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ON SOME ADVANCED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract

In this paper, we obtain a generalization of advanced integral inequality and by means of examples we show the usefulness of our results.

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Contents

1. Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations. Many retarded inequalities have been discovered (see [2], [3], [5], [7]). However, we almost neglect the importance of advanced inequalities. After all, it does great benefit to solve the bound of certain integral equations, which help us to fulfill a diversity of desired goals. In this paper we establish two advanced integral inequalities and an application of our results is also given.

2. Preliminaries and Lemmas

In this paper, we assume throughout that $\mathbb{R}_+ = [0, \infty)$, is a subset of the set of real numbers $\mathbb{R}$. The following lemmas play an important role in this paper.

Lemma 2.1. *Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$. Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function and let c be a nonnegative constant. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If $u, f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and*

$$\varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} f(s)\psi(u(s))ds, \qquad t \in \mathbb{R}_+, \tag{2.1}$$

then for $0 \leq T \leq t < \infty$,

$$u(t) \leq \varphi^{-1}\left\{G^{-1}[G(c) + \int_{\alpha(t)}^{\infty} f(s)ds]\right\}, \tag{2.2}$$

where $G(z) = \int_{z_0}^{z} \frac{ds}{\psi[\varphi^{-1}(s)]}$, $z \geq z_0 > 0$, φ^{-1}, G^{-1} are respectively the inverse of φ and G, $T \in \mathbb{R}_+$ is chosen so that

$$G(c) + \int_{\alpha(t)}^{\infty} f(s)ds \in \mathrm{Dom}(G^{-1}), \qquad t \in [T, \infty). \tag{2.3a}$$

$$G^{-1}\left[G(c) + \int_{\alpha(t)}^{\infty} f(s)ds\right] \in \mathrm{Dom}(\varphi^{-1}), \qquad t \in [T, \infty). \tag{2.3b}$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$z(t) = c + \varepsilon + \int_{\alpha(t)}^{\infty} f(s)\psi(u(s))ds, \qquad t \in \mathbb{R}_+, \tag{2.4}$$

where ε is an arbitrary small positive number. From inequality (2.1), we have

$$u(t) \leq \varphi^{-1}[z(t)]. \tag{2.5}$$

Differentiating (2.4) and using (2.5) and the monotonicity of φ^{-1}, ψ, we deduce that

$$\begin{aligned} z'(t) &= -f\big(\alpha(t)\big)\psi\left[u\big(\alpha(t)\big)\right]\alpha'(t) \\ &\geq -f\big(\alpha(t)\big)\psi\left[\varphi^{-1}\big(z(\alpha(t))\big)\right]\alpha'(t) \\ &\geq -f\big(\alpha(t)\big)\psi\left[\varphi^{-1}\big(z(t)\big)\right]\alpha'(t). \end{aligned}$$

For

$$\psi[\varphi^{-1}(z(t))] \geq \psi[\varphi^{-1}(z(\infty))] = \psi[\varphi^{-1}(c+\varepsilon)] > 0,$$

from the definition of G, the above relation gives

$$\frac{d}{dt}G(z(t)) = \frac{z'(t)}{\psi[\varphi^{-1}(z(t))]} \geq -f\big(\alpha(t)\big)\alpha'(t).$$

Setting $t = s$, and integrating it from t to ∞ and letting $\varepsilon \to 0$ yields

$$G\big(z(t)\big) \leq G(c) + \int_{\alpha(t)}^{\infty} f(s)ds, \qquad t \in \mathbb{R}_+.$$

From (2.3), (2.5) and the above relation, we obtain the inequality (2.2). □

In fact, we can regard Lemma 2.1 as a generalized form of an Ou-Iang type inequality with advanced argument.

