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ON HYERS-ULAM STABILITY OF A SPECIAL CASE OF O'CONNOR'S AND GAJDA'S FUNCTIONAL EQUATIONS

[1]BELAID BOUIKHALENE, [2]ELHOUCIEN ELQORACHI AND [2]AHMED REDOUANI

[1]University of Ibn Tofail
Faculty of Sciences
Department of Mathematics
Kenitra, Morocco
EMail: bbouikhalene@yahoo.fr

[2]University of Ibn Zohr
Faculty of Sciences
Department of Mathematics
Agadir, Morocco
EMail: elqorachi@hotmail.com

EMail: redouani_ahmed@yahoo.fr

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Abstract

In this paper, we obtain the Hyers-Ulam stability for the following functional equation

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|a(x)\overline{a(y)},\ \ x,y\in G,$$

where G is a locally compact group, K is a compact subgroup of G, ω_K is the normalized Haar measure of K, Φ is a finite group of K-invariant morphisms of G and $f, a : G \longrightarrow \mathbb{C}$ are continuous complex-valued functions such that f satisfies the Kannappan type condition

$$(*)\quad \int_K\int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyk^{-1}hxh^{-1})d\omega_K(k)d\omega_K(h),$$

for all $x, y, z \in G$.

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Contents

1. Introduction

Let G be a locally compact group. Let K be a compact subgroup of G. Let ω_K be the normalized Haar measure of K. A mapping $\varphi : G \to G$ is a morphism of G if φ is a homeomorphism of G onto itself which is either a group-homomorphism, (i.e. $\varphi(xy) = \varphi(x)\varphi(y)$, $x, y \in G$), or a group-antihomomorphism, (i.e. $\varphi(xy) = \varphi(y)\varphi(x)$, $x, y \in G$). We denote by $Mor(G)$ the group of morphisms of G and Φ a finite subgroup of $Mor(G)$ which is K-invariant (i.e. $\varphi(K) \subset K$, for all $\varphi \in \Phi$). The number of elements of a finite group Φ will be designated by $|\Phi|$. The Banach algebra of the complex bounded measures on G is denoted by $M(G)$, it is the topological dual of $C_0(G)$: Banach space of continuous functions vanishing at infinity. Finally the Banach space of all complex measurable and essentially bounded functions on G is denoted by $L_\infty(G)$ and $\mathcal{C}(G)$ designates the space of all continuous complex valued functions on G.

The stability problem for functional equations are strongly related to the question of S.M. Ulam concerning the stability of group homomorphisms [26], [16]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of mathematicians ([16], [2], [3], [22], [23], [24], [25], [20], ...). The main purpose of this paper is to generalize the Hyers-Ulam stability problem for the following functional equation

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|a(x)\overline{a(y)}, \quad x, y \in G, \tag{1.1}$$

where G is a locally compact group, and $f, a \in \mathcal{C}(G)$ with the assumption that

f satisfies the Kannappan type condition: (*)

$$\int_K \int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyk^{-1}hxh^{-1})d\omega_K(k)d\omega_K(h),$$

for all $x, y, z \in G$.

In the case where G is a locally compact abelian group, O'Connor [19], Gajda [14] and Stetkær [21] studied respectively the functional equation

$$f(x-y) = \sum_{i=1}^{n} a_i(x)\overline{a_i(y)}, \qquad x, y \in G,\ n \in \mathbb{N}, \tag{1.2}$$

$$f(x+y) + f(x-y) = 2\sum_{i=1}^{n} a_i(x)\overline{a_i(y)}, \qquad x, y \in G,\ n \in \mathbb{N}, \tag{1.3}$$

and

$$\int_H f(xh \cdot y)dh = a(x)\overline{a(y)}, \quad x, y \in \tilde{G}, \tag{1.4}$$

where $\tilde{G}$ is a locally compact group and H is a compact subgroup of $Aut(\tilde{G})$.

In the case $n = 1$ equations (1.2) and (1.3) are special cases of (1.1). Moreover, taking $G = \tilde{G} \times_s H$ the semi direct product of $\tilde{G}$ and H, and $K = \{e\} \times H$, we observe that equation (1.4) is also a special case of (1.1).

This equation may be considered as a common generalization of functional equations

$$f(xy^{-1}) = a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.5}$$

$$f(xy) + f(xy^{-1}) = 2a(x)\overline{a(y)}, \qquad x, y \in G. \tag{1.6}$$

It is also a generalization of the equations

$$\int_K f(xky^{-1}k^{-1})d\omega_K(k) = a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.7}$$

$$\int_K f(xkyk^{-1})d\omega_K(k) + \int_K f(xky^{-1}k^{-1})d\omega_K(k) = 2a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.8}$$

$$\int_K f(xky^{-1})\overline{\chi}(k)d\omega_K(k) = a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.9}$$

$$\int_K f(xky)\overline{\chi}(k)d\omega_K(k) + \int_K f(xky^{-1})\overline{\chi}(k)d\omega_K(k) = 2a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.10}$$

$$\int_K f(xky^{-1})d\omega_K(k) = a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.11}$$

$$\int_K f(xky)d\omega_K(k) + \int_K f(xk\varphi(y^{-1}))d\omega_K(k) = 2a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.12}$$ ([10], [11]).

