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A BOUND FOR CERTAIN BIBASIC SUMS AND APPLICATIONS

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ABSTRACT. In this paper, we use the terminating case of the Euler formula, the limiting case of the q-Gauss sum and the Grüss inequality to derive a bound for certain bibasic sums. Applications of the inequality are also given.

Key words and phrases: Basic hypergeometric function; q-binomial coefficient; Euler formula; q-Gauss sum; Grüss inequality.

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1. INTRODUCTION AND SOME KNOWN RESULTS

q-Series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequality techniques are useful tools in the study of q-series, see [1, 7, 8]. In [1], the authors gave some inequalities for hypergeometric functions. In this paper, we give a new inequality about q-series. First, we recall some definitions, notations and known results which will be used in this paper. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n, \tag{1.2}$$

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where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.3}$$

Although it is not obvious from (1.3), it is a well-known fact [2] that the q-binomial coefficient is a polynomial in q of degree $k(n-k)$ with nonnegative integer coefficients. The q-Gauss sum [2, 3, 4]

$$\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \tag{1.4}$$

which has the limiting case [4]

$$\frac{1}{(x;q)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)} \frac{x^k}{(x;q)_k}. \tag{1.5}$$

We also need the following Euler formula [2]

$$(x;q)_\infty = \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{x^k}{(q;q)_k}, \tag{1.6}$$

which has the terminating form [2, 4]

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^k. \tag{1.7}$$

The following is well known in the literature as the Grüss inequality [6]:

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \cdot \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leq \frac{(M-m)(N-n)}{4}, \tag{1.8}$$

provided that $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and $m \leq f(x) \leq M,\ n \leq g(x) \leq N$ for all $x \in [a,b]$, where m, M, n, N are given constants.

The discrete version of the Grüss inequality can be stated as: If $a \leq a_i \leq A$ and $b \leq b_i \leq B$ ($i = 1, 2, \ldots, n$), then we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) \cdot \left(\frac{1}{n} \sum_{i=1}^{n} b_i \right) \right| \leq \frac{(A-a)(B-b)}{4}, \tag{1.9}$$

where a, A, b, B are given real constants.

2. A BOUND FOR BIBASIC SUMS

In this section, by means of the terminating form of the Euler formula, the limiting case of the q-Gauss sum and the Grüss inequality, we derive a bound of the following bibasic sums

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x;p)_k}. \tag{2.1}$$

For any real number x, let $[x]$ denote the greatest integer less than or equal x. The main result of this paper is the following theorem, which gives an upper bound of (2.1). It is obvious that, under the conditions of the following theorem, the lower bound of (2.1) is zero.

Theorem 2.1. *Let $0 < p < 1$, $0 < q < 1$, $0 < x < 1$ and $0 < y < 1$, then for any positive integer n we have*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x;p)_k} \leq \frac{(-y;q)_n}{(n+1)(x;p)_n} + \frac{n+1}{4(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q, \tag{2.2}$$

where $k_0 = [\frac{n-1}{2}]$.

Proof. Let $k_0 = [\frac{n-1}{2}]$. Since,

$$\frac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \frac{(q;q)_n/(q;q)_{k+1}(q;q)_{n-k-1}}{(q;q)_n/(q;q)_k(q;q)_{n-k}} = \frac{1-q^{n-k}}{1-q^{k+1}},$$

we get

$$\begin{cases} \dfrac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} \geq 1, & \text{when } k \leq k_0, \\[2ex] \dfrac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} < 1, & \text{when } k > k_0. \end{cases}$$

So we have

$$1 \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \leq \begin{bmatrix} n \\ k_0 \end{bmatrix}_q, \quad k = 0, 1, \dots, n. \tag{2.3}$$

Similarly,

$$1 \leq \begin{bmatrix} n \\ k \end{bmatrix}_p \leq \begin{bmatrix} n \\ k_0 \end{bmatrix}_p, \quad k = 0, 1, \dots, n. \tag{2.4}$$

Under the conditions of the theorem, we also have

$$0 < p^{k(k-1)} < 1, \qquad 0 < q^{\frac{1}{2}k(k-1)} < 1, \tag{2.5}$$

$$0 < \frac{1}{(x;p)_k} = \frac{1}{1-x} \cdot \frac{1}{1-xp} \cdots \frac{1}{1-xp^{k-1}} \leq \frac{1}{(x;p)_n}, \tag{2.6}$$

$$0 < x^k < 1, \qquad 0 < y^k < 1, \tag{2.7}$$

where $k = 0, 1, \dots, n$.