Lemma 2.2. *Let u f and g be nonnegative continuous functions defined on $\mathbb{R}_+$, and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ $(i = 1, 2)$ on $(0, \infty)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If*

$$\varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} f(s) w_1(u(s)) ds + \int_t^{\infty} g(s) w_2(u(s)) ds, \quad t \in \mathbb{R}_+, \tag{2.6}$$

then for $0 \leq T \leq t < \infty$,

(i) *For the case $w_2(u) \leq w_1(u)$,*

$$u(t) \leq \varphi^{-1}\left\{G_1^{-1}\left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s) ds + \int_t^{\infty} g(s) ds\right]\right\}. \tag{2.7}$$

(ii) *For the case $w_1(u) \leq w_2(u)$,*

$$u(t) \leq \varphi^{-1}\left\{G_2^{-1}\left[G_2(c) + \int_{\alpha(t)}^{\infty} f(s) ds + \int_t^{\infty} g(s) ds\right]\right\}, \tag{2.8}$$

where

$$G_i(r) = \int_{r_0}^{r} \frac{ds}{w_i(\varphi^{-1}(s))}, \quad r \geq r_0 > 0, \quad (i = 1, 2)$$

and φ^{-1}, G_i^{-1} $(i = 1, 2)$ are respectively the inverse of φ, G_i, $T \in \mathbb{R}_+$ is chosen so that

$$G_i(c) + \int_{\alpha(t)}^{\infty} f(s) ds + \int_t^{\infty} g(s) ds \in \mathrm{Dom}(G_i^{-1}), \tag{2.9}$$

$$(i = 1, 2), \qquad t \in [T, \infty).$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$z(t) = c + \varepsilon + \int_{\alpha(t)}^{\infty} f(s) w_1(u(s))ds + \int_{t}^{\infty} g(s) w_2(u(s))ds, \quad 0 \le T \le t < \infty, \tag{2.10}$$

where ε is an arbitrary small positive number. From inequality (2.6), we have

$$u(t) \le \varphi^{-1}[z(t)], \quad t \in \mathbb{R}_+. \tag{2.11}$$

Differentiating (2.10) and using (2.11) and the monotonicity of φ^{-1}, w_1, w_2, we deduce that

$$\begin{aligned} z'(t) &= -f\big(\alpha(t)\big) w_1 \left[u\big(\alpha(t)\big)\right] \alpha'(t) - g(t) w_2\big[u(t)\big], \\ &\ge -f\big(\alpha(t)\big) w_1 \left[\varphi^{-1}\big(z(\alpha(t))\big)\right] \alpha'(t) - g(t) w_2 \left[\varphi^{-1}\big(z(t)\big)\right], \\ &\ge -f\big(\alpha(t)\big) w_1 \left[\varphi^{-1}\big(z(t)\big)\right] \alpha'(t) - g(t) w_2 \left[\varphi^{-1}\big(z(t)\big)\right]. \end{aligned}$$

(i) When $w_2(u) \le w_1(u)$

$$z'(t) \ge -f\big(\alpha(t)\big) w_1 \left[\varphi^{-1}\big(z(t)\big)\right] \alpha'(t) - g(t) w_1 \left[\varphi^{-1}\big(z(t)\big)\right], \quad t \in \mathbb{R}_+.$$

For

$$w_1[\varphi^{-1}(z(t))] \ge w_1 \left[\varphi^{-1}(z(\infty))\right] = w_1[\varphi^{-1}(c+\varepsilon)] > 0,$$

from the definition of $G_1(r)$, the above relation gives

$$\frac{d}{dt} G_1(z(t)) = \frac{z'(t)}{w_1[\varphi^{-1}(z(t))]} \ge -f\big(\alpha(t)\big)\alpha'(t) - g(t), \quad t \in \mathbb{R}_+.$$

Setting $t = s$ and integrating it from t to ∞ and let $\varepsilon \to 0$ yields

$$G_1\big(z(t)\big) \le G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds, \quad t \in \mathbb{R}_+,$$

so,

$$z(t) \le G_1^{-1}\left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right], \quad 0 \le T \le t < \infty.$$

Using (2.11), we have

$$u(t) \le \varphi^{-1}\left\{G_1^{-1}\left[G_1(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right]\right\}, \quad 0 \le T \le t < \infty.$$

(ii) When $w_1(u) \le w_2(u)$, the proof can be completed similarly. □

3. Main Results

In this section, we obtain our main results as follows:

Theorem 3.1. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+$ and let c be a nonnegative constant. Moreover, let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$, $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If*

$$\varphi(u(t)) \leq c + \int_{\alpha(t)}^{\infty} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds, \qquad t \in \mathbb{R}_+ \tag{3.1}$$

then for $0 \leq T \leq t < \infty$,

$$u(t) \leq \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G[\Omega(c) + \int_{\alpha(t)}^{\infty} g(s)ds] + \int_{\alpha(t)}^{\infty} f(s)ds\right)\right]\right\}, \tag{3.2}$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(s)}, \quad r \geq r_0 > 0,$$

$$G(z) = \int_{z_0}^{z} \frac{ds}{\psi\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad z \geq z_0 > 0,$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ are respectively the inverse of Ω, φ, G and $T \in \mathbb{R}_+$ is chosen so that

$$G\left[\Omega(c) + \int_{\alpha(t)}^{\infty} g(s)ds\right] + \int_{\alpha(t)}^{\infty} f(s)ds \in \mathrm{Dom}(G^{-1})$$

and

$$G^{-1}\left\{G\left[\Omega(c)+\int_{\alpha(t)}^{\infty}g(s)ds\right]+\int_{\alpha(t)}^{\infty}f(s)ds\right\}\in \mathrm{Dom}(\Omega^{-1})$$

for $t\in[T,\infty)$.

Proof. Let us first assume that $c>0$. Define the nonincreasing positive function $z(t)$ by the right-hand side of (3.1). Then $z(\infty)=c,\ u(t)\le\varphi^{-1}[z(t)]$ and

$$\begin{aligned} z'(t) &= -\left[f\big(\alpha(t)\big)u\big(\alpha(t)\big)\psi\big[u\big(\alpha(t)\big)\big]-g\big(\alpha(t)\big)u\big(\alpha(t)\big)\right]\alpha'(t)\\ &\ge -\left[f\big(\alpha(t)\big)\varphi^{-1}\big(z(\alpha(t))\big)\psi\big[\varphi^{-1}\big(z(\alpha(t))\big)\big]-g\big(\alpha(t)\big)\varphi^{-1}\big(z(\alpha(t))\big)\right]\alpha'(t)\\ &\ge -\left[f\big(\alpha(t)\big)\varphi^{-1}\big(z(t)\big)\psi\big[\varphi^{-1}\big(z(\alpha(t))\big)\big]-g\big(\alpha(t)\big)\varphi^{-1}\big(z(t)\big)\right]\alpha'(t). \end{aligned}$$

Since $\varphi^{-1}(z(t))\ge\varphi^{-1}(c)>0$,

$$\frac{z'(t)}{\varphi^{-1}\big(z(t)\big)}\ge -\left\{f\big(\alpha(t)\big)\psi\left[\varphi^{-1}\big(z(\alpha(t))\big)\right]+g\big(\alpha(t)\big)\right\}\alpha'(t).$$

Setting $t=s$ and integrating it from t to ∞ yields

$$\Omega(z(t))\le\Omega(c)+\int_{\alpha(t)}^{\infty}g(s)ds+\int_{\alpha(t)}^{\infty}f(s)\psi[\varphi^{-1}(z(s))]ds.$$

Let $T\le T_1$ be an arbitrary number. We denote $p(t)=\Omega(c)+\int_{\alpha(t)}^{\infty}g(s)ds$. From the above relation, we deduce that

$$\Omega(z(t))\le p(T_1)+\int_{\alpha(t)}^{\infty}f(s)\psi[\varphi^{-1}(z(s))]ds,\qquad T_1\le t<\infty.$$

Now an application of Lemma 2.1 gives

$$z(t) \leq \Omega^{-1}\left\{G^{-1}\left[G(p(T_1)) + \int_{\alpha(t)}^{\infty} f(s)ds\right]\right\}, \quad T_1 \leq t < \infty,$$

so,

$$u(t) \leq \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G(p(T_1)) + \int_{\alpha(t)}^{\infty} f(s)ds\right)\right]\right\}, \quad T_1 \leq t < \infty.$$

Taking $t = T_1$ in the above inequality, since T_1 is arbitrary, we can prove the desired inequality (3.2).

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. □

Corollary 3.2. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+$ and let c be a nonnegative constant. Moreover, let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If*

$$u^2(t) \leq c^2 + \int_{\alpha(t)}^{\infty} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds, \quad t \in \mathbb{R}_+,$$

then for $0 \leq T \leq t < \infty$,

$$u(t) \leq \Omega^{-1}\left[\Omega\left(c + \frac{1}{2}\int_{\alpha(t)}^{\infty} g(s)ds\right) + \frac{1}{2}\int_{\alpha(t)}^{\infty} f(s)ds\right],$$

where

$$\Omega(r) = \int_1^r \frac{ds}{\psi(s)}, \quad r > 0,$$

Ω^{-1} is the inverse of Ω, and $T \in \mathbb{R}_+$ is chosen so that

$$\Omega\left(c + \frac{1}{2}\int_{\alpha(t)}^{\infty} g(s)ds\right) + \frac{1}{2}\int_{\alpha(t)}^{\infty} f(s)ds \in \mathrm{Dom}(\Omega^{-1})$$

for all $t \in [T, \infty)$.