If G is a compact group, equation (1.1) may be considered as a generalization of the equations

$$\int_G f(xty^{-1}t^{-1})dt = a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.13}$$

$$\int_G f(xtyt^{-1})dt + \int_G f(xty^{-1}t^{-1})dt = 2a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.14}$$

$$\sum_{\varphi\in\Phi} \int_G f(xt\varphi(y^{-1})t^{-1})dt = |\Phi|a(x)\overline{a(y)}, \qquad x, y \in G. \tag{1.15}$$

Furthermore the following equations are also a particular case of (1.1).

$$\sum_{\varphi\in\Phi} f(x\varphi(y^{-1})) = |\Phi|a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.16}$$

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y^{-1}))d\omega_K(k) = |\Phi|a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.17}$$

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y^{-1}))\overline{\chi}(k)d\omega_K(k) = |\Phi|a(x)\overline{a(y)}, \qquad x, y \in G, \tag{1.18}$$

where χ is a character of K. For more information about the equations (1.1) – (1.18) (see [1], [4], [6], [7], [11], [12], [14], [15], [19], [21]).

In the next section, we note some results for later use.

2. Generalized Stability Results of Cauchy's and Wilson's Equations

Let G, K and Φ be given as above. One can prove (see [4]) the following two propositions.

Proposition 2.1. *For an arbitrary fixed $\tau \in \Phi$, the mapping*

$$\Phi \ni \varphi \mapsto \varphi \circ \tau \in \Phi$$

is a bijection and for all $x, y \in G$, we have

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) = \sum_{\psi\in\Phi} \int_K f(xk\psi(y)k^{-1})d\omega_K(k).$$

Proposition 2.2. *Let $\varphi \in \Phi$ and $f \in \mathcal{C}(G)$. Then*

i)

$$\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) = \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k), \quad x, y \in G,\ h \in K.$$

ii) Moreover, if f satisfies the Kannappan type condition $(*)$, then we have

$$\int_K \int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k)$$
$$= \int_K \int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k),$$

for all $z, y, x \in G$.

The next results extend the ones obtained in [4], [8], [9], [10] and [13].

Theorem 2.3. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be a continuous function. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions such that f satisfies the Kannappan type condition $(*)$ and*

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)g(y)\right| \leq \varepsilon(y), \qquad x, y \in G. \tag{2.1}$$

If f is unbounded, then g satisfies the functional equation

$$\sum_{\varphi\in\Phi}\int_K g(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|g(x)g(y), \qquad x, y \in G. \tag{2.2}$$

Proof. Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be a continuous function, and let $f, g \in C(G)$ satisfying inequality (2.1). Let $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ (resp. Φ^-) is a set of group-homomorphisms (resp. of group-antihomomorphisms). By using Propositions 2.1, 2.2 and the fact that f satisfies the condition (*), for all $x, y, z \in G$, we get

