

Journal of Inequalities in Pure and Applied Mathematics

ON SOME ADVANCED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

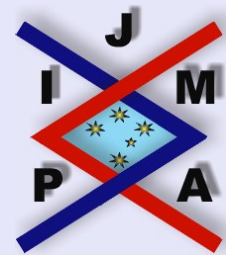
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ISSN (electronic): 1443-5756
214-04



volume 6, issue 3, article 60,
2005.

*Received 08 November, 2004;
accepted 03 April, 2005.*

Communicated by: S.S. Dragomir

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Abstract

In this paper, we obtain a generalization of advanced integral inequality and by means of examples we show the usefulness of our results.

2000 Mathematics Subject Classification: 26D15, 26D10.

Key words: Advanced integral inequality; Integral equation.

Project supported by a grant from NECC and NSF of Shandong Province, China (Y2001A03).

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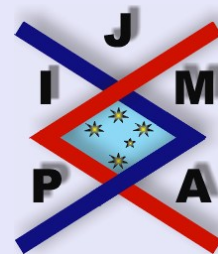
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1. Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations. Many retarded inequalities have been discovered (see [2], [3], [5], [7]). However, we almost neglect the importance of advanced inequalities. After all, it does great benefit to solve the bound of certain integral equations, which help us to fulfill a diversity of desired goals. In this paper we establish two advanced integral inequalities and an application of our results is also given.



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2. Preliminaries and Lemmas

In this paper, we assume throughout that $\mathbb{R}_+ = [0, \infty)$, is a subset of the set of real numbers \mathbb{R} . The following lemmas play an important role in this paper.

Lemma 2.1. *Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$. Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function and let c be a nonnegative constant. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If $u, f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and*

$$(2.1) \quad \varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} f(s)\psi(u(s))ds, \quad t \in \mathbb{R}_+,$$

then for $0 \leq T \leq t < \infty$,

$$(2.2) \quad u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[G(c) + \int_{\alpha(t)}^{\infty} f(s)ds \right] \right\},$$

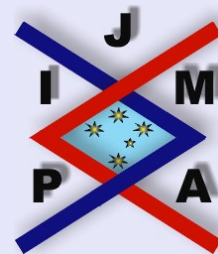
where $G(z) = \int_{z_0}^z \frac{ds}{\psi[\varphi^{-1}(s)]}$, $z \geq z_0 > 0$, φ^{-1}, G^{-1} are respectively the inverse of φ and G , $T \in \mathbb{R}_+$ is chosen so that

$$(2.3a) \quad G(c) + \int_{\alpha(t)}^{\infty} f(s)ds \in \text{Dom}(G^{-1}), \quad t \in [T, \infty).$$

$$(2.3b) \quad G^{-1} \left[G(c) + \int_{\alpha(t)}^{\infty} f(s)ds \right] \in \text{Dom}(\varphi^{-1}), \quad t \in [T, \infty).$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$(2.4) \quad z(t) = c + \varepsilon + \int_{\alpha(t)}^{\infty} f(s)\psi(u(s))ds, \quad t \in \mathbb{R}_+,$$



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where ε is an arbitrary small positive number. From inequality (2.1), we have

$$(2.5) \quad u(t) \leq \varphi^{-1}[z(t)].$$

Differentiating (2.4) and using (2.5) and the monotonicity of φ^{-1} , ψ , we deduce that

$$\begin{aligned} z'(t) &= -f(\alpha(t))\psi[u(\alpha(t))] \alpha'(t) \\ &\geq -f(\alpha(t))\psi[\varphi^{-1}(z(\alpha(t)))] \alpha'(t) \\ &\geq -f(\alpha(t))\psi[\varphi^{-1}(z(t))] \alpha'(t). \end{aligned}$$

For

$$\psi[\varphi^{-1}(z(t))] \geq \psi[\varphi^{-1}(z(\infty))] = \psi[\varphi^{-1}(c + \varepsilon)] > 0,$$

from the definition of G , the above relation gives

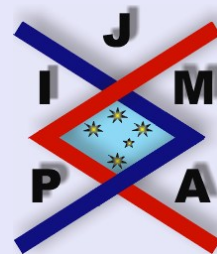
$$\frac{d}{dt}G(z(t)) = \frac{z'(t)}{\psi[\varphi^{-1}(z(t))]} \geq -f(\alpha(t))\alpha'(t).$$

