ON A WEIGHTED INTERPOLATION OF FUNCTIONS WITH CIRCULAR MAJORANT

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Abstract: Denote by L_n the projection operator obtained by applying the Lagrange interpolation method, weighted by $(1-x^2)^{1/2}$, at the zeros of the Chebyshev polynomial of the second kind of degree $n+1$. The norm $\|L_n\| = \max\limits_{\|f\|_\infty \le 1} \|L_n f\|_\infty$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[-1,1]$, is known to be asymptotically the same as the minimum possible norm over all choices of interpolation nodes for unweighted Lagrange interpolation. Because the projection forces the interpolating function to vanish at ± 1, it is appropriate to consider a modified projection norm $\|L_n\|_\psi = \max\limits_{|f(x)| \le \psi(x)} \|L_n f\|_\infty$, where $\psi \in C[-1,1]$ is a given function (a *curved majorant*) that satisfies $0 \le \psi(x) \le 1$ and $\psi(\pm 1) = 0$. In this paper the asymptotic behaviour of the modified projection norm is studied in the case when $\psi(x)$ is the circular majorant $w(x) = (1-x^2)^{1/2}$. In particular, it is shown that asymptotically $\|L_n\|_w$ is smaller than $\|L_n\|$ by the quantity $2\pi^{-1}(1-\log 2)$.

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1. Introduction

Suppose $n \geq 1$ is an integer, and for any s, let $\theta_s = \theta_{s,n} = (s+1)\pi/(n+2)$. For $i = 0, 1, \ldots, n$, put $x_i = \cos\theta_i$. The x_i are the zeros of the Chebyshev polynomial of the second kind of degree $n+1$, defined by $U_{n+1}(x) = [\sin(n+2)\theta]/\sin\theta$ where $x = \cos\theta$ and $0 \leq \theta \leq \pi$. Also let w be the weight function $w(x) = \sqrt{1-x^2}$, and denote the set of all polynomials of degree n or less by P_n.

In the paper [5], J.C. Mason and G.H. Elliott introduced the interpolating projection L_n of $C[-1,1]$ on $\{wp_n : p_n \in P_n\}$ that is defined by

$$(L_n f)(x) = w(x) \sum_{i=0}^{n} \ell_i(x) \frac{f(x_i)}{w(x_i)}, \tag{1.1}$$

where $\ell_i(x)$ is the fundamental Lagrange polynomial

$$\ell_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k} = \frac{U_{n+1}(x)}{U'_{n+1}(x_i)(x - x_i)}. \tag{1.2}$$

Mason and Elliott studied the projection norm

$$\|L_n\| = \max_{\|f\|_\infty \leq 1} \|L_n f\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the uniform norm $\|g\|_\infty = \max_{-1\leq x\leq 1} |g(x)|$, and obtained results that led to the conjecture

$$\|L_n\| = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + o(1) \quad \text{as } n \to \infty, \tag{1.3}$$

where $\gamma = 0.577\ldots$ is Euler's constant. This result (1.3) was proved later by Smith [8].

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As pointed out by Mason and Elliott, the projection norm for the much-studied Lagrange interpolation method based on the zeros of the Chebyshev polynomial of the first kind $T_{n+1}(x) = \cos(n+1)\theta$, where $x = \cos\theta$ and $0 \le \theta \le \pi$, is

$$\frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma\right) + o(1).$$

(See Luttmann and Rivlin [4] for a short proof of this result based on a conjecture that was later established by Ehlich and Zeller [3].) Therefore the norm of the weighted interpolation method (1.1) is smaller by a quantity asymptotic to $2\pi^{-1}\log 2$. In addition, (1.3) means that L_n, which is based on a simple node system, has (to within $o(1)$ terms) the same norm as the Lagrange method of minimal norm over all possible choices of nodes — and the optimal nodes for Lagrange interpolation are not known explicitly. (See Brutman [2, Section 3] for further discussion and references on the optimal choice of nodes for Lagrange interpolation.)

