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IMPROVEMENT OF THE NON-UNIFORM VERSION OF BERRY-ESSEEN INEQUALITY VIA PADITZ-SIGANOV THEOREMS

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ABSTRACT. We improve the constant in a non-uniform bound of the Berry-Esseen inequality without assuming the existence of the absolute third moment by using the method obtained from the Paditz-Siganov theorems. Our bound is better than the results of Thongtha and Neammanee in 2007 ([14]).

Key words and phrases: Berry-Esseen inequality, Paditz-Siganov theorems, central limit theorem, uniform and non-uniform bounds.

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1. INTRODUCTION AND MAIN RESULTS

The Berry-Esseen inequality is one of the most important inequalities in the theory of probability. This inequality was independently discovered by two mathematicians, Andrew C. Berry ([2]) and Carl-Gustav Esseen ([5]) in 1941 and 1945 respectively. Let $X_1, X_2, \ldots, X_n$ be independent random varibles with zero mean and $\sum_{i=1}^n EX_i^2 = 1$. Define $W_n = X_1+X_2+\cdots+X_n$. Then $\operatorname{Var} W_n = 1$. Let F_n be the distribution function of W_n and Φ the standard normal distribution function, i.e.,

$$F_n(x) = P(W_n \le x) \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

The central limit theorem shows that F_n converges pointwise to Φ as $n \to \infty$ and the bounds of this convergence are,

$$\sup_{x\in\mathbb{R}} |P(W_n \le x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} E|X_i|^3 \tag{1.1}$$

and

$$|P(W_n \le x) - \Phi(x)| \le \frac{C_1}{1+|x|^3} \sum_{i=1}^{n} E|X_i|^3 \tag{1.2}$$

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for uniform and non-uniform versions respectively, where both C_0 ans C_1 are positive constants and stated under the assumption that $E|X_i|^3 < \infty$ for $i = 1, 2, \dots, n$.

In the case of identical X_i's, Siganov ([11]) and Chen ([5]) improved the constant down to 0.7655 and 0.7164, respectively. For non-uniform bounds, Nageav ([7]) was the first to obtain (1.2) and Michel ([6]) calculated the constant to be 30.84.

Without assuming identically distributed $X_i's$, Beek ([15]) sharpened the constant down to 0.7975 in 1972 for the uniform version. The best bound was found by Siganov ([11]) in 1986.

Theorem 1.1 (Siganov,1986)**.** *Let $X_1, X_2, \dots, X_n$ be independent random variables such that $EX_i = 0$ and $E|X_i|^3 < \infty$ for $i = 1, 2, \dots, n$. Assume that $\sum_{i=1}^n EX_i^2 = 1$. Then*

$$\sup_{x\in\mathbb{R}} |P(W_n \le x) - \Phi(x)| \le 0.7915 \sum_{i=1}^n E|X_i|^3,$$

where $W_n = X_1 + X_2 + \cdots + X_n$.

For the non-uniform version, Bikelis ([1]) generalized (1.2) to this case and Paditz ([9]) calculated C_1 to be 114.7 in 1977. He also improved his result down to 31.935 in 1989.

Theorem 1.2 (Paditz ([10]),1989)**.** *Under the assumptions of Theorem 1.1, we have*

$$|P(W_n \le x) - \Phi(x)| \le \frac{31.935}{1 + |x|^3} \sum_{i=1}^n E|X_i|^3.$$

In 2001, Chen and Shao ([3]) gave new versions of (1.1) and (1.2) without assuming the existence of third moments. Their results are

$$\text{(1.3)} \qquad \sup_{x\in\mathbb{R}} |P(W_n \le x) - \Phi(x)| \le 4.1 \sum_{i=1}^n \{E|X_i|^2 I(|X_i| \ge 1|) + E|X_i|^3 I(|X_i| < 1)\}$$

and

$$\text{(1.4)} \quad |P(W_n \le x) - \Phi(x)| \le C_2 \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |x|)}{(1 + |x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1 + |x|)^3} \right\},$$

where C_2 is a positive constant and $I(A)$ is an indicator random variable such that

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

In 2005, Neammanee ([8]) combined the concentration inequality in ([3]) with a coupling approach to calculate the constant in (1.4), giving,

$$\text{(1.5)} \qquad |P(W_n \le x) - \Phi(x)| \le C_3 \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \ge 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\},$$

where C_3 is 21.44 for large values of x such that $|x| \ge 14.$

Thongtha and Neammanee ([14]) improved the concentration inequality used in ([8]) and gave a better constant, i.e., 9.7 for $|x| \ge 14$. The method which was used in ([8]) is Stein's method which was first introduced by Stein ([12]) in 1972. In this work, we provide a better constant by using Paditz-Siganov theorems. The results are as follows.

