) \right)^{p'} \nu(x) dx \right)^{\frac{1}{p'}}$$

and

$$G(r) = \left(\int_r^\infty \left(\varphi \left(\frac{r}{x} \right) \right)^q \mu(x) dx \right)^{\frac{1}{q}} \left(\int_0^r \nu(x) dx \right)^{\frac{1}{p'}}$$

are bounded on $]0, \infty[$, where $p' = \frac{p}{p-1}$.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 24 of 40

Theorem 4.2. Let p and q be two real numbers such that

$$1 < p \leq q < \infty$$

and μ, ν two measurable non-negative functions on $]0, \infty[$, satisfying the hypothesis of Theorem 4.1.

Let

$$\varphi :]0, 1[\longrightarrow]0, \infty[$$

be a measurable non-decreasing function.

If there exists $\beta \in [0, 1]$ such that the function

$$r \longrightarrow \left(\int_r^\infty \left(\varphi \left(\frac{r}{x} \right) \right)^{\beta q} \mu(x) dx \right)^{\frac{1}{q}} \left(\int_0^r \left(\varphi \left(\frac{x}{r} \right) \right)^{p'(1-\beta)} \nu(x) dx \right)^{\frac{1}{p'}}$$

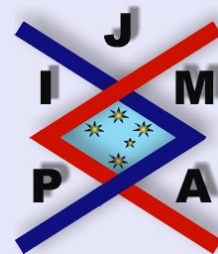
is bounded on $]0, \infty[$, then there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f , we have

$$\left(\int_0^\infty (T_\varphi(f(x)))^q \mu(x) dx \right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x) dx \right)^{\frac{1}{p}}$$

where $p' = \frac{p}{p-1}$.

The last result that we need is:

Corollary 4.3. With the hypothesis of Theorem 4.1 and $\varphi = 1$, the following assertions are equivalent:



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 25 of 40

1. there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f we have

$$\left(\int_0^\infty (\mathcal{H}(f)(x))^q \mu(x) dx \right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x) dx \right)^{\frac{1}{p}},$$

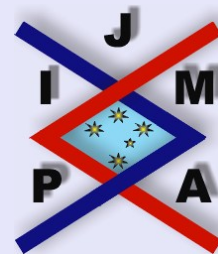
2. The function

$$I(r) = \left(\int_r^\infty \mu(x) dx \right)^{\frac{1}{q}} \left(\int_0^r \nu(x) dx \right)^{\frac{1}{p'}}$$

is bounded on $]0, \infty[$,

where \mathcal{H} is the Hardy operator defined by

$$\forall x > 0, \quad \mathcal{H}(f)(x) = \int_0^x f(t) \nu(t) dt.$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 26 of 40

5. The Riemann - Liouville and Weyl Transforms Associated with the Operator Δ

This section deals with the proof of the Hardy type inequalities (1.1) and (1.2) mentioned in the introduction.

We denote by

- $L^p([0, \infty[, A(x)dx)$; $1 < p < \infty$, the space of measurable functions on $[0, \infty[$, satisfying

$$\|f\|_{p,A} = \left(\int_0^\infty (f(x))^p A(x) dx \right)^{\frac{1}{p}} < \infty.$$

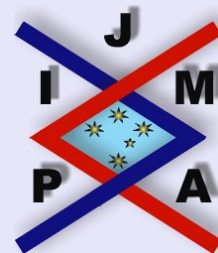
- \mathcal{R}_0 the operator defined for all non-negative measurable functions f by

$$\forall x > 0, \quad \mathcal{R}_0(f)(x) = \int_0^x h(x,t)f(t)dt,$$

where h is the kernel studied in the third section.

- \mathcal{R}_1 the operator defined for all non-negative measurable functions f by

$$\forall x > 0, \quad \mathcal{R}_1(f)(x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\Pi}\Gamma(\alpha + \frac{1}{2})} x^{\alpha - \frac{1}{2}} A^{-\frac{1}{2}}(x) \int_0^x (x^2 - t^2)^{\alpha - \frac{1}{2}} f(t) dt.$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 27 of 40

Definition 5.1.

1. The Riemann-Liouville transform associated with the operator Δ is defined for all non-negative measurable functions f on $]0, \infty[$ by

$$\mathcal{R}(f)(x) = \int_0^x k(x, t) f(t) dt.$$

2. The Weyl transform associated with operator Δ is defined for all non-negative measurable functions f by

$$\mathcal{W}(f)(t) = \int_t^\infty k(x, t) f(x) A(x) dx$$

where k is the kernel given by the relation (2.5).