Now, an immediate consequence of (1.1) is that for all f, $(L_n f)(\pm 1) = 0$. Thus L_n is particularly appropriate for approximations of those f for which $f(\pm 1) = 0$. This leads naturally to a study of the norm

$$\|L_n\|_\psi = \max_{|f(x)| \le \psi(x)} \|L_n f\|_\infty, \tag{1.4}$$

where $\psi \in C[-1,1]$ is a given function (a *curved majorant*) that satisfies $0 \le \psi(x) \le 1$ and $\psi(\pm 1) = 0$. Evidently $\|L_n\|_\psi \le \|L_n\|$. In this paper we will look at the particular case when $\psi(x)$ is the circular majorant $w(x) = \sqrt{1-x^2}$. Note that studies of this nature were initiated by P. Turán in the early 1970s, in the context of obtaining Markov and Bernstein type estimates for p' if $p \in P_n$ satisfies $|p(x)| \le w(x)$ for $x \in [-1,1]$ — see Rahman [6] for a key early paper in this area.

Our principal result is the following theorem, the proof of which will be developed in Sections 2 and 3.

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Theorem 1.1. *The modified projection norm $\|L_n\|_w$, defined by (1.4) with $w(x) = \sqrt{1-x^2}$, satisfies*

$$\|L_n\|_w = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma - 1\right) + o(1) \quad \textit{as } n \to \infty. \tag{1.5}$$

Observe that (1.5) shows $\|L_n\|_w$ is smaller than $\|L_n\|$ by an amount that is asymptotic to $2\pi^{-1}(1-\log 2)$.

Before proving the theorem, we make a few remarks about the method to be used. By (1.1),

$$\|L_n\|_w = \max_{-1\le x\le 1}\left(w(x)\sum_{i=0}^{n}|\ell_i(x)|\right).$$

Since the x_i are arranged symmetrically about 0, then $w(x)\sum_{i=0}^{n}|\ell_i(x)|$ is even, and so by (1.2),

$$\|L_n\|_w = \max_{0\le\theta\le\pi/2} F_n(\theta),$$

where

$$F_n(\theta) = \frac{|\sin(n+2)\theta|}{n+2}\sum_{i=0}^{n}\frac{\sin^2\theta_i}{|\cos\theta - \cos\theta_i|}. \tag{1.6}$$

Figure 1 shows the graph of a typical $F_n(\theta)$ if n is even, and it suggests that the local maximum values of $F_n(\theta)$ are monotonic increasing as θ moves from left to right, so that the maximum of $F_n(\theta)$ occurs close to $\pi/2$. For n odd, similar graphs suggest that the maximum occurs precisely at $\pi/2$. These observations help to motivate the strategy used in Sections 2 and 3 to prove the theorem — the approach is akin to that used by Brutman [1] in his investigation of the Lebesgue function for Lagrange interpolation based on the zeros of Chebyshev polynomials of the first kind.

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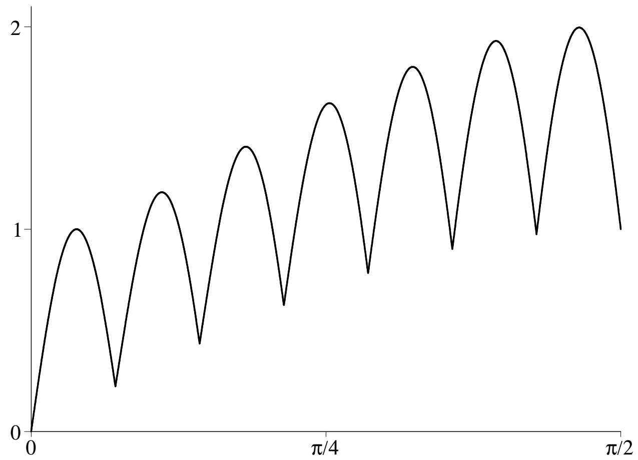

Figure 1: Plot of $F_{12}(\theta)$ for $0 \le \theta \le \pi/2$

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2. Some Lemmas

This section contains several lemmas that will be needed to prove the theorem. The first such lemma provides alternative representations of the function $F_n(\theta)$ that was defined in (1.6).

Lemma 2.1. *If j is an integer with $0 \le j \le n+1$, and $\theta_{j-1} \le \theta \le \theta_j$, then*

$$F_n(\theta) = (-1)^j \frac{\sin(n+2)\theta}{n+2} \left(\sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta} + \sum_{i=j}^{n} \frac{\sin^2\theta_i}{\cos\theta - \cos\theta_i} \right) \tag{2.1}$$

$$= (-1)^j \left[\sin(n+1)\theta + \frac{2\sin(n+2)\theta}{n+2} \sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta} \right]. \tag{2.2}$$