$$\begin{aligned}
&|\Phi||f(z)|\left|\sum_{\varphi\in\Phi}\int_K g(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|g(x)g(y)\right| \\
&= \left|\sum_{\varphi\in\Phi}\int_K |\Phi|f(z)g(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|^2 f(z)g(x)g(y)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \left| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1})d\omega_K(k)d\omega_K(h) \right.\\
&\qquad \left. - \sum_{\varphi\in\Phi} \int_K |\Phi| f(z)g(xk\varphi(y)k^{-1})d\omega_K(k) \right|\\
&\quad + \left| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1})d\omega_K(k)d\omega_K(h) \right.\\
&\qquad \left. - |\Phi| g(y) \sum_{\tau\in\Phi} \int_K f(zh\tau(x)h^{-1})d\omega_K(h) \right|\\
&\quad + |\Phi||g(y)| \left| \sum_{\tau\in\Phi} \int_K f(zk\tau(x)k^{-1})d\omega_K(k) - |\Phi| f(z)g(x) \right|\\
&= \left| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1})d\omega_K(k)d\omega_K(h) \right.\\
&\qquad \left. - \sum_{\varphi\in\Phi} \int_K |\Phi| f(z)g(xk\varphi(y)k^{-1})d\omega_K(k) \right|\\
&\quad + \left| \sum_{\psi\in\Phi} \int_K \sum_{\tau\in\Phi^+} \int_K f(zh\tau(x)k\psi(y)k^{-1})h^{-1})d\omega_K(k)d\omega_K(h) \right.\\
&\qquad + \sum_{\psi\in\Phi} \int_K \sum_{\tau\in\Phi^-} \int_K f(zhk^{-1}\psi(y)k\tau(x)h^{-1})d\omega_K(k)d\omega_K(h)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -|\Phi| g(y) \sum_{\tau\in\Phi} \int_K f(zh\tau(x)h^{-1}) d\omega_K(h) \Bigg| \\
&\quad + |\Phi||g(y)| \left| \sum_{\tau\in\Phi} \int_K f(zk\tau(x)k^{-1}) d\omega_K(k) - |\Phi| f(z) g(x) \right| \\
&= \Bigg| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(k) d\omega_K(h) \\
&\qquad - \sum_{\varphi\in\Phi} \int_K |\Phi| f(z) g(xk\varphi(y)k^{-1}) d\omega_K(k) \Bigg| \\
&\quad + \Bigg| \sum_{\psi\in\Phi} \int_K \sum_{\tau\in\Phi^+} \int_K f(zh\tau(x)h^{-1}k\psi(y)k^{-1}) d\omega_K(k) d\omega_K(h) \\
&\qquad + \sum_{\psi\in\Phi} \int_K \sum_{\tau\in\Phi^-} \int_K f(zh\tau(x)h^{-1}k\psi(y)k^{-1}) d\omega_K(k) d\omega_K(h) \\
&\qquad -|\Phi| g(y) \sum_{\tau\in\Phi} \int_K f(zh\tau(x)h^{-1}) d\omega_K(h) \Bigg| \\
&\quad + |\Phi||g(y)| \left| \sum_{\tau\in\Phi} \int_K f(zk\tau(x)k^{-1}) d\omega_K(k) - |\Phi| f(z) g(x) \right| \\
&= \Bigg| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(k) d\omega_K(h)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \sum_{\varphi\in\Phi}\int_K |\Phi| f(z) g(xk\varphi(y)k^{-1}) d\omega_K(k)\Bigg| \\
&+ \left|\sum_{\psi\in\Phi}\int_K \sum_{\tau\in\Phi}\int_K f(zh\tau(x)h^{-1}k\psi(x)k^{-1}) d\omega_K(k) d\omega_K(h)\right. \\
&\qquad \left. -|\Phi| g(y) \sum_{\tau\in\Phi}\int_K f(zh\tau(x)h^{-1}) d\omega_K(h)\right| \\
&+ |\Phi||g(y)| \left|\sum_{\tau\in\Phi}\int_K f(zk\tau(x)k^{-1}) d\omega_K(k) - |\Phi| f(z) g(x)\right| \\
\le & \sum_{\varphi\in\Phi}\int_K \left|\sum_{\tau\in\Phi}\int_K f(zh\tau(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h)\right. \\
&\qquad \left. - |\Phi| f(z) g(xk\varphi(y)k^{-1})\right| d\omega_K(k) \\
&+ \sum_{\tau\in\Phi}\int_K \left|\sum_{\psi\in\Phi}\int_K f(zh\tau(x)h^{-1}k\psi(y)k^{-1})) d\omega_K(h)\right. \\
&\qquad \left. - |\Phi| f(zh\tau(x)h^{-1}) g(y)\right| d\omega_K(h) \\
&+ |\Phi||g(y)| \left|\sum_{\tau\in\Phi}\int_K f(zh\tau(x)h^{-1}) d\omega_K(h) - |\Phi| f(z) g(x)\right| \\
\le & \sum_{\varphi\in\Phi}\int_K \varepsilon(xk\varphi(y)k^{-1}) d\omega_K(k) + |\Phi|\varepsilon(y) + |\Phi||g(y)|\varepsilon(x).
\end{aligned}
$$

Since f is unbounded, then it follows that g is a solution of (2.1). This ends the proof of our theorem. □

The next results extend the ones obtained in [8] and [13].

Theorem 2.4. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions such that f satisfies the condition $(*)$ and*

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi| f(x)g(y)\right| \le \varepsilon(y), \qquad x, y \in G. \tag{2.3}$$

Suppose furthermore there exists $x_0 \in G$ such that $|g(x_0)| > 1$. Then there exists exactly one solution $F \in C(G)$ of the equation

$$\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi| F(x)g(y), \qquad x, y \in G, \tag{2.4}$$

such that $F - f$ is bounded and one has

$$|F(x) - f(x)| \le \frac{\varepsilon(x_0)}{|\Phi|(|g(x_0)| - 1)}, \qquad x \in G. \tag{2.5}$$

Proof. In the proof, we use the ideas and methods that are analogous to the ones used in [8], [13] and [20]. Let $\beta = |\Phi| g(x_0)$, for all $x \in G$, one has

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(x_0)k^{-1})d\omega_K(k) - \beta f(x)\right| \le \varepsilon(x_0), \qquad x, y \in G. \tag{2.6}$$

We define the following functions sequence

$$G_1(x) = \sum_{\varphi\in\Phi}\int_K f(xk\varphi(x_0)k^{-1})d\omega_K(k), \qquad x \in G, \tag{2.7}$$

$$G_{n+1}(x) = \sum_{\varphi\in\Phi}\int_K G_n(xk\varphi(x_0)k^{-1})d\omega_K(k), \qquad x \in G \text{ and } n \in \mathbb{N}. \tag{2.8}$$