Let

$$\begin{cases} a_k = \begin{bmatrix} n \\ k \end{bmatrix}_p p^{k(k-1)} \frac{x^k}{(x;p)_k} \\[2ex] b_k = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} y^k \end{cases} \tag{2.8}$$

in the discrete version of the Grüss inequality (1.9). Combining (2.3), (2.4), (2.5), (2.6) and (2.7) one gets

$$0 < a_k < \frac{1}{(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \quad \text{and} \quad 0 < b_k < \begin{bmatrix} n \\ k_0 \end{bmatrix}_q.$$

Substituting a_k and b_k into the discrete version of the Grüss inequality (1.9), gives

$$\begin{aligned}(2.9)\quad &\left|\frac{1}{n+1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_p\begin{bmatrix}n\\k\end{bmatrix}_q p^{k(k-1)}q^{\frac{1}{2}k(k-1)}\frac{x^ky^k}{(x;p)_k}\right.\\ &\qquad\left.-\left\{\frac{1}{n+1}\sum_{k=0}^{n}p^{k(k-1)}\begin{bmatrix}n\\k\end{bmatrix}_p\frac{x^k}{(x;p)_k}\right\}\left\{\frac{1}{n+1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{1}{2}k(k-1)}y^k\right\}\right|\\ &\qquad\qquad\le\frac{1}{4(x;p)_n}\begin{bmatrix}n\\k_0\end{bmatrix}_p\begin{bmatrix}n\\k_0\end{bmatrix}_q.\end{aligned}$$

Using (1.5) and (1.7) one gets

$$(2.10)\qquad \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_p p^{k(k-1)}\frac{x^k}{(x;p)_k}=\frac{1}{(x;p)_n},$$

$$(2.11)\qquad \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{1}{2}k(k-1)}y^k=(-y;q)_n.$$

Substituting (2.10) and (2.11) into (2.9), we have

$$\begin{aligned}(2.12)\quad &\left|\frac{1}{n+1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_p\begin{bmatrix}n\\k\end{bmatrix}_q p^{k(k-1)}q^{\frac{1}{2}k(k-1)}\frac{x^ky^k}{(x;p)_k}-\frac{(-y;q)_n}{(n+1)^2(x;p)_n}\right|\\ &\qquad\qquad\le\frac{1}{4(x;p)_n}\begin{bmatrix}n\\k_0\end{bmatrix}_p\begin{bmatrix}n\\k_0\end{bmatrix}_q,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(2.13)\quad &\frac{(-y;q)_n}{(n+1)(x;p)_n}-\frac{n+1}{4(x;p)_n}\begin{bmatrix}n\\k_0\end{bmatrix}_p\begin{bmatrix}n\\k_0\end{bmatrix}_q\\ &\qquad\le\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_p\begin{bmatrix}n\\k\end{bmatrix}_q p^{k(k-1)}q^{\frac{1}{2}k(k-1)}\frac{x^ky^k}{(x;p)_k}\\ &\qquad\le\frac{(-y;q)_n}{(n+1)(x;p)_n}+\frac{n+1}{4(x;p)_n}\begin{bmatrix}n\\k_0\end{bmatrix}_p\begin{bmatrix}n\\k_0\end{bmatrix}_q.\end{aligned}$$

The proof is thus completed. □

In the proof, we used the Euler formula and the limiting case of the q-Gauss sum. We wish to point out, that there may be other pairs of summation theorems which also lead to interesting results. As an application of the inequality, we can easily obtain the following one.

Corollary 2.2. *Under the conditions of Theorem 2.1, we have*

$$(2.14)\qquad \sum_{k=0}^{n}\binom{n}{k}^2\frac{x^ky^k}{(1-x)^k}\le\frac{(1+y)^n}{(n+1)(1-x)^n}+\frac{n+1}{4(1-x)^n}\binom{n}{k_0}^2,$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

Proof. From [5], we know

$$\lim_{q\to1}\begin{bmatrix}n\\i\end{bmatrix}_q=\binom{n}{i}.$$

Letting $p\to1$ and $q\to1$ on both sides of inequality (2.2), we get (2.14). □

3. APPLICATION OF THE INEQUALITY

Convergence is an important problem in the study of q-series. In this section, we use the inequality obtained in this paper to give a sufficient condition for the convergence of a bibasic series.

Theorem 3.1. *Suppose* $0 < p < 1$, $0 < q < 1$, $0 < x < 1$ *and* $0 < y < 1$. *Let* $\{c_n\}$ *be any sequence of numbers. If*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1,$$

then the bibasic series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} c_n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x;p)_k} \tag{3.1}$$

converges absolutely.

Proof. Multiplying both sides of (2.2) by $|c_n|$, one gets

$$\begin{aligned} |c_n| \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x;p)_k} &\le \frac{(-y;q)_n |c_n|}{(n+1)(x;p)_n} + \frac{(n+1)|c_n|}{4(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q \\ &\le \frac{(-y;q)_n |c_n|}{(n+1)(x;p)_n} + \frac{(n+1)|c_n|}{4(x;p)_n (p;p)_\infty (q;q)_\infty}. \end{aligned} \tag{3.2}$$

The ratio test shows that both

$$\sum_{n=0}^{\infty} \frac{(-y;q)_n c_n}{(n+1)(x;p)_n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(n+1)c_n}{4(x;p)_n (p;p)_\infty (q;q)_\infty}$$

are absolutely convergent. Together with (3.2), this immediately yields that the series in (3.1) is absolutely convergent. □

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