Proof. The result (2.1) follows immediately from (1.6). For (2.2), note that the Lagrange interpolation polynomial for $U_n(x)$ based on the zeros of $U_{n+1}(x)$ is simply $U_n(x)$ itself, so, with $\ell_i(x)$ defined by (1.2),

$$U_n(x) = \sum_{i=0}^{n} \ell_i(x) U_n(x_i) = \frac{U_{n+1}(x)}{n+2} \sum_{i=0}^{n} \frac{1-x_i^2}{x-x_i}.$$

(This formula appears in Rivlin [7, p. 23, Exercise 1.3.2].) Therefore

$$\sin(n+1)\theta = \frac{\sin(n+2)\theta}{n+2} \sum_{i=0}^{n} \frac{\sin^2\theta_i}{\cos\theta - \cos\theta_i}.$$

If this expression is used to rewrite the second sum in (2.1), the result (2.2) is obtained. □

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We now show that on the interval $[0, \pi/2]$, the values of $F_n(\theta)$ at the midpoints between consecutive θ-nodes are increasing — this result is established in the next two lemmas.

Lemma 2.2. *If j is an integer with $0 \le j \le n$, then*

$$\Delta_{n,j} := (n+2)\left(F_n(\theta_{j+1/2}) - F_n(\theta_{j-1/2})\right) = 2\sin\theta_j \sin\theta_{-1/2} \times \Delta^*_{n,j},$$

where

$$\Delta^*_{n,j} := (j-n-1) + \sum_{i=1}^{j} \cot\theta_{(2j+2i-3)/4} \cot\theta_{(2j-2i-1)/4} + \cot\theta_{j-1/4}\cot\theta_{-1/2} + \frac{1}{2}\csc\theta_{j-1/4}\csc\theta_{j+1/4}. \tag{2.3}$$

Proof. By (2.2),

$$\Delta_{n,j} = -2(n+2)\sin\theta_j \sin\theta_{-1/2} + 2\left[\sum_{i=0}^{j} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta_{j+1/2}} - \sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta_{j-1/2}}\right].$$

From the trigonometric identity

$$\frac{\sin^2 A}{\cos A - \cos B} = \frac{1}{2}\sin B\left[\cot\left(\frac{B-A}{2}\right) + \cot\left(\frac{B+A}{2}\right)\right] - \cos A - \cos B, \tag{2.4}$$

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it follows that

$$\sum_{i=0}^{j} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta_{j+1/2}} - \sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{\cos\theta_i - \cos\theta_{j-1/2}}$$

$$= \frac{1}{2}\sin\theta_{j+1/2} \sum_{\substack{i=0 \\ i\neq j+1}}^{2j+2} \cot\theta_{(2i-3)/4} - \frac{1}{2}\sin\theta_{j-1/2} \sum_{\substack{i=0 \\ i\neq j}}^{2j} \cot\theta_{(2i-3)/4}$$

$$- \cos\theta_j - (j+1)\cos\theta_{j+1/2} + j\cos\theta_{j-1/2}$$

$$= \cos\theta_j \sin\theta_{-1/2} \sum_{i=1}^{2j} \cot\theta_{(2i-3)/4} + (2j+2)\sin\theta_j \sin\theta_{-1/2}$$

$$+ \frac{1}{2}\sin\theta_{j+1/2}\left(\cot\theta_{j-1/4} + \cot\theta_{j+1/4}\right).$$

Therefore

$$\Delta_{n,j} = (4j-2n)\sin\theta_j \sin\theta_{-1/2} + 2\cos\theta_j \sin\theta_{-1/2} \sum_{i=1}^{2j} \cot\theta_{(2i-3)/4} + \sin\theta_{j+1/2}\left(\cot\theta_{j-1/4} + \cot\theta_{j+1/4}\right). \tag{2.5}$$

Next consider

$$j\sin\theta_j + \cos\theta_j \sum_{i=1}^{2j} \cot\theta_{(2i-3)/4}$$

$$= \sum_{i=1}^{j} \left[\sin\theta_j + \cos\theta_j\left(\cot\theta_{(2i-3)/4} + \cot\theta_{(4j-2i-1)/4}\right)\right]$$

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$$= \sin\theta_j \sum_{i=1}^{j} \left[1 + \frac{\cos\theta_j}{\sin\theta_{(2i-3)/4} \sin\theta_{(4j-2i-1)/4}} \right]$$

$$= \sin\theta_j \sum_{i=1}^{j} \cot\theta_{(4j-2i-1)/4} \cot\theta_{(2i-3)/4}$$

$$= \sin\theta_j \sum_{i=1}^{j} \cot\theta_{(2j+2i-3)/4} \cot\theta_{(2j-2i-1)/4}. \tag{2.6}$$