Next, we will prove the uniform convergence of the function sequence $(\beta^{-n}G_n)_{n\geq 1}$, therefore we need to show by induction the following inequalities

$$|G_{n+1}(x) - \beta G_n(x)| \leq |\Phi|^n \varepsilon(x_0), \qquad x \in G, n \geq 1, \tag{2.9}$$

$$|G_n(x) - \beta^n f(x)| \leq \varepsilon(x_0)(|\Phi|^{n-1} + |\Phi|^{n-2}|\beta| + \cdots + |\beta|^{n-1}), \tag{2.10}$$

and

$$|\beta^{-(n+1)}G_{n+1}(x) - \beta^{-n}G_n(x)| \leq |\beta|^{-(n+1)}|\Phi|^n\varepsilon(x_0). \tag{2.11}$$

In view of (2.5) one has for all $x \in G$

$$\begin{aligned}|G_2(x) - \beta G_1(x)|\\ &= \left|\sum_{\varphi\in\Phi}\int_K G_1(xk\varphi(x_0)k^{-1})d\omega_K(k) - \beta\sum_{\varphi\in\Phi}\int_K f(xk\varphi(x_0)k^{-1})d\omega_K(k)\right|\end{aligned}$$

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$$
\begin{aligned}
&= \left| \sum_{\varphi\in\Phi} \int_K \sum_{\tau\in\Phi} \int_K f(xk\varphi(x_0)k^{-1}h\tau(x_0)h^{-1}) d\omega_K(h) d\omega_K(k) \right. \\
&\qquad \left. -\beta \sum_{\tau\in\Phi} \int_K f(xk\tau(x_0)k^{-1}) d\omega_K(k) \right| \\
&\leq \sum_{\tau\in\Phi} \int_K \left| \sum_{\varphi\in\Phi} \int_K f(xk\tau(x_0)k^{-1}h\varphi(x_0)h^{-1}) d\omega_K(h) \right. \\
&\qquad \left. - \beta f(xk\tau(x_0)k^{-1}) \right| d\omega_K(k) \\
&\leq |\Phi| \varepsilon(x_0).
\end{aligned}
$$

Assume (2.8) holds for $n \geq 1$, then for $n+1$, one has

$$
\begin{aligned}
|G_{n+2}(x) - \beta G_{n+1}(x)| &= \left| \sum_{\varphi\in\Phi} \int_K G_{n+1}(xk\varphi(x_0)k^{-1}) d\omega_K(k) \right. \\
&\qquad \left. -\beta \sum_{\varphi\in\Phi} \int_K G_n(xk\varphi(x_0)k^{-1}) d\omega_K(k) \right| \\
&\leq \sum_{\varphi\in\Phi} \int_K |G_{n+1}(xk\varphi(x_0)k^{-1}) d\omega_K(k) \\
&\qquad - \beta G_n(xk\varphi(x_0)k^{-1})| d\omega_K(k) \\
&\leq |\Phi|^{n+1} \varepsilon(x_0).
\end{aligned}
$$

In view of (2.5) we have for all $x \in G$

$$|G_1(x) - \beta f(x)| = \left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(x_0)k^{-1}) d\omega_K(k) - \beta f(x) \right|$$
$$\leq \varepsilon(x_0).$$

Suppose (2.9) is true for $n \geq 1$. For $n+1$ one has

$$\begin{aligned} &|G_{n+1}(x) - \beta^{n+1} f(x)| \\ &\qquad \leq |G_{n+1}(x) - \beta G_n(x)| + |\beta||G_n(x) - \beta^n f(x)| \\ &\qquad \leq |\Phi|^n \varepsilon(x_0) + |\beta|\varepsilon(x_0)(|\Phi|^{n-1} + |\Phi|^{n-2}|\beta| + \cdots + |\beta|^{n-1}) \\ &\qquad = \varepsilon(x_0)(|\Phi|^n + |\Phi|^{n-1}|\beta| + \cdots + |\beta|^n). \end{aligned}$$

For inequality (2.10), using (2.8), for all $x \in G$ we get

$$\begin{aligned} |\beta^{-(n+1)} G_{n+1}(x) - \beta^{-n} G_n(x)| &= |\beta^{-(n+1)}||G_{n+1}(x) - \beta G_n(x)| \\ &\leq |\beta|^{-(n+1)} |\Phi|^n \varepsilon(x_0). \end{aligned}$$

So by using inequality (2.10) we deduce the uniform convergence of the sequence $(\beta^{-n} G_n)_{n \geq 1}$. Let F be a continuous function defined by

$$F(x) = \lim_{n \longrightarrow +\infty} \beta^{-n} G_n(x), \qquad x \in G.$$

Since

$$\beta^{-(n+1)} G_{n+1}(x) = \beta^{-1} \sum_{\varphi \in \Phi} \int_K \beta^{-n} G_n(xk\varphi(x_0)k^{-1}) d\omega_K(k),$$

then one has

$$\beta F(x)=\sum_{\varphi\in\Phi}\int_K F(xk\varphi(x_0)k^{-1})d\omega_K(k),\qquad x\in G.$$

In view of (2.9), one has for all $x\in G$

$$|\beta^{-n}G_n(x)-f(x)|\le|\beta|^{-n}\varepsilon(x_0)(|\Phi|^{n-1}+|\Phi|^{n-2}|\beta|+\cdots+|\beta|^{n-1}),$$

which proves that

$$|F(x)-f(x)|<\frac{\varepsilon(x_0)}{|\Phi|(|g(x_0)|-1)},\qquad x\in G.$$

Now we are going to show that F satisfies the equation

$$\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y)k^{-1})d\omega_K(k)=|\Phi|F(x)g(y),\qquad x,y\in G.$$