Setting $t = s$, and integrating it from t to ∞ and letting $\varepsilon \rightarrow 0$ yields

$$G(z(t)) \leq G(c) + \int_{\alpha(t)}^{\infty} f(s)ds, \quad t \in \mathbb{R}_+.$$

From (2.3), (2.5) and the above relation, we obtain the inequality (2.2). \square

In fact, we can regard Lemma 2.1 as a generalized form of an Ou-Iang type inequality with advanced argument.



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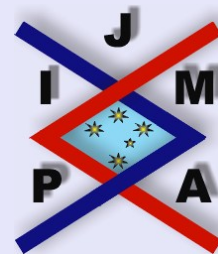


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Lemma 2.2. Let u , f and g be nonnegative continuous functions defined on \mathbb{R}_+ , and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ ($i = 1, 2$) on $(0, \infty)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If

$$(2.6) \quad \varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} f(s)w_1(u(s))ds + \int_t^{\infty} g(s)w_2(u(s))ds, \quad t \in \mathbb{R}_+,$$

then for $0 \leq T \leq t < \infty$,

(i) For the case $w_2(u) \leq w_1(u)$,

$$(2.7) \quad u(t) \leq \varphi^{-1} \left\{ G_1^{-1} \left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right] \right\}.$$

(ii) For the case $w_1(u) \leq w_2(u)$,

$$(2.8) \quad u(t) \leq \varphi^{-1} \left\{ G_2^{-1} \left[G_2(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right] \right\},$$

where

$$G_i(r) = \int_{r_0}^r \frac{ds}{w_i(\varphi^{-1}(s))}, \quad r \geq r_0 > 0, \quad (i = 1, 2)$$

and φ^{-1} , G_i^{-1} ($i = 1, 2$) are respectively the inverse of φ , G_i , $T \in \mathbb{R}_+$ is chosen so that

$$(2.9) \quad G_i(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \in \text{Dom}(G_i^{-1}),$$

$$(i = 1, 2), \quad t \in [T, \infty).$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$(2.10) \quad z(t) = c + \varepsilon + \int_{\alpha(t)}^{\infty} f(s)w_1(u(s))ds + \int_t^{\infty} g(s)w_2(u(s))ds,$$

$$0 \leq T \leq t < \infty,$$

where ε is an arbitrary small positive number. From inequality (2.6), we have

$$(2.11) \quad u(t) \leq \varphi^{-1}[z(t)], \quad t \in \mathbb{R}_+.$$

Differentiating (2.10) and using (2.11) and the monotonicity of φ^{-1} , w_1 , w_2 , we deduce that

$$\begin{aligned} z'(t) &= -f(\alpha(t))w_1[u(\alpha(t))] \alpha'(t) - g(t)w_2[u(t)], \\ &\geq -f(\alpha(t))w_1[\varphi^{-1}(z(\alpha(t)))] \alpha'(t) - g(t)w_2[\varphi^{-1}(z(t))], \\ &\geq -f(\alpha(t))w_1[\varphi^{-1}(z(t))] \alpha'(t) - g(t)w_2[\varphi^{-1}(z(t))]. \end{aligned}$$

(i) When $w_2(u) \leq w_1(u)$

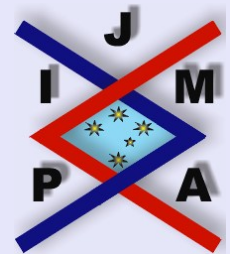
$$z'(t) \geq -f(\alpha(t))w_1[\varphi^{-1}(z(t))] \alpha'(t) - g(t)w_1[\varphi^{-1}(z(t))], \quad t \in \mathbb{R}_+.$$