Also

$$\begin{aligned} &\sin\theta_{j+1/2} \left(\cot\theta_{j-1/4} + \cot\theta_{j+1/4}\right) \\ &\quad = 2\sin\theta_j \frac{\cos\theta_j \sin\theta_{j+1/2}}{\sin\theta_{j-1/4} \sin\theta_{j+1/4}} \\ &\quad = \sin\theta_j \frac{2\cos\theta_{j+1/4} \sin\theta_{j+1/4} + \sin\theta_{-1/2}}{\sin\theta_{j-1/4} \sin\theta_{j+1/4}} \\ &\quad = \sin\theta_j \sin\theta_{-1/2} \left[\frac{2\cos\theta_{j+1/4}}{\sin\theta_{j-1/4} \sin\theta_{-1/2}} + \csc\theta_{j-1/4} \csc\theta_{j+1/4} \right] \end{aligned}$$

$$\quad = \sin\theta_j \sin\theta_{-1/2} \left[-2 + 2\cot\theta_{j-1/4} \cot\theta_{-1/2} + \csc\theta_{j-1/4} \csc\theta_{j+1/4} \right]. \tag{2.7}$$

The lemma is now established by substituting (2.6) and (2.7) into (2.5). □

Lemma 2.3. *If j is an integer with $0 \le j \le (n-1)/2$, then*

$$F_n(\theta_{j+1/2}) > F_n(\theta_{j-1/2}).$$

Proof. By Lemma 2.2 we need to show that $\Delta^*_{n,j} > 0$, where $\Delta^*_{n,j}$ is defined by (2.3). Now, if $0 < a < \pi/4$ and $0 < b < a$, then

$$\cot(a+b)\cot(a-b) > \cot^2 a.$$

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Also, $x\csc^2 x$ is decreasing on $(0,\pi/4)$, so $\csc^2 x > \pi/(2x)$ if $0 < x < \pi/4$. Thus

$$\begin{aligned} j + \sum_{i=1}^{j} \cot\theta_{(2j+2i-3)/4}\cot\theta_{(2j-2i-1)/4} &> j + \sum_{i=1}^{j}\cot^2\theta_{(j-1)/2} \\ &= j\csc^2\theta_{(j-1)/2} > \frac{(n+2)j}{j+1}, \end{aligned}$$

and so

$$\begin{aligned} \Delta^*_{n,j} &> 1 + \cot\theta_{j-1/4}\cot\theta_{-1/2} + \frac{1}{2}\csc\theta_{j-1/4}\csc\theta_{j+1/4} - \frac{n+2}{j+1} \\ &> \csc^2\theta_{(4j-3)/8} + \frac{1}{2}\csc\theta_{j-1/4}\csc\theta_{j+1/4} - \frac{n+2}{j+1}. \end{aligned}$$

Because $\theta_{(4j-3)/8} < \pi/4$, the first term in this expression can be estimated using $\csc^2 x > \pi/(2x)$, while the second term can be estimated using $\csc x > 1/x$. Therefore

$$\begin{aligned} \Delta^*_{n,j} &> \left[\frac{n+2}{j+5/4} - \frac{n+2}{j+1}\right] + \frac{(n+2)^2}{2\pi^2(j+3/4)(j+5/4)} \\ &> \frac{n+2}{(j+1)(j+5/4)}\left[-\frac{1}{4} + \frac{n+2}{2\pi^2}\right]. \end{aligned}$$

This latter quantity is positive if $n \geq 3$. Since $0 \leq j \leq (n-1)/2$, the only unresolved cases are when $j = 0$ and $n = 1, 2$, and it is a trivial calculation using (2.3) to show that $\Delta^*_{n,j} > 0$ in these cases as well. $\square$

We next show that in any interval between successive θ-nodes, $F_n(\theta)$ achieves its maximum in the right half of the interval.