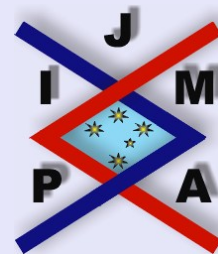
Proposition 5.1.

1. For $\rho > 0$, $\alpha > -\frac{1}{2}$ and $p > \max(2, 2\alpha + 2)$ there exists a positive constant $C_1(\alpha, p)$ such that for all $f \in L^p([0, \infty[, A(x)dx)$,

$$\|\mathcal{R}_0(f)\|_{p,A} \leq C_1(\alpha, p) \|f\|_{p,A}.$$

2. For $\rho = 0$, $\alpha > \frac{1}{2}$ and $p > 2\alpha + 2$, there exists a positive constant $C_2(\alpha, p)$ such that for all $f \in L^p([0, \infty[, A(x)dx)$

$$\|\mathcal{R}_0(f)\|_{p,A} \leq C_2(\alpha, p) \|f\|_{p,A}.$$



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 28 of 40

Proof. 1. Suppose that $\rho > 0$ and $p > \max(2, 2\alpha + 2)$. Let

$$\nu(x) = A^{1-p'}(x)$$

and

$$\mu(x) = C_1(\alpha, a)x^{p(\alpha-\frac{1}{2})}A^{1-\frac{p}{2}}(x)1_{]0,a[}(x) + C_2(\alpha, a)x^{p\gamma}A^{1-\frac{p}{2}}(x)1_{[a,\infty[}(x),$$

with $a > 1$, $C_1(\alpha, a)$, $C_2(\alpha, a)$ and γ are the constants given in Theorem 3.3 and Theorem 3.7.

Then

$$\nu(x) \leq m_1(\alpha, p)x^{(2\alpha+1)(1-p')}$$

and

$$\mu(x) \leq m_2(\alpha, p)x^{2\alpha+1-p}.$$

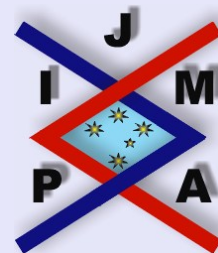
These inequalities imply that

$$\forall b > 0; \quad \int_0^b \nu(x)dx < \infty,$$

$$\forall 0 < b_1 < b_2; \quad \int_{b_1}^{b_2} \mu(x)dx < \infty$$

and

$$\begin{aligned} I(r) &= \left(\int_r^\infty \mu(x)dx \right)^{\frac{1}{p}} \left(\int_0^r \nu(x)dx \right)^{\frac{1}{p'}} \\ &\leq \left(m_2(\alpha, p) \int_r^\infty x^{2\alpha+1-p}dx \right)^{\frac{1}{p}} \left(m_1(\alpha, p) \int_0^r x^{(2\alpha+1)(1-p')}dx \right)^{\frac{1}{p'}} \end{aligned}$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 29 of 40

$$\begin{aligned} &\leq \frac{(m_2(\alpha, p))^{\frac{1}{p}}(m_1(\alpha, p))^{\frac{1}{p'}}}{(p - 2\alpha - 2)^{\frac{1}{p}}((2\alpha + 1)(1 - p') + 1)^{\frac{1}{p'}}} \\ &= \frac{(m_2(\alpha, p))^{\frac{1}{p}} \times ((p - 1)m_1(\alpha, p))^{\frac{1}{p'}}}{p - 2\alpha - 2}. \end{aligned}$$

From Corollary 4.3, there exists a positive constant $C_{p,\alpha}$ such that for all non-negative measurable functions g we have

$$(5.1) \quad \left(\int_0^\infty (\mathcal{H}(g)(x))^p \mu(x) dx \right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty (g(x))^p \nu(x) dx \right)^{\frac{1}{p}},$$

with

$$\mathcal{H}(g)(x) = \int_0^x g(t)\nu(t)dt.$$

Now let us put

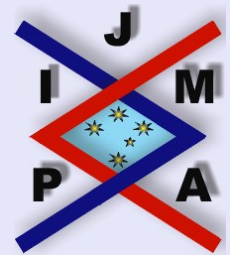
$$T(f)(x) = \left(\frac{\mu(x)}{A(x)} \right)^{\frac{1}{p}} \int_0^x f(t)dt,$$