Corollary 3.3. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+$ and let c be a nonnegative constant. Moreover, let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If*

$$u^2(t) \leq c^2 + \int_{\alpha(t)}^{\infty} [f(s)u^2(s) + g(s)u(s)]ds, \quad t \geq 0,$$

then

$$u(t) \leq \left(c + \frac{1}{2}\int_{\alpha(t)}^{\infty} g(s)ds\right) \exp\left[\frac{1}{2}\int_{\alpha(t)}^{\infty} f(s)ds\right], \quad t \geq 0.$$

Corollary 3.4. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+$ and let c be a nonnegative constant. Moreover, let p, q be positive constants with $p \geq q$, $p \neq 1$. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \geq t$ on $\mathbb{R}_+$. If*

$$u^p(t) \leq c + \int_{\alpha(t)}^{\infty} [f(s)u^q(s) + g(s)u(s)]ds, \quad t \in \mathbb{R}_+,$$

then for $t \in \mathbb{R}_+$,

$$u(t) \le \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{p-1}{p}\int_{\alpha(t)}^{\infty} g(s)ds\right)^{\frac{p}{p-1}} \exp\left[\frac{1}{p}\int_{\alpha(t)}^{\infty} f(s)ds\right], & \text{when } p = q, \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{p-1}{p}\int_{\alpha(t)}^{\infty} g(s)ds\right)^{\frac{p-q}{p-1}} + \frac{p-q}{p}\int_{\alpha(t)}^{\infty} f(s)ds\right]^{\frac{1}{p-q}}, & \text{when } p > q. \end{cases}$$

Theorem 3.5. *Let* u, f *and* g *be nonnegative continuous functions defined on* $\mathbb{R}_+$, *and let* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be an increasing function with* $\varphi(\infty) = \infty$ *and let* c *be a nonnegative constant. Moreover, let* $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $w_i(u) > 0\,(i = 1, 2)$ *on* $(0, \infty)$ *and* $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing with* $\alpha(t) \ge t$ *on* $\mathbb{R}_+$. *If*

$$\varphi(u(t)) \le c + \int_{\alpha(t)}^{\infty} f(s)u(s)w_1(u(s))ds + \int_t^{\infty} g(s)u(s)w_2(u(s))ds, \tag{3.3}$$

then for $0 \le T \le t < \infty$,

(i) *For the case* $w_2(u) \le w_1(u)$,

$$u(t) \le \varphi^{-1}\left\{\Omega^{-1}\left[G_1^{-1}\left(G_1(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right)\right]\right\}, \tag{3.4}$$

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(ii) *For the case* $w_1(u) \le w_2(u)$,

(3.5) $$u(t) \le \varphi^{-1}\left\{\Omega^{-1}\left[G_2^{-1}\left(G_2(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right)\right]\right\},$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(s)}, \quad r \ge r_0 > 0,$$
$$G_i(z) = \int_{z_0}^{z} \frac{ds}{w_i\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad z \ge z_0 > 0 \quad (i = 1, 2),$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ *are respectively the inverse of* Ω, φ, G, *and* $T \in \mathbb{R}_+$ *is chosen so that*

$$G_i\left(\Omega(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right) \in \mathrm{Dom}(G_i^{-1}),$$
$$G_i^{-1}\left[G_i\left(\Omega(c) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right)\right] \in \mathrm{Dom}(\Omega^{-1}),$$

for all $t \in [T, \infty)$.