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Lemma 2.4. *If j is an integer with $0 \le j \le (n+1)/2$, and $0 < t < 1/2$, then*

$$F_n(\theta_{j-1/2+t}) \ge F_n(\theta_{j-1/2-t}). \tag{2.8}$$

Proof. If $j = (n+1)/2$, then $\theta_{j-1/2} = \pi/2$, so equality holds in (2.8) because $F_n(\theta)$ is symmetric about $\pi/2$. Thus we can assume $j \le n/2$. For convenience, write $a = j - 1/2 - t$, $b = j - 1/2 + t$. Since $\sin(n+2)\theta_a = \sin(n+2)\theta_b = (-1)^j \cos t\pi$, it follows from (2.1) that $F_n(\theta_b) - F_n(\theta_a)$ has the same sign as

$$G_{n,j}(t) := \sum_{i=j}^{n} \frac{\sin^2\theta_i}{(\cos\theta_b - \cos\theta_i)(\cos\theta_a - \cos\theta_i)} - \sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{(\cos\theta_i - \cos\theta_b)(\cos\theta_i - \cos\theta_a)}.$$

If $j = 0$ this is clearly positive, and otherwise

$$\begin{aligned} G_{n,j}(t) &> \sum_{i=j}^{2j-1} \frac{\sin^2\theta_i}{(\cos\theta_b - \cos\theta_i)(\cos\theta_a - \cos\theta_i)} \\ &\qquad - \sum_{i=0}^{j-1} \frac{\sin^2\theta_i}{(\cos\theta_i - \cos\theta_b)(\cos\theta_i - \cos\theta_a)} \\ &= \sum_{i=0}^{j-1} \left[\frac{\sin^2\theta_{2j-i-1}}{(\cos\theta_b - \cos\theta_{2j-i-1})(\cos\theta_a - \cos\theta_{2j-i-1})} \right. \\ &\qquad \left. - \frac{\sin^2\theta_i}{(\cos\theta_i - \cos\theta_b)(\cos\theta_i - \cos\theta_a)} \right]. \end{aligned}$$

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We will show that each term in this sum is positive. Because $\sin\theta_{2j-i-1} > \sin\theta_i$, this will be true if for $0 \le i \le j-1$,

$$\begin{aligned}\sin\theta_{2j-i-1}(\cos\theta_i - \cos\theta_b)(\cos\theta_i - \cos\theta_a)\\ - \sin\theta_i(\cos\theta_b - \cos\theta_{2j-i-1})(\cos\theta_a - \cos\theta_{2j-i-1}) > 0.\end{aligned}$$

By rewriting each difference of cosine terms as a product of sine terms, it follows that we require

$$\begin{aligned}\sin\theta_{2j-i-1}\sin\theta_{(j+i-1/2+t)/2}\sin\theta_{(j+i-1/2-t)/2}\\ - \sin\theta_i\sin\theta_{(3j-i-3/2+t)/2}\sin\theta_{(3j-i-3/2-t)/2} > 0.\end{aligned}$$

To establish this inequality, note that

$$\begin{aligned}&\sin\theta_{2j-i-1}\sin\theta_{(j+i-1/2+t)/2}\sin\theta_{(j+i-1/2-t)/2}\\ &\qquad - \sin\theta_i\sin\theta_{(3j-i-3/2+t)/2}\sin\theta_{(3j-i-3/2-t)/2}\\ &= \frac{1}{2}\left[\cos\theta_{t-1}(\sin\theta_{2j-i-1} - \sin\theta_i) - \sin\theta_{2j-i-1}\cos\theta_{j+i+1/2} + \sin\theta_i\cos\theta_{3j-i-1/2}\right]\\ &= \cos\theta_{t-1}\sin\theta_{j-i-3/2}\cos\theta_{j-1/2} - \frac{1}{4}\left[\sin\theta_{j-2i-5/2} + \sin\theta_{3j-2i-3/2}\right]\\ &= \cos\theta_{j-1/2}\left[\cos\theta_{t-1}\sin\theta_{j-i-3/2} - \frac{1}{2}\sin\theta_{2j-2i-2}\right]\\ &= \cos\theta_{j-1/2}\sin\theta_{j-i-3/2}\left[\cos\theta_{t-1} - \cos\theta_{j-i-3/2}\right] > 0,\end{aligned}$$

and so the lemma is proved. □

The final major step in the proof of the theorem is to show that in each interval between successive θ-nodes, the maximum value of $F_n(\theta)$ is achieved essentially at the midpoint of the interval.

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Lemma 2.5. *If n, j are integers with $n \geq 2$ and $0 \leq j \leq (n+1)/2$, then*

$$\max_{\theta_{j-1} \leq \theta \leq \theta_j} F_n(\theta) = F_n(\theta_{j-1/2}) + \mathcal{O}\left((\log n)^{-1}\right), \tag{2.9}$$

where the $\mathcal{O}\left((\log n)^{-1}\right)$ term is independent of j.