Thus we need to show by induction the inequality

$$\left|\sum_{\varphi\in\Phi}\int_K\beta^{-n}G_n(xk\varphi(y)k^{-1})d\omega_K(k)-|\Phi|\beta^{-n}G_n(x)g(y)\right|\le\frac{\varepsilon(y)}{|g(x_0)|^n}. \tag{2.12}$$

For $n = 1$, one has, by using the fact that f satisfies the condition $(*)$

$$\frac{1}{\beta}\left|\sum_{\varphi\in\Phi}\int_K G_1(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|G_1(x)g(y)\right|$$

$$= \frac{1}{\beta}\left|\sum_{\varphi\in\Phi}\int_K\sum_{\tau\in\Phi}\int_K f(xk\varphi(y)k^{-1}h\tau(x_0)h^{-1})d\omega_K(k)d\omega_K(h)\right.$$

$$\left. -|\Phi|g(y)\sum_{\tau\in\Phi}\int_K f(xk\tau(x_0)k^{-1})d\omega_K(k)\right|$$

$$= \frac{1}{\beta}\left|\sum_{\tau\in\Phi}\int_K\sum_{\varphi\in\Phi}\int_K f(xk\tau(x_0)k^{-1}h\varphi(y)h^{-1})d\omega_K(k)d\omega_K(h)\right.$$

$$\left. -|\Phi|g(y)\sum_{\tau\in\Phi}\int_K f(xk\tau(x_0)k^{-1})d\omega_K(k)\right|$$

$$\leq \frac{1}{\beta}\sum_{\tau\in\Phi}\int_K\left|\sum_{\varphi\in\Phi}\int_K f(xk\tau(x_0)k^{-1}h\varphi(y)h^{-1})d\omega_K(h)\right.$$

$$\left. - |\Phi|g(y)f(xk\tau(x_0)k^{-1})g(y)\right|d\omega_K(k)$$

$$\leq \frac{|\Phi|\varepsilon(y)}{|\beta|} = \frac{\varepsilon(y)}{|g(x_0)|}.$$

Assume (2.12) holds for some $n \geq 1$. For $n + 1$, one has by using the fact that

f satisfies the condition $(*)$

$$\left|\sum_{\varphi\in\Phi}\int_K \beta^{-(n+1)}G_{n+1}(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|\beta^{-(n+1)}G_{n+1}(x)g(y)\right|$$

$$= \frac{1}{\beta}\Bigg|\sum_{\varphi\in\Phi}\int_K \sum_{\tau\in\Phi}\int_K \beta^{-n}G_n(xk\varphi(y)k^{-1}h\tau(x_0)h^{-1})d\omega_K(h)d\omega_K(k)$$

$$- |\Phi|g(y)\sum_{\tau\in\Phi}\int_K \beta^{-n}G_n(xk\tau(x_0)k^{-1})d\omega_K(k)\Bigg|$$

$$\leq \frac{1}{\beta}\sum_{\tau\in\Phi}\int_K \Bigg|\sum_{\varphi\in\Phi}\int_K \beta^{-n}G_n(xk\tau(x_0)k^{-1}h\varphi(y)h^{-1})d\omega_K(h)$$

$$- |\Phi|g(y)\beta^{-n}G_n(xk\tau(x_0)k^{-1})\Bigg|d\omega_K(k)\Bigg|$$

$$\leq \frac{|\Phi|}{|\Phi|\ |\ g(x_0)\ ||}\frac{\varepsilon(y)}{|g(x_0)|^n} = \frac{\varepsilon(y)}{|g(x_0)|^{n+1}}.$$

□

3. The Main Results

In the next proposition, we investigate the stability of the functional equation (1.1).

Proposition 3.1. *Let* $\delta > 0$. *Let* $f, a \in C(G)$ *such that*

$$\left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y^{-1})k^{-1}) d\omega_K(k) - |\Phi| a(x)\overline{a}(y) \right| \leq \delta, \qquad x, y \in G. \tag{3.1}$$

Then

i) If f *is bounded then* a *is bounded and one has,*

$$|a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \qquad x \in G,$$

$$|f(x)| \leq |a(e)| \sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x \in G.$$

ii) If f *is unbounded then* $a(e) \neq 0$. *Furthermore there exists* $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$.

Proof. i) Let f be a continuous bounded solution of (3.1), then by taking $x = y$ in (3.1) we get

$$|\Phi||a(x)|^2 \leq |\Phi| \sup|f| + \delta, \qquad x \in G,$$

i.e.

$$|a(x)| \le \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \qquad x \in G.$$

For $y = e$ in (3.1) we get

$$|f(x)| \le |a(x)||a(e)| + \frac{\delta}{|\Phi|},$$

i.e.

$$|f(x)| \le |a(e)|\sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x \in G.$$

We will prove (ii) by contradiction. If $a(e) = 0$ then $|f(x)| < \frac{\delta}{|\Phi|}$.