then we have

$$\mathcal{H}(g)(x) = \left(\frac{\mu(x)}{A(x)} \right)^{-\frac{1}{p}} T(f)(x),$$

where

$$g(x) = f(x)A^{p'-1}(x).$$



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents

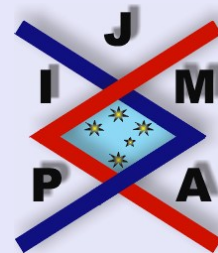


Go Back

Close

Quit

Page 30 of 40



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 31 of 40

From inequality (5.1), we deduce that for all non-negative measurable functions f , we have

$$(5.2) \quad \left(\int_0^\infty (T(f)(x))^p A(x) dx \right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty (f(x))^p A(x) dx \right)^{\frac{1}{p}}.$$

On the other hand from Theorems 3.3 and 3.7 we deduce that the function

$$\mathcal{R}_0(f)(x) = \int_0^x h(x,t)f(t)dt$$

is well defined and we have

$$(5.3) \quad |\mathcal{R}_0(f)(x)| \leq T(|f|)(x).$$

Thus, the relations (5.2) and (5.3) imply that

$$\left(\int_0^\infty |\mathcal{R}_0(f)(x)|^p A(x) dx \right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty |f(x)|^p A(x) dx \right)^{\frac{1}{p}},$$

which proves 1).

2. Suppose that $\rho = 0$ and $\alpha > \frac{1}{2}$. From Theorems 3.3 and 3.6 we have

$$\forall 0 < t < x; \quad |h(t,x)| \leq Cx^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x).$$

Therefore if we take

$$\mu(x) = x^{(\alpha-\frac{1}{2})p}A^{1-\frac{p}{2}}(x)$$

and

$$\nu(x) = A^{1-p'}(x),$$

we obtain the result in the same manner as 1). □

Proposition 5.2. *Suppose that $-\frac{1}{2} < \alpha \leq \frac{1}{2}$, $\rho = 0$ and that there exists a positive constant a such*

$$\forall 0 < t < x, \quad x > a, \quad h(x, t) = 0.$$

Then for all $p > 2\alpha + 2$, we can find a positive constant $C_{\alpha,a}$ satisfying

$$\forall f \in L^p([0, \infty[, A(x)dx); \quad \|\mathcal{R}_0(f)\|_{p,A} \leq C_{\alpha,a} \|f\|_{p,A}.$$

Proof. The hypothesis and Theorem 3.3 imply that there exists a positive constant a such that

$$\forall 0 < t < x; \quad |h(t, x)| \leq C(\alpha, a)x^{\alpha-\frac{1}{2}}A^{-\frac{1}{2}}(x)\mathbf{1}_{]0,a]}(x).$$

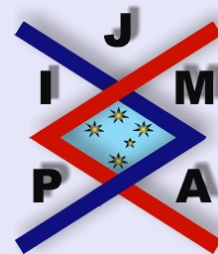
Therefore, if we take

$$\mu(x) = C(\alpha, a)x^{p(\alpha-\frac{1}{2})}A^{1-\frac{p}{2}}(x)\mathbf{1}_{]0,a]}(x)$$

and

$$\nu(x) = A^{1-p'}(x)$$

then, we obtain the result using a similar procedure to that in Proposition 1, 2). □



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 32 of 40

Now, let us study the operator \mathcal{R}_1 defined for all measurable non-negative functions f by

$$\mathcal{R}_1(f)(x) = C_\alpha x^{\frac{1}{2}-\alpha} A^{-\frac{1}{2}}(x) \int_0^x (x^2 - t^2)^{\alpha-\frac{1}{2}} f(t) dt,$$

where

$$C_\alpha = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}.$$

Proposition 5.3.

1. For $\alpha > -\frac{1}{2}$, $\rho > 0$ and $p > \max(2, 2\alpha + 2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0, +\infty[, A(x)dx)$, we have

$$\|\mathcal{R}_1(f)\|_{p,A} \leq C_{p,\alpha} \|f\|_{p,A}.$$

2. For $\alpha > -\frac{1}{2}$, $\rho = 0$ and $p > 2\alpha + 2$ there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0, +\infty[, A(x)dx)$, we have

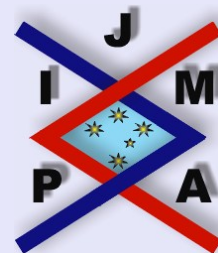
$$\|\mathcal{R}_1(f)\|_{p,A} \leq C_{p,\alpha} \|f\|_{p,A}.$$