Proof. By Lemma 2.4, it is sufficient to show that $G_{n,j,t} := F_n(\theta_{j-1/2+t}) - F_n(\theta_{j-1/2})$ is bounded above by an $\mathcal{O}\left((\log n)^{-1}\right)$ term that is independent of j and t for $0 \leq t \leq 1/2$.

Now, by (2.2) we have

$$\begin{aligned} G_{n,j,t} &= \frac{2}{n+2} \sum_{i=0}^{j-1} \left[\frac{\cos t\pi \sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2+t}} - \frac{\sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2}} \right] \\ &\quad + 2 \sin \frac{(n+1)t\pi}{2(n+2)} \sin \left(\frac{(2j+1)\pi}{2(n+2)} - \frac{(n+1)t\pi}{2(n+2)} \right). \end{aligned} \tag{2.10}$$

Since $\cos t\pi \leq 1 - 4t^2$ if $0 \leq t \leq 1/2$, then each summation term can be estimated by

$$\frac{\cos t\pi \sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2+t}} - \frac{\sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2}} \leq \frac{-4t^2 \sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2+t}}.$$

From $(2x)/\pi \leq \sin x \leq x$ for $0 \leq x \leq \pi/2$, it follows that

$$\begin{aligned} \frac{\sin^2 \theta_i}{\cos \theta_i - \cos \theta_{j-1/2+t}} &= \frac{\sin^2 \theta_i}{2 \sin \theta_{(j+i-1/2+t)/2} \sin \theta_{(j-i-5/2+t)/2}} \\ &\geq \frac{8(i+1)^2}{\pi^2 (j-i)(j+i+2)}, \end{aligned}$$

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and so

$$\sum_{i=0}^{j-1}\left[\frac{\cos t\pi \sin^2\theta_i}{\cos\theta_i-\cos\theta_{j-1/2+t}}-\frac{\sin^2\theta_i}{\cos\theta_i-\cos\theta_{j-1/2}}\right]$$
$$\leq -\frac{32t^2}{\pi^2}\sum_{i=0}^{j-1}\frac{(i+1)^2}{(j-i)(j+i+2)}$$
$$= -\frac{32t^2}{\pi^2}\left[-j-\frac{1}{2}+\frac{j+1}{2}\sum_{k=1}^{2j+1}\frac{1}{k}\right]$$
$$\leq -\frac{16t^2}{\pi^2}(j+1)\left(\log(j+1)-1\right), \tag{2.11}$$

where the final inequality follows from

$$\sum_{k=1}^{2j+1}\frac{1}{k}\geq 1+\log(j+1).$$

Also,

$$\sin\frac{(n+1)t\pi}{2(n+2)}\sin\left(\frac{(2j+1)\pi}{2(n+2)}-\frac{(n+1)t\pi}{2(n+2)}\right)\leq \sin\frac{t\pi}{2}\sin\frac{(2j+1)\pi}{2(n+2)} \leq \frac{t\pi^2(j+1)}{2(n+2)}. \tag{2.12}$$

We now return to the characterization (2.10) of $G_{n,j,t}$. By (2.12), $G_{n,0,t}\leq \pi^2/(2(n+2))$. For $j\geq 1$, it follows from (2.11) and (2.12) that

$$G_{n,j,t}\leq \frac{2\pi^2 t(j+1)}{n+2}\left[1-\frac{16t}{\pi^4}\log(j+1)\right]\leq \frac{\pi^6}{32(n+2)}\left[\frac{j+1}{\log(j+1)}\right], \tag{2.13}$$

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where the latter inequality follows by maximizing the quadratic in t. On the interval $1 \leq j \leq (n+1)/2$, the maximum of $(j+1)/\log(j+1)$ occurs at an endpoint, so

$$\frac{j+1}{\log(j+1)} \leq \max\left\{\frac{2}{\log 2}, \frac{n+3}{2\log((n+3)/2)}\right\}. \tag{2.14}$$

The result (2.9) then follows from (2.13) and (2.14). □

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3. Proof of the Theorem

Since $\|L_n\|_w = \max_{0\le\theta\le\pi/2} F_n(\theta)$, it follows from Lemmas 2.3 and 2.5 that

$$\|L_n\|_w = \begin{cases} F_n\left(\frac{\pi}{2}\right) + \mathcal{O}\left((\log n)^{-1}\right) & \text{if } n \text{ is odd,} \\ F_n\left(\frac{\pi(n+1)}{2(n+2)}\right) + \mathcal{O}\left((\log n)^{-1}\right) & \text{if } n \text{ is even.} \end{cases}$$

To obtain the asymptotic result (1.5) for $\|L_n\|_w$ we use a method that was introduced by Luttmann and Rivlin [4, Theorem 3], and used also by Mason and Elliott [5, Section 9].