For

$$w_1[\varphi^{-1}(z(t))] \geq w_1[\varphi^{-1}(z(\infty))] = w_1[\varphi^{-1}(c + \varepsilon)] > 0,$$

from the definition of $G_1(r)$, the above relation gives

$$\frac{d}{dt}G_1(z(t)) = \frac{z'(t)}{w_1[\varphi^{-1}(z(t))]} \geq -f(\alpha(t))\alpha'(t) - g(t), \quad t \in \mathbb{R}_+.$$



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Setting $t = s$ and integrating it from t to ∞ and let $\varepsilon \rightarrow 0$ yields

$$G_1(z(t)) \leq G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds, \quad t \in \mathbb{R}_+,$$

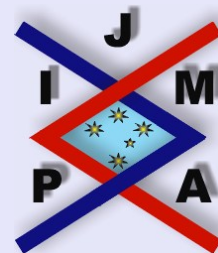
so,

$$z(t) \leq G_1^{-1} \left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right], \quad 0 \leq T \leq t < \infty.$$

Using (2.11), we have

$$u(t) \leq \varphi^{-1} \left\{ G_1^{-1} \left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right] \right\}, \quad 0 \leq T \leq t < \infty.$$

(ii) When $w_1(u) \leq w_2(u)$, the proof can be completed similarly. □



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3. Main Results

In this section, we obtain our main results as follows:

Theorem 3.1. *Let u , f and g be nonnegative continuous functions defined on \mathbb{R}_+ and let c be a nonnegative constant. Moreover, let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$, $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If*

$$(3.1) \quad \varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds, \quad t \in \mathbb{R}_+$$

then for $0 \leq T \leq t < \infty$,

$$(3.2) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G^{-1} \left(G[\Omega(c) + \int_{\alpha(t)}^{\infty} g(s)ds] + \int_{\alpha(t)}^{\infty} f(s)ds \right) \right] \right\},$$

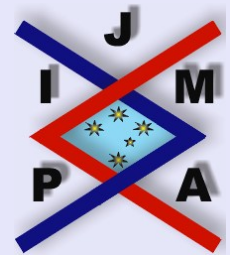
where

$$\Omega(r) = \int_{r_0}^r \frac{ds}{\varphi^{-1}(s)}, \quad r \geq r_0 > 0,$$

$$G(z) = \int_{z_0}^z \frac{ds}{\psi\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad z \geq z_0 > 0,$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ are respectively the inverse of Ω, φ, G and $T \in \mathbb{R}_+$ is chosen so that

$$G \left[\Omega(c) + \int_{\alpha(t)}^{\infty} g(s)ds \right] + \int_{\alpha(t)}^{\infty} f(s)ds \in \text{Dom}(G^{-1})$$



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and

$$G^{-1} \left\{ G \left[\Omega(c) + \int_{\alpha(t)}^{\infty} g(s) ds \right] + \int_{\alpha(t)}^{\infty} f(s) ds \right\} \in \text{Dom}(\Omega^{-1})$$

for $t \in [T, \infty)$.

Proof. Let us first assume that $c > 0$. Define the nonincreasing positive function $z(t)$ by the right-hand side of (3.1). Then $z(\infty) = c$, $u(t) \leq \varphi^{-1}[z(t)]$ and

$$\begin{aligned} z'(t) &= - [f(\alpha(t))u(\alpha(t))\psi[u(\alpha(t))] - g(\alpha(t))u(\alpha(t))] \alpha'(t) \\ &\geq - [f(\alpha(t))\varphi^{-1}(z(\alpha(t)))\psi[\varphi^{-1}(z(\alpha(t)))] - g(\alpha(t))\varphi^{-1}(z(\alpha(t)))] \alpha'(t) \\ &\geq - [f(\alpha(t))\varphi^{-1}(z(t))\psi[\varphi^{-1}(z(\alpha(t)))] - g(\alpha(t))\varphi^{-1}(z(t))] \alpha'(t). \end{aligned}$$

Since $\varphi^{-1}(z(t)) \geq \varphi^{-1}(c) > 0$,

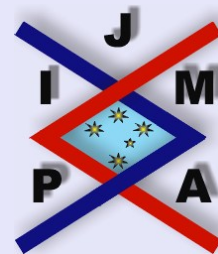
$$\frac{z'(t)}{\varphi^{-1}(z(t))} \geq - \{f(\alpha(t))\psi[\varphi^{-1}(z(\alpha(t)))] + g(\alpha(t))\} \alpha'(t).$$