If $|a(x)| \le |a(e)|$, for all $x \in G$, then by taking $y = e$ in (3.1) one has

$$|f(x)| \le |a(e)|^2 + \frac{\delta}{|\Phi|}, \qquad x \in G,$$

i.e. f is bounded, which is the desired contradiction. □

The main results are the following theorems.

Theorem 3.2. *Let $\delta > 0$. Assume that $f, a \in C(G)$ satisfy inequality (3.1) and f fulfills $(*)$. Then*

i) f, a are bounded
or

ii) f is unbounded and

$$\overline{a}(e)\sum_{\varphi\in\Phi}\int_K \check{\overline{a}}(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|\check{\overline{a}}(x)\check{\overline{a}}(y), \qquad x,y\in G, \tag{3.2}$$

where $\check{\overline{a}}(x) = \overline{a(x^{-1})}$*, for* $x \in G$.

Proof. ii) Since f is unbounded then by Proposition 3.1 we have $a(e) \neq 0$. By using the fact that f and a satisfy inequality (3.1) one has

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|a(x)\check{\overline{a}}(y) + \theta(x,y), \qquad x,y\in G,$$

where $|\theta(x,y)| < \delta$. By taking $y = e$ we get for all $x \in G$

$$|\Phi|f(x) = |\Phi|a(x)\overline{a}(e) + \theta(x,e),$$

so

$$(f(x) - a(x)\overline{a}(e)) = \frac{1}{|\Phi|}\theta(x,e),$$

then we get

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)g(y)\right| < \varepsilon(y), \qquad x,y\in G,$$

where

$$g(y) = \frac{\check{\overline{a}}(y)}{\overline{a}(e)}, \quad \text{and} \quad \varepsilon(y) = \delta(1 + |g(y)|).$$

In view of Theorem 2.3, we deduce that

$$\overline{a}(e)\sum_{\varphi\in\Phi}\int_K \check{\overline{a}}(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|\check{\overline{a}}(x)\check{\overline{a}}(y), \qquad x,y\in G.$$

The cases of f bounded follows from Proposition 3.1. □

Theorem 3.3. *Let $\delta > 0$. Assume that $f, a \in C(G)$ satisfy inequality (3.1) and f fulfills $(*)$. Then either*

$$|a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \qquad x \in G, \tag{3.3}$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x\in G, \tag{3.4}$$

or there exist $x_0 \in G$ such that $|a(x_0)| > |a(e)|$ and a unique continuous function $F : G \longrightarrow \mathbb{C}$ such that

a)

$$\overline{a}(e)\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|F(x)\check{\overline{a}}(y), \qquad x,y\in G,$$

b) $F - f$ is bounded and one has

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{|\Phi|(|a(x_0)| - |a(e)|)}, \qquad x\in G.$$

Proof. If f is bounded, by using Theorem 3.2 and Proposition 3.1, we obtain the first case of the theorem.

Now, let f be unbounded. Since $a(e) \neq 0$ it follows that

$$\left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi| f(x) g(y) \right| < \varepsilon(y), \qquad x, y \in G,$$

where

$$g(y) = \frac{\check{\bar{a}}(y)}{\overline{\overline{a(e)}}}, \quad \text{and} \quad \varepsilon(y) = \delta(1 + |g(y)|).$$

Finally, by using Proposition 3.1 and Theorem 2.4 we get the rest of the proof. □

4. Applications

The following theorems are a particular case of Theorem 3.3.

If $K \subset Z(G)$, then we have

Theorem 4.1. *Let $\delta > 0$. Let f, a be a complex-valued functions on G such that f satisfies the Kannappan condition (see [18])*

$$f(zxy) = f(zyx), \qquad x, y \in G \tag{4.1}$$

and the functional inequality

$$\left| \sum_{\varphi \in \Phi} f(x\varphi(y)) - |\Phi| a(x)\overline{a}(y) \right| \leq \delta, \qquad x, y \in G. \tag{4.2}$$

Then either

$$|a(x)| \leq \sqrt{\sup |f| + \frac{\delta}{|\Phi|}}, \qquad x \in G, \tag{4.3}$$

$$|f(x)| \leq |a(e)| \sqrt{\sup |f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x \in G, \tag{4.4}$$

or there exist $x_0 \in G$ such that $|a(x_0)| > |a(e)|$ and a unique function $F : G \longrightarrow \mathbb{C}$ such that *a)*

$$\overline{a}(e)\sum_{\varphi\in\Phi}F(x\varphi(y)) = |\Phi|F(x)\check{\overline{a}}(y), \qquad x,y\in G,$$

b) $F-f$ *is bounded and one has*

$$|F(x)-f(x)| \leq \frac{\delta(|a(e)|+|a(x_0)|)}{|\Phi|(|a(x_0)|-|a(e)|)}, \qquad x\in G.$$

If G is abelian then condition (4.1) holds. By taking $\Phi=\{I\}$ (resp. $\Phi=\{I,-I\}$), we get the following corollaries.