If n is odd, then by (2.2) with $n = 2m-1$,

$$\begin{aligned} F_n\left(\frac{\pi}{2}\right) &= \frac{2}{2m+1}\sum_{i=0}^{m-1}\frac{\sin^2\theta_i}{\cos\theta_i} \\ &= \frac{2}{2m+1}\sum_{k=1}^{m}\left[\csc\frac{(k-1/2)\pi}{2m+1} - \sin\frac{(k-1/2)\pi}{2m+1}\right], \end{aligned} \tag{3.1}$$

where the second equality follows by reversing the order of summation. Now,

$$\begin{aligned} &\frac{\pi}{2m+1}\sum_{k=1}^{m}\csc\frac{(k-1/2)\pi}{2m+1} \\ &\qquad = \frac{\pi}{2m+1}\sum_{k=1}^{m}\left[\csc\frac{(k-1/2)\pi}{2m+1} - \frac{2m+1}{(k-1/2)\pi}\right] + \sum_{k=1}^{m}\frac{1}{k-1/2}. \end{aligned}$$

The asymptotic behaviour as $m\to\infty$ of each of the sums in this expression is given

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by

$$\lim_{m\to\infty} \frac{\pi}{2m+1} \sum_{k=1}^{m} \left[\csc \frac{(k-1/2)\pi}{2m+1} - \frac{2m+1}{(k-1/2)\pi}\right] = \int_0^{\pi/2} \left[\csc x - \frac{1}{x}\right] dx$$
$$= \log \frac{4}{\pi}$$

and

$$\sum_{k=1}^{m} \frac{1}{k-1/2} = 2\sum_{k=1}^{2m} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} = \log(4m) + \gamma + o(1).$$

Also,

$$\sum_{k=1}^{m} \sin \frac{(k-1/2)\pi}{2m+1} = \csc \frac{\pi}{4m+2} \sin^2 \frac{m\pi}{4m+2} = \frac{2m+1}{\pi} + \mathcal{O}(1).$$

Substituting these asymptotic results into (3.1) yields the desired result (1.5) if n is odd.

On the other hand, if $n = 2m$ is even, then by (2.2) and (2.4),

$$F_n\left(\frac{\pi(n+1)}{2(n+2)}\right)$$
$$= \sin \frac{\pi}{4m+4} + \frac{1}{m+1} \sum_{i=0}^{m-1} \frac{\sin^2 \theta_i}{\cos \theta_i - \cos \frac{(2m+1)\pi}{4m+4}}$$
$$= \frac{1}{m+1} \left(\frac{1}{2} \cos \frac{\pi}{4m+4} \sum_{i=1}^{2m+2} \cot \frac{(2i-1)\pi}{8m+8} - \sum_{i=0}^{m-1} \cos \theta_i\right) + \mathcal{O}(m^{-1}). \tag{3.2}$$

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The sum of the cotangent terms can be estimated by a similar argument to that above, using

$$\int_0^{\pi/2} (\cot x - x^{-1})\, dx = \log \frac{2}{\pi},$$

to obtain

$$\frac{1}{2m+2} \sum_{i=1}^{2m+2} \cot \frac{(2i-1)\pi}{8m+8} = \frac{2}{\pi}\left(\log \frac{16m}{\pi} + \gamma\right) + o(1).$$

Also,

$$\begin{aligned}\frac{1}{m+1} \sum_{i=0}^{m-1} \cos \theta_i &= \frac{1}{\sqrt{2}(m+1)}\left(\cos \frac{m\pi}{4m+4} \csc \frac{\pi}{4m+4} - \sqrt{2}\right)\\ &= \frac{2}{\pi} + \mathcal{O}(m^{-1}).\end{aligned}$$

If these asymptotic results are substituted into (3.2), the result (1.5) is obtained if n is even, and so the proof of Theorem 1.1 is completed.

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