Proof. Let T_φ the Hardy type operator defined for all non-negative measurable functions f by

$$T_\varphi(f)(x) = \int_0^x \varphi\left(\frac{t}{x}\right) f(t) \nu(t) dt,$$

where

$$\varphi(x) = (1 - x^2)^{\alpha-\frac{1}{2}}$$



Hardy Type Inequalities For Integral Transforms Associated With A Singular Second Order Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents

⏪

⏩

◀

▶

Go Back

Close

Quit

Page 33 of 40

and

$$\nu(x) = A^{1-p'}(x).$$

Then for all non-negative measurable functions f , we have

$$(5.4) \quad \mathcal{R}_1(f)(x) = C_\alpha x^{-\frac{1}{2}+\alpha} A^{-\frac{1}{2}}(x) T_\varphi(g)(x),$$

where

$$g(x) = f(x) A^{p'-1}(x).$$

Let

$$\mu(x) = x^{p(\alpha-\frac{1}{2})} A^{1-\frac{p}{2}}(x),$$

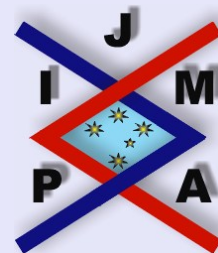
then, according to the hypothesis satisfied by the function A , it follows that there exist positive constants C_1, C_2 such that for all $\alpha > -\frac{1}{2}$ and $\rho > 0$ we have

$$(5.5) \quad \forall x > 0; \quad 0 \leq \mu(x) \leq C_1 x^{2\alpha+1-p}$$

$$(5.6) \quad \forall x > 0; \quad 0 \leq \nu(x) \leq C_2 x^{(2\alpha+1)(1-p')}.$$

Thus from the relations (5.5) and (5.6) we deduce that for $\alpha \geq \frac{1}{2}$, $\rho > 0$ and $p > 2\alpha + 2$, we have

- the function φ is continuous and non-increasing on $]0, 1[$.
- the functions φ , ν and μ satisfy the hypothesis of Theorem 4.1.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 34 of 40

- the functions

$$F(r) = \left(\int_r^\infty \mu(x) dx \right)^{\frac{1}{p}} \left(\int_0^r \left(\varphi \left(\frac{x}{r} \right) \right)^{p'} \nu(x) dx \right)^{\frac{1}{p'}}$$

and

$$G(r) = \left(\int_r^\infty \left(\varphi \left(\frac{r}{x} \right) \right)^p \mu(t) dt \right)^{\frac{1}{p}} \left(\int_0^r \nu(t) dt \right)^{\frac{1}{p'}}$$

are bounded on $[0, \infty[$.

Hence from Theorem 4.1, there exists $C_{p,\alpha} > 0$ such that for all measurable non-negative functions f we have

$$\left(\int_0^\infty (T_\varphi(f(x)))^p \mu(x) dx \right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty (f(x))^p \nu(x) dx \right)^{\frac{1}{p}}.$$

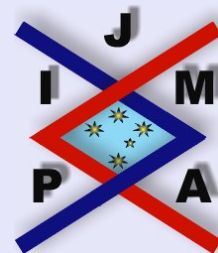
This inequality together with the relation (5.4) lead to

$$\left(\int_0^\infty (\mathcal{R}_1(f(x)))^p A(x) dx \right)^{\frac{1}{p}} \leq C_{p,\alpha} \left(\int_0^\infty (f(x))^p A(x) dx \right)^{\frac{1}{p}}$$

which proves the Proposition 1, 1) in the case $\alpha \geq \frac{1}{2}$.

For $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $p > 2$ we have

- the function φ is continuous and non-decreasing on $]0, 1[$.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 35 of 40

- if we pick

$$\beta \in \left] \max \left(0, \frac{1 - p(\frac{1}{2} + \alpha)}{p(\frac{1}{2} - \alpha)} \right), \min \left(1, \frac{1}{p(\frac{1}{2} - \alpha)} \right) \right[$$

and using inequalities (5.5) and (5.6), we deduce that the function

$$H(r) = \left(\int_r^\infty \left(\varphi \left(\frac{r}{x} \right) \right)^{\beta p} \mu(x) dx \right)^{\frac{1}{p}} \left(\int_0^r \left(\varphi \left(\frac{x}{r} \right) \right)^{(1-\beta)p'} \nu(x) dx \right)^{\frac{1}{p'}}$$

is bounded on $]0, \infty[$.