Corollary 4.2. *Let* $\delta>0$. *Let* f,a *be complex-valued functions on* G *such that*

(4.5) $$|f(x-y)-a(x)\overline{a}(y)|\leq\delta, \qquad x,y\in G.$$

Then either

(4.6) $$|a(x)|\leq\sqrt{\sup|f|+\delta}, \qquad x\in G,$$

(4.7) $$|f(x)|\leq|a(e)|\sqrt{\sup|f|+\delta}+\delta, \qquad x\in G,$$

or there exist $x_0\in G$ *such that* $|a(x_0)|>|a(e)|$ *and a unique function* $F: G\longrightarrow\mathbb{C}$ *such that*

a)

$$\overline{a}(e)F(x+y)=F(x)\check{\overline{a}}(y), \qquad x,y\in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Corollary 4.3. *Let* $\delta > 0$. *Let* f, a *be a complex-valued functions on* G *such that*

$$|f(x+y) + f(x-y) - 2a(x)\overline{a}(y)| \leq \delta, \qquad x, y \in G. \tag{4.8}$$

Then either

$$|a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{2}}, \qquad x \in G, \tag{4.9}$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f| + \frac{\delta}{2}} + \frac{\delta}{2}, \qquad x \in G,$$

or there exist $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$ *and a unique function* $F : G \longrightarrow \mathbb{C}$ *such that*

a)
$$\overline{a}(e)F(x+y) + \overline{a}(e)F(x-y) = 2F(x)\check{\overline{a}}(y), \qquad x, y \in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{2(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

If $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$ and $x \in G$, where χ is a character of K, then we have

Theorem 4.4. *Let $\delta > 0$ and let χ be a character of K. Assume that $(f, a) \in \mathcal{C}(G)$ satisfy $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$,*

$$\int_K \int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) \tag{4.10}$$

and

$$\left| \sum_{\varphi\in\Phi} \int_K f(xk\varphi(y^{-1}))\overline{\chi}(k)d\omega_K(k) - |\Phi|a(x)\overline{a}(y) \right| \le \delta, \qquad x, y \in G. \tag{4.11}$$

Then either

$$|a(x)| \le \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \qquad x \in G, \tag{4.12}$$

$$|f(x)| \le |a(e)|\sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x \in G, \tag{4.13}$$

or there exist $x_0 \in G$ such that $|a(x_0)| > |a(e)|$ and a unique continuous function $F : G \longrightarrow \mathbb{C}$ such that *a)*

$$\overline{a}(e)\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y))\overline{\chi}(k)d\omega_K(k)=|\Phi|F(x)\check{\overline{a}}(y),\quad x,y\in G,$$

b) $F-f$ *is bounded and one has*

$$|F(x)-f(x)|\leq\frac{\delta(|a(e)|+|a(x_0)|)}{|\Phi|(|a(x_0)|-|a(e)|)},\qquad x\in G.$$

Corollary 4.5. *Let* $\delta>0$ *and let* χ *be a character of* K. *Assume that* $(f,a)\in\mathcal{C}(G)$ *satisfy* $f(kxh)=\chi(k)f(x)\chi(h)$, $k,h\in K$, $x\in G$,

$$\int_K\int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)=\int_K\int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)$$

and

(4.14) $$\left|\int_K f(xky)\overline{\chi}(k)d\omega_K(k)+\int_K f(xky^{-1}))\overline{\chi}(k)d\omega_K(k)-2a(x)\overline{a}(y)\right|\leq\delta,\quad x,y\in G.$$

Then either

(4.15) $$|a(x)|\leq\sqrt{\sup|f|+\frac{\delta}{2}},\qquad x\in G,$$

$$(4.16)\qquad |f(x)| \le |a(e)|\sqrt{\sup|f| + \frac{\delta}{2}} + \frac{\delta}{2}, \qquad x \in G,$$

or there exist $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$ *and a unique continuous function* $F : G \longrightarrow \mathbb{C}$ *such that*

a)

$$\overline{a}(e)\int_K F(xky)\overline{\chi}(k)d\omega_K(k) + \overline{a}(e)\int_K F(xky^{-1}))\overline{\chi}(k)d\omega_K(k) = 2F(x)\check{\overline{a}}(y), \qquad x, y \in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \le \frac{\delta(|a(e)| + |a(x_0)|)}{2(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Corollary 4.6. *Let* $\delta > 0$ *and let* χ *be a character of* K. *Assume that* $(f, a) \in \mathcal{C}(G)$ *satisfy* $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$, *and*

$$(4.17)\qquad \left|\int_K f(xky^{-1}))\overline{\chi}(k)d\omega_K(k) - a(x)\overline{a}(y)\right| \le \delta, \qquad x, y \in G.$$

Then either

$$(4.18)\qquad |a(x)| \le \sqrt{\sup|f| + \delta}, \qquad x \in G,$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f|+\delta} + \delta, \; x \in G,$$

or there exist $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$ *and a unique continuous function* $F : G \longrightarrow \mathbb{C}$ *such that*

a)

$$\overline{a}(e) \int_K F(xky)\overline{\chi}(k)d\omega_K(k) = F(x)\check{\overline{a}}(y), \qquad x, y \in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |\, a(x_0)\,|)}{(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Remark 1. *If the algebra* $\overline{\chi}\omega_K * M(G) * \overline{\chi}\omega_K$ *is commutative then the condition* $(*)$ *holds [4]. Furthermore in the case where* $\Phi = \{I\}$, *we do not need the condition* $(*)$.