Theorem 1.3. *We have*

$$|P(W_n \le x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x|)}{(1+|x|)^3} \right\},$$

where

$$C = \begin{cases} 49.89 & \textit{if} \quad 0 \le |x| < 1.3, \\ 59.45 & \textit{if} \quad 1.3 \le |x| < 2, \\ 73.52 & \textit{if} \quad 2 \le |x| < 3, \\ 76.17 & \textit{if} \quad 3 \le |x| < 7.98, \\ 45.80 & \textit{if} \quad 7.98 \le |x| < 14, \\ 39.39 & \textit{if} \quad |x| \ge 14. \end{cases}$$

To compare Theorem 1.3 with the result of Thongtha and Neammanee ([14]) in (1.5), we give Corollary 1.4.

Corollary 1.4. *We have*

$$|P(W_n \le x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|\frac{x}{4}|)}{(1+|\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|\frac{x}{4}|)}{(1+|\frac{x}{4}|)^3} \right\},$$

where

$$C = \begin{cases} 9.54 & \textit{if} \quad 0 \le |x| < 1.3, \\ 19.74 & \textit{if} \quad 1.3 \le |x| < 2, \\ 18.38 & \textit{if} \quad 2 \le |x| < 3, \\ 14.63 & \textit{if} \quad 3 \le |x| < 7.98, \\ 5.13 & \textit{if} \quad 7.98 \le |x| < 14, \\ 3.55 & \textit{if} \quad |x| \ge 14. \end{cases}$$

We note from Corollary 1.4 that our result is better than a bound from Thongtha and Neammanee in ([14]).

2. PROOF OF THE MAIN RESULTS

In this section, we will prove Theorem 1.3 by using the Paditz-Siganov theorems. Corollary 1.4 can be obtained easily from Theorem 1.3. To prove these results, let

$$Y_{i,x} = X_i I(|X_i| < 1+x), \qquad S_x = \sum_{i=1}^{n} Y_{i,x},$$

$$\alpha_x = \sum_{i=1}^{n} EX_j^2 I(|X_j| \ge 1+x), \qquad \beta_x = \sum_{i=1}^{n} E|X_j|^3 I(|X_j| < 1+x),$$

$$\gamma_x = \frac{\beta_x}{2} \quad \text{and} \quad \delta_x = \frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \text{ for } x > 0.$$

Proposition 2.1. *For each $n \in \mathbb{N}$, we have*

(1) $\sum_{i=1}^{n} E|Y_{i,x} - EY_{i,x}|^3 \le \beta_x + \frac{7\alpha_x}{1+x}$,
(2) $1 - 2\alpha_x \le \operatorname{Var} S_x \le 1$, *and*
(3) *If* $\alpha_x \le 0.11$, *then* $0 < \frac{1}{\sqrt{\operatorname{Var} S_x}} \le 1 + 1.452\alpha_x$.

Proof. 1. By the fact that

$$|EX_iI(|X_i|<1+x)| = |EX_iI(|X_i|\geq 1+x)|, \tag{2.1}$$

$$E|X_i|^2 \leq \sum_{i=1}^n EX_i^2 = 1 \quad \text{and} \quad E^2X_i \leq EX_i^2,$$