Setting $t = s$ and integrating it from t to ∞ yields

$$\Omega(z(t)) \leq \Omega(c) + \int_{\alpha(t)}^{\infty} g(s) ds + \int_{\alpha(t)}^{\infty} f(s)\psi[\varphi^{-1}(z(s))] ds.$$

Let $T \leq T_1$ be an arbitrary number. We denote $p(t) = \Omega(c) + \int_{\alpha(t)}^{\infty} g(s) ds$. From the above relation, we deduce that

$$\Omega(z(t)) \leq p(T_1) + \int_{\alpha(t)}^{\infty} f(s)\psi[\varphi^{-1}(z(s))] ds, \quad T_1 \leq t < \infty.$$



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Now an application of Lemma 2.1 gives

$$z(t) \leq \Omega^{-1} \left\{ G^{-1} \left[G(p(T_1)) + \int_{\alpha(t)}^{\infty} f(s) ds \right] \right\}, \quad T_1 \leq t < \infty,$$

so,

$$u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G^{-1} \left(G(p(T_1)) + \int_{\alpha(t)}^{\infty} f(s) ds \right) \right] \right\}, \quad T_1 \leq t < \infty.$$

Taking $t = T_1$ in the above inequality, since T_1 is arbitrary, we can prove the desired inequality (3.2).

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \rightarrow 0$. \square

Corollary 3.2. *Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+ and let c be a nonnegative constant. Moreover, let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If*

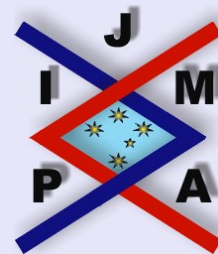
$$u^2(t) \leq c^2 + \int_{\alpha(t)}^{\infty} [f(s)u(s)\psi(u(s)) + g(s)u(s)] ds, \quad t \in \mathbb{R}_+,$$

then for $0 \leq T \leq t < \infty$,

$$u(t) \leq \Omega^{-1} \left[\Omega \left(c + \frac{1}{2} \int_{\alpha(t)}^{\infty} g(s) ds \right) + \frac{1}{2} \int_{\alpha(t)}^{\infty} f(s) ds \right],$$

where

$$\Omega(r) = \int_1^r \frac{ds}{\psi(s)}, \quad r > 0,$$



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Ω^{-1} is the inverse of Ω , and $T \in \mathbb{R}_+$ is chosen so that

$$\Omega \left(c + \frac{1}{2} \int_{\alpha(t)}^{\infty} g(s) ds \right) + \frac{1}{2} \int_{\alpha(t)}^{\infty} f(s) ds \in \text{Dom}(\Omega^{-1})$$

for all $t \in [T, \infty)$.

Corollary 3.3. Let u , f and g be nonnegative continuous functions defined on \mathbb{R}_+ and let c be a nonnegative constant. Moreover, let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If

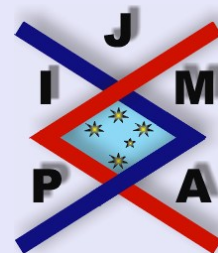
$$u^2(t) \leq c^2 + \int_{\alpha(t)}^{\infty} [f(s)u^2(s) + g(s)u(s)] ds, \quad t \geq 0,$$

then

$$u(t) \leq \left(c + \frac{1}{2} \int_{\alpha(t)}^{\infty} g(s) ds \right) \exp \left[\frac{1}{2} \int_{\alpha(t)}^{\infty} f(s) ds \right], \quad t \geq 0.$$

Corollary 3.4. Let u , f and g be nonnegative continuous functions defined on \mathbb{R}_+ and let c be a nonnegative constant. Moreover, let p, q be positive constants with $p \geq q$, $p \neq 1$. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If

$$u^p(t) \leq c + \int_{\alpha(t)}^{\infty} [f(s)u^q(s) + g(s)u(s)] ds, \quad t \in \mathbb{R}_+,$$