In the next theorem we assume that f is bi-K-invariant (i.e. $f(hxk) = f(x), h, k \in K, x \in G$), then we have

Theorem 4.7. *Let* $\delta > 0$ *and assume that* $(f, a) \in \mathcal{C}(G)$ *satisfy* $f(kxh) = f(x)$, $k, h \in K$, $x \in G$,

$$\int_K \int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)d\omega_K(k)d\omega_K(h) \tag{4.19}$$

and

$$\left| \sum_{\varphi\in\Phi} \int_K f(xk\varphi(y^{-1}))d\omega_K(k) - |\Phi| a(x)\overline{a}(y) \right| \leq \delta, \qquad x, y \in G. \tag{4.20}$$

Then either

$$|a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \qquad x \in G, \tag{4.21}$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|}, \qquad x \in G, \tag{4.22}$$

or there exist $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$ *and a unique continuous function* $F : G \longrightarrow \mathbb{C}$ *such that*

a)

$$\overline{a}(e)\sum_{\varphi\in\Phi}\int_K F(xk\varphi(y))d\omega_K(k) = |\Phi|F(x)\check{\overline{a}}(y), \qquad x, y \in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{|\Phi|(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Corollary 4.8. *Let* $\delta > 0$ *and assume that* $(f, a) \in \mathcal{C}(G)$ *satisfy* $f(kxh) = f(x)$*,* $k, h \in K$*,* $x \in G$*,*

$$\int_K\int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyhx)d\omega_K(k)d\omega_K(h)$$

and

$$(4.23)\qquad \left|\int_K f(xky)d\omega_K(k) + \int_K f(xky^{-1}))d\omega_K(k) - 2a(x)\overline{a}(y)\right| \leq \delta, \qquad x, y \in G.$$

Then either

$$(4.24)\qquad |a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{2}}, \qquad x \in G,$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f| + \frac{\delta}{2}} + \frac{\delta}{2}, \qquad x \in G,$$

or there exist $x_0 \in G$ *such that* $|a(x_0)| > |a(e)|$ *and a unique continuous function* $F : G \longrightarrow \mathbb{C}$ *such that*

a)

$$\overline{a}(e)\int_K F(xky)d\omega_K(k) + \overline{a}(e)\int_K F(xky^{-1})d\omega_K(k) = 2F(x)\check{\overline{a}}(y), \quad x, y \in G,$$

b) $F - f$ *is bounded and one has*

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{2(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Corollary 4.9. *Let $\delta > 0$ and assume that $(f, a) \in \mathcal{C}(G)$ satisfy $f(kxh) = f(x)$, $k, h \in K$, $x \in G$, and*

$$\left|\int_K f(xky^{-1}))d\omega_K(k) - a(x)\overline{a}(y)\right| \leq \delta, \qquad x, y \in G. \tag{4.25}$$

Then either

$$|a(x)| \leq \sqrt{\sup|f| + \delta}, \qquad x \in G, \tag{4.26}$$

$$|f(x)| \leq |a(e)|\sqrt{\sup|f| + \delta} + \delta, \qquad x \in G,$$

or there exist $x_0 \in G$ such that $|a(x_0)| > |a(e)|$ and a unique continuous function $F : G \longrightarrow \mathbb{C}$ such that

a)

$$\overline{a}(e) \int_K F(xky)d\omega_K(k) = F(x)\check{\overline{a}}(y), \qquad x, y \in G,$$

b) $F - f$ is bounded and one has

$$|F(x) - f(x)| \leq \frac{\delta(|a(e)| + |a(x_0)|)}{(|a(x_0)| - |a(e)|)}, \qquad x \in G.$$

Remark 2. *If the algebra $\omega_K * M(G) * \omega_K$ is commutative then the condition $(*)$ holds [4].*

In the next corollary, we assume that $G = K$ is a compact group.

Theorem 4.10. *Let $\delta > 0$ and let f, a be complex measurable and essentially bounded functions on G such that f is a central function and (f, a) satisfy the inequality*

$$\left| \sum_{\varphi\in\Phi} \int_G f(xt\varphi(y)t^{-1})dt - |\Phi| a(x)\overline{a}(y) \right| \leq \delta, \qquad x, y \in G. \tag{4.27}$$

Then

$$|a(x)| \leq \sqrt{\sup|f| + \frac{\delta}{|\Phi|}}, \tag{4.28}$$

and

$$|f(x)| \leq |a(e)| \sqrt{\sup|f| + \frac{\delta}{|\Phi|}} + \frac{\delta}{|\Phi|},$$

for all $x \in G$.

Proof. Let $f, a \in L^\infty(G)$. Since f is central, then it satisfies the condition $(*)$ ([4], [6]). If f is unbounded then a is a solution of the functional equation (3.2). In view of [15], we get the fact that a is continuous. Since G is compact then a is bounded. Consequently f is bounded, which is the desired property. □

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