we have

$$\begin{aligned}
&\sum_{i=1}^n E|Y_{i,x} - EY_{i,x}|^3 \\
&= \sum_{i=1}^n E|X_iI(|X_i|<1+x) - EX_iI(|X_i|<1+x)|^3 \\
&\leq \sum_{i=1}^n [E|X_i|^3 I(|X_i|<1+x) + 3EX_i^2 I(|X_i|<1+x)|EX_iI(|X_i|<1+x)| \\
&\qquad + 3E|X_iI\left(|X_i|<1+x\right)|E^2X_iI\left(|X_i|<1+x\right)| + |EX_iI\left(|X_i|<1+x\right)|^3] \\
&\leq \sum_{i=1}^n E|X_i|^3 I(|X_i|<1+x) + 3\sum_{i=1}^n |EX_iI(|X_i|<1+x)| \\
&\qquad + 3\sum_{i=1}^n E|X_i||EX_iI\left(|X_i|<1+x\right)||EX_iI\left(|X_i|<1+x\right)| \\
&\qquad + \sum_{i=1}^n E|X_i|^2 I\left(|X_i|<1+x\right)|EX_iI\left(|X_i|<1+x\right)| \\
&\leq \beta_x + 3\sum_{i=1}^n |EX_iI(|X_i|\geq 1+x)| + 3\sum_{i=1}^n E|X_i|^2|EX_iI(|X_i|\geq 1+x)| \\
&\qquad + \sum_{i=1}^n |EX_iI\left(|X_i|\geq 1+x\right)| \\
&\leq \beta_x + 3\sum_{i=1}^n E|X_i|I(|X_i|\geq 1+x) + 3\sum_{i=1}^n E|X_i|I(|X_i|\geq 1+x) \\
&\qquad + \sum_{i=1}^n E|X_i|I\left(|X_i|\geq 1+x\right) \\
&= \beta_x + 7\sum_{i=1}^n E|X_i|I(|X_i|\geq 1+x) \\
&\leq \beta_x + 7\sum_{i=1}^n \frac{E|X_i|^2 I(|X_i|\geq 1+x)}{(1+x)} = \beta_x + \frac{7\alpha_x}{(1+x)}.
\end{aligned}$$

2. By (2.1), we note that

$$\begin{aligned}
\operatorname{Var} S_x &= \sum_{i=1}^n \operatorname{Var} Y_{i,x} = \sum_{i=1}^n (EY_{i,x}^2 - E^2Y_{i,x}) \\
&= \sum_{i=1}^n EX_i^2 I(|X_i|<1+x) - \sum_{i=1}^n E^2X_iI(|X_i|<1+x)
\end{aligned}$$

$$= 1 - \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1+x) - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x)$$

$$= 1 - \alpha_x - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x). \tag{2.2}$$

From this and the fact that $\alpha_x \geq 0$, we have $\operatorname{Var} S_x \leq 1$.

By (2.2), we have

$$\begin{aligned} \operatorname{Var} S_x &= 1 - \alpha_x - \sum_{i=1}^{n} E^2 X_i I(|X_i| \geq 1+x) \\ &\geq 1 - \alpha_x - \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq 1+x) \\ &= 1 - 2\alpha_x. \end{aligned}$$

Hence, $1 - 2\alpha_x \leq \operatorname{Var} S_x \leq 1$.

3. For $0 < t \leq 0.11$, by using Taylor's formula, we have

$$\begin{aligned} \frac{1}{\sqrt{1-2t}} &= 1 + \frac{t}{(1-2c)^{\frac{3}{2}}} \text{ for some } c \in (0, 0.11] \\ &\leq 1 + \frac{t}{(1-2(0.11))^{\frac{3}{2}}} \\ &\leq 1 + 1.452t. \end{aligned}$$

From this fact and 2., we have

$$0 < \frac{1}{\sqrt{\operatorname{Var} S_x}} \leq \frac{1}{\sqrt{1-2\alpha_x}} \leq 1 + 1.452\alpha_x$$

for $\alpha_x \leq 0.11$. □

Proposition 2.2. *For each $x > 0$, let $\bar{Y}_{i,x} = \frac{Y_{i,x} - EY_{i,x}}{\sqrt{\operatorname{Var} S_x}}$ and $\bar{S}_x = \sum_{i=1}^{n} \bar{Y}_{i,x}$.*

(1) *If $\alpha_x \leq 0.099$ and $1.3 \leq x \leq 2$, then*

$$\left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| \leq \frac{54.513\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3}.$$

(2) *If $(1+x)^2\alpha_x < \frac{1}{5}$, then*

$$\left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi(\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| \leq \frac{C_1\alpha_x}{(1+x)^2} + \frac{C_2\beta_x}{(1+x)^3}$$

where $C_1 = 57.186$ $C_2 = 73.515$ for $2 \leq x < 3$,

$C_1 = 33.318$ $C_2 = 76.17$ for $3 \leq x < 7.98$,

$C_1 = 3.976$ $C_2 = 45.8$ for $7.98 \leq x < 14$, and

$C_1 = 1.226$ $C_2 = 39.382$ for $x \geq 14$.