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then for $t \in \mathbb{R}_+$,

$$u(t) \leq \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{p-1}{p} \int_{\alpha(t)}^{\infty} g(s) ds \right)^{\frac{p}{p-1}} \exp \left[\frac{1}{p} \int_{\alpha(t)}^{\infty} f(s) ds \right], & \text{when } p = q, \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{p-1}{p} \int_{\alpha(t)}^{\infty} g(s) ds \right)^{\frac{p-q}{p-1}} + \frac{p-q}{p} \int_{\alpha(t)}^{\infty} f(s) ds \right]^{\frac{1}{p-q}}, & \text{when } p > q. \end{cases}$$

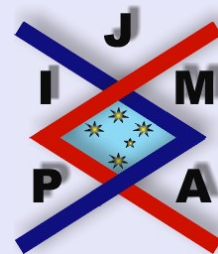
Theorem 3.5. Let u , f and g be nonnegative continuous functions defined on \mathbb{R}_+ , and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ ($i = 1, 2$) on $(0, \infty)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on \mathbb{R}_+ . If

$$(3.3) \quad \varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} f(s)u(s)w_1(u(s))ds + \int_t^{\infty} g(s)u(s)w_2(u(s))ds,$$

then for $0 \leq T \leq t < \infty$,

(i) For the case $w_2(u) \leq w_1(u)$,

$$(3.4) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G_1^{-1} \left(G_1(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right) \right] \right\},$$



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(ii) For the case $w_1(u) \leq w_2(u)$,

$$(3.5) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G_2^{-1} \left(G_2(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right) \right] \right\},$$

where

$$\Omega(r) = \int_{r_0}^r \frac{ds}{\varphi^{-1}(s)}, \quad r \geq r_0 > 0,$$

$$G_i(z) = \int_{z_0}^z \frac{ds}{w_i\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad z \geq z_0 > 0 \quad (i = 1, 2),$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ are respectively the inverse of Ω, φ, G , and $T \in \mathbb{R}_+$ is chosen so that

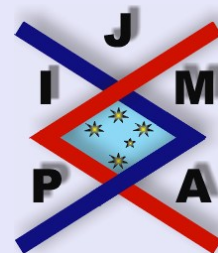
$$G_i \left(\Omega(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right) \in \text{Dom}(G_i^{-1}),$$

$$G_i^{-1} \left[G_i \left(\Omega(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right) \right] \in \text{Dom}(\Omega^{-1}),$$

for all $t \in [T, \infty)$.

Proof. Let $c > 0$, define the nonincreasing positive function $z(t)$ and make

$$(3.6) \quad z(t) = c + \int_{\alpha(t)}^{\infty} f(s)u(s)w_1(u(s))ds + \int_t^{\infty} g(s)u(s)w_2(u(s))ds.$$



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From inequality (3.3), we have

$$(3.7) \quad u(t) \leq \varphi^{-1}[z(t)].$$

Differentiating (3.6) and using (3.7) and the monotonicity of φ^{-1} , w_1 , w_2 , we deduce that

$$\begin{aligned} z'(t) &= -f(\alpha(t))u(\alpha(t))w_1 [u(\alpha(t))] \alpha'(t) - g(t)u(t)w_2 [u(t)], \\ &\geq -f(\alpha(t))\varphi^{-1}(z(\alpha(t)))w_1 [\varphi^{-1}(z(\alpha(t)))] \alpha'(t) \\ &\quad - g(t)\varphi^{-1}(z(t))w_2 [\varphi^{-1}(z(t))], \\ &\geq -f(\alpha(t))\varphi^{-1}(z(t))w_1 [\varphi^{-1}(z(t))] \alpha'(t) \\ &\quad - g(t)\varphi^{-1}(z(t))w_2 [\varphi^{-1}(z(t))]. \end{aligned}$$

(i) When $w_2(u) \leq w_1(u)$

$$\frac{z'(t)}{\varphi^{-1}(z(t))} \geq -f(\alpha(t))w_1 [\varphi^{-1}(z(t))] \alpha'(t) - g(t)w_1 [\varphi^{-1}(z(t))].$$