Proof. Let $c > 0$, define the nonincreasing positive function $z(t)$ and make

(3.6) $$z(t) = c + \int_{\alpha(t)}^{\infty} f(s)u(s)w_1(u(s))ds + \int_t^{\infty} g(s)u(s)w_2(u(s))ds.$$

From inequality (3.3), we have

$$u(t) \le \varphi^{-1}[z(t)]. \tag{3.7}$$

Differentiating (3.6) and using (3.7) and the monotonicity of φ^{-1}, w_1, w_2, we deduce that

$$\begin{aligned} z'(t) &= -f(\alpha(t))u(\alpha(t))w_1\left[u(\alpha(t))\right]\alpha'(t) - g(t)u(t)w_2[u(t)], \\ &\ge -f(\alpha(t))\varphi^{-1}(z(\alpha(t)))w_1\left[\varphi^{-1}(z(\alpha(t)))\right]\alpha'(t) \\ &\qquad - g(t)\varphi^{-1}(z(t))w_2\left[\varphi^{-1}(z(t))\right], \\ &\ge -f(\alpha(t))\varphi^{-1}(z(t))w_1\left[\varphi^{-1}(z(t))\right]\alpha'(t) \\ &\qquad - g(t)\varphi^{-1}(z(t))w_2\left[\varphi^{-1}(z(t))\right]. \end{aligned}$$

(i) When $w_2(u) \le w_1(u)$

$$\frac{z'(t)}{\varphi^{-1}(z(t))} \ge -f(\alpha(t))w_1\left[\varphi^{-1}(z(t))\right]\alpha'(t) - g(t)w_1\left[\varphi^{-1}(z(t))\right].$$

For

$$w_1[\varphi^{-1}(z(t))] \ge w_1[\varphi^{-1}(z(\infty))] = w_1[\varphi^{-1}(c+\varepsilon)] > 0,$$

setting $t = s$ and integrating from t to ∞ yields

$$\Omega(z(t)) \le \Omega(c) + \int_{\alpha(t)}^{\infty} f(s)w_1\left[\varphi^{-1}(z(t))\right]ds + \int_t^{\infty} g(s)w_1\left[\varphi^{-1}(z(t))\right]ds.$$

From Lemma 2.2, we obtain

$$z(t) \le \Omega^{-1}\left\{G_1^{-1}\left[G_1(\Omega(c)) + \int_{\alpha(t)}^{\infty} f(s)ds + \int_t^{\infty} g(s)ds\right]\right\}, \quad 0 \le T \le t < \infty.$$

Using $u(t) \le \varphi^{-1}[z(t)]$, we get the inequality in (3.4)

If $c = 0$, we can carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$.

(ii) When $w_1(u) \le w_2(u)$, the proof can be completed similarly. □

4. An Application

We consider an integral equation

$$x^p(t) = a(t) + \int_t^\infty F[s, x(s), x(\phi(s))]ds. \tag{4.1}$$

Assume that:

$$|F(x, y, u)| \le f(x)|u|^q + g(x)|u| \tag{4.2}$$

and

$$|a(t)| \le c, \quad c > 0 \quad p \ge q > 0, \quad p \ne 1, \tag{4.3}$$

where f, g are nonnegative continuous real-valued functions, and $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\phi(t) \ge t$ on $\mathbb{R}_+$. From (4.1), (4.2) and (4.3) we have

$$|x(t)|^p \le c + \int_t^\infty f(s)|x(\phi(s))|^q + g(s)|x(\phi(s))|ds.$$

Making the change of variables from the above inequality and taking

$$M = \sup_{t \in R_+} \frac{1}{\phi'(t)},$$

we have

$$|x(t)|^p \le c + M \int_{\phi(t)}^\infty \bar{f}(s)|x(s)|^q + \bar{g}(s)|x(s)|ds,$$

in which $\bar{f}(s) = f(\phi^{-1}(s)),\ \bar{g}(s) = g(\phi^{-1}(s))$. From Corollary 3.4, we obtain

$$|x(t)| \leq \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{M(p-1)}{p}\int_{\phi(t)}^{\infty}\bar{g}(s)ds\right)^{\frac{p}{p-1}} \exp\left[\frac{M}{p}\int_{\phi(t)}^{\infty}\bar{f}(s)ds\right], \\ \qquad\qquad\qquad\qquad \text{when} \quad p = q \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{M(p-1)}{p}\int_{\phi(t)}^{\infty}\bar{g}(s)ds\right)^{\frac{p-q}{p-1}} + \frac{M(p-q)}{p}\int_{\phi(t)}^{\infty}\bar{f}(s)ds\right]^{\frac{1}{p-q}}, \\ \qquad\qquad\qquad\qquad \text{when} \quad p > q. \end{cases}$$

If the integrals of $f(s), g(s)$ are bounded, then we have the bound of the solution of (4.1).

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