Consequently, the result follows from Theorem 4.2 and relation (5.4).
2) can be obtained in the same fashion as 1). □

Now we will give the main results of this paper.

Theorem 5.4.

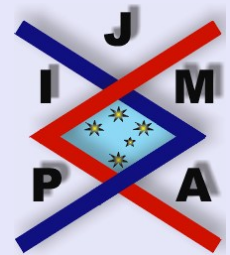
1. For $\alpha > -\frac{1}{2}$, $\rho > 0$ and $p > \max(2, 2\alpha + 2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0, \infty[, A(x)dx)$,

$$\|\mathcal{R}(f)\|_{p,A} \leq C_{p,\alpha} \|f\|_{p,A}.$$

2. For $\alpha > -\frac{1}{2}$, $\rho > 0$ and $p > \max(2, 2\alpha + 2)$, there exists a positive constant $C_{p,\alpha}$ such that for all $g \in L^{p'}([0, \infty[, A(x)dx)$,

$$\left\| \frac{1}{A(x)} \mathcal{W}(g) \right\|_{p',A} \leq C_{p,\alpha} \|g\|_{p',A}$$

where $p' = \frac{p}{p-1}$.



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 36 of 40

Proof. 1) follows from Proposition 1, 1) and Proposition 1, 1), and the fact that

$$\mathcal{R}(f) = \mathcal{R}_0(f) + \mathcal{R}_1(f).$$

2) follows from 1) and the relations

$$(5.7) \quad \|g\|_{p',A} = \max_{\|f\|_{p,A} \leq 1} \int_0^\infty f(x)g(x)A(x)dx,$$

for all measurable non-negative functions f and g

$$(5.8) \quad \int_0^\infty \mathcal{R}(f)(x)g(x)A(x)dx = \int_0^\infty \mathcal{W}(g)(x)f(x)dx.$$

□

Theorem 5.5.

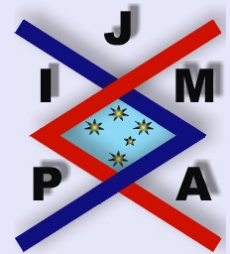
1. For $\alpha > \frac{1}{2}$, $\rho = 0$ and $p > 2\alpha + 2$ there exists a positive constant $C_{p,\alpha}$ such that for all $f \in L^p([0, \infty[, A(x)dx)$

$$\|\mathcal{R}(f)\|_{p,A} \leq C_{p,\alpha} \|f\|_{p,A}.$$

2. For $\alpha > \frac{1}{2}$, $\rho = 0$ and $p > 2\alpha + 2$ there exists a positive constant $C_{p,\alpha}$ such that for all $g \in L^{p'}([0, \infty[, A(x)dx)$

$$\left\| \frac{1}{A(x)} \mathcal{W}(g) \right\|_{p',A} \leq C_{p,\alpha} \|g\|_{p',A}$$

where $p' = \frac{p}{p-1}$.



Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

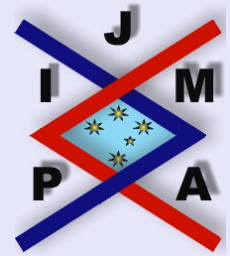
Close

Quit

Page 37 of 40

3. For $-\frac{1}{2} < \alpha \leq \frac{1}{2}$, $\rho = 0$, $p > 2\alpha + 2$ and under the hypothesis of Proposition 5.2, the previous results hold.

Proof. This theorem is obtained by using Propositions 1, 2), 5.2 and 1, 2). \square



**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

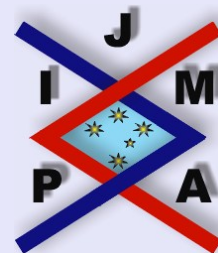
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Quit

Page 38 of 40

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Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator

M. Dziri and L.T. Rachdi

Title Page

Contents

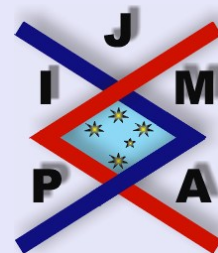


Go Back

Close

Quit

Page 39 of 40



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**Hardy Type Inequalities For
Integral Transforms Associated
With A Singular Second Order
Differential Operator**

M. Dziri and L.T. Rachdi

Title Page

Contents



Go Back

Close

Quit

Page 40 of 40