For

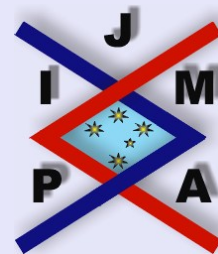
$$w_1[\varphi^{-1}(z(t))] \geq w_1[\varphi^{-1}(z(\infty))] = w_1[\varphi^{-1}(c + \varepsilon)] > 0,$$

setting $t = s$ and integrating from t to ∞ yields

$$\Omega(z(t)) \leq \Omega(c) + \int_{\alpha(t)}^{\infty} f(s)w_1 [\varphi^{-1}(z(t))] ds + \int_t^{\infty} g(s)w_1 [\varphi^{-1}(z(t))] ds.$$

From Lemma 2.2, we obtain

$$z(t) \leq \Omega^{-1} \left\{ G_1^{-1} \left[G_1(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds \right] \right\}, \quad 0 \leq T \leq t < \infty.$$



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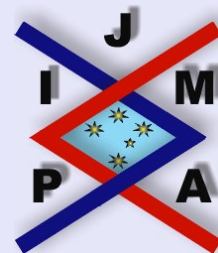
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Using $u(t) \leq \varphi^{-1}[z(t)]$, we get the inequality in (3.4)

If $c = 0$, we can carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \rightarrow 0$.

(ii) When $w_1(u) \leq w_2(u)$, the proof can be completed similarly. □



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4. An Application

We consider an integral equation

$$(4.1) \quad x^p(t) = a(t) + \int_t^\infty F[s, x(s), x(\phi(s))]ds.$$

Assume that:

$$(4.2) \quad |F(x, y, u)| \leq f(x)|u|^q + g(x)|u|$$

and

$$(4.3) \quad |a(t)| \leq c, \quad c > 0 \quad p \geq q > 0, \quad p \neq 1,$$

where f, g are nonnegative continuous real-valued functions, and $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\phi(t) \geq t$ on \mathbb{R}_+ . From (4.1), (4.2) and (4.3) we have

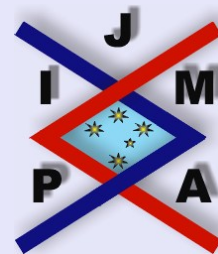
$$|x(t)|^p \leq c + \int_t^\infty f(s)|x(\phi(s))|^q + g(s)|x(\phi(s))|ds.$$

Making the change of variables from the above inequality and taking

$$M = \sup_{t \in \mathbb{R}_+} \frac{1}{\phi'(t)},$$

we have

$$|x(t)|^p \leq c + M \int_{\phi(t)}^\infty \bar{f}(s)|x(s)|^q + \bar{g}(s)|x(s)|ds,$$



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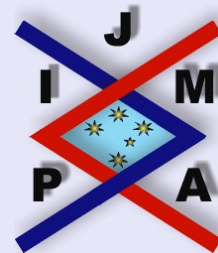
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in which $\bar{f}(s) = f(\phi^{-1}(s))$, $\bar{g}(s) = g(\phi^{-1}(s))$. From Corollary 3.4, we obtain

$$|x(t)| \leq \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{M(p-1)}{p} \int_{\phi(t)}^{\infty} \bar{g}(s) ds \right)^{\frac{p}{p-1}} \exp \left[\frac{M}{p} \int_{\phi(t)}^{\infty} \bar{f}(s) ds \right], & \text{when } p = q \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{M(p-1)}{p} \int_{\phi(t)}^{\infty} \bar{g}(s) ds \right)^{\frac{p-q}{p-1}} + \frac{M(p-q)}{p} \int_{\phi(t)}^{\infty} \bar{f}(s) ds \right]^{\frac{1}{p-q}}, & \text{when } p > q. \end{cases}$$

If the integrals of $f(s)$, $g(s)$ are bounded, then we have the bound of the solution of (4.1).



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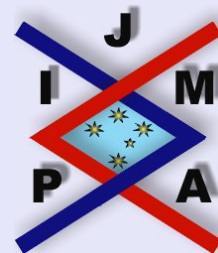
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