Proof. 1. By Proposition 2.1(1) of ([14]) and Proposition 2.1(2), we have

$$|ES_x| \leq \frac{\alpha_x}{1+x} \leq 0.043 \text{ and } 1 \geq \operatorname{Var} S_x \geq 0.802 \tag{2.3}$$

which imply

$$0 \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}} \leq \frac{2 + 0.043}{\sqrt{0.802}} = 2.2813. \tag{2.4}$$

By Proposition 2.1(1) and (2.3),

$$\begin{aligned}\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 &= \sum_{i=1}^{n} E\left|\frac{Y_{i,x}-EY_{i,x}}{\sqrt{\operatorname{Var} S_x}}\right|^3\\ &= \frac{1}{(\operatorname{Var} S_x)^{\frac{3}{2}}}\sum_{i=1}^{n} E|Y_{i,x}-EY_{i,x}|^3\\ &\leq \frac{1}{(\operatorname{Var} S_x)^{\frac{3}{2}}}\left(\beta_x+\frac{7\alpha_x}{1+x}\right)\\ &= 1.3923\beta_x + 4.2375\alpha_x. \qquad (2.5)\end{aligned}$$

Note that $\bar{S}_x = \sum_{i=1}^{n} \bar{Y}_{i,x}$ is the sum of independent random variables whose

$$E\bar{Y}_{i,x} = 0 \text{ and } \operatorname{Var} \bar{S}_x = 1.$$

By (2.5) and Theorem 1.1,

$$\begin{aligned}\left|P\left(\bar{S}_x \leq z\right) - \Phi(z)\right| &\leq 0.7915\sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3\\ &\leq 0.7915(1.3923\beta_x + 4.2375\alpha_x)\\ &\leq 1.102\beta_x + 3.354\alpha_x\end{aligned}$$

for all $z \in \mathbb{R}$. From this fact, (2.3) and (2.4), we have

$$\begin{aligned}&\left|P\left(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right|\\ &\qquad\leq \frac{\left(1+\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right)^3(1.102\beta_x+3.354\alpha_x)}{\left(1+\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right)^3}\\ &\qquad\leq \frac{(3.2813)^3(1.102\beta_x+3.354\alpha_x)}{\left(1+\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right)^3}\\ &\qquad\leq \frac{38.933\beta_x+118.495\alpha_x}{(0.957+x)^3}\\ &\qquad\leq \frac{41.195\beta_x}{(1+x)^3}+\frac{125.379\alpha_x}{(1+x)^3}\\ &\qquad\leq \frac{41.195\beta_x}{(1+x)^3}+\frac{54.513\alpha_x}{(1+x)^2}\end{aligned}$$

where we use the fact that

$$\frac{1+x}{0.957+x} \leq 1.019 \text{ for all } 1.3 < x < 2$$

in the fourth inequality.

2. **Case** $2 \leq x < 3$**.**

We can prove the result of this case by using the same argument as 1.

Case $3 \leq x < 7.98$.

To bound $\left|P\left(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right|$ in 1., we used Theorem 1.1.

But in this case, we will use Theorem 1.2.
We note that

$$0 \le \alpha_x \le 0.0125, \qquad 1 \ge \operatorname{Var} S_x \ge 0.975, \tag{2.6}$$

and, by Proposition 2.1(1) of ([14]), $|ES_x| \le 0.00313$.
Then, for $3 \le x \le 7.98$,

$$\frac{1}{1+\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)^3} \le \frac{2.29}{\left(1+\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)^3} \qquad \text{and} \qquad \sum_{i=1}^{n} E|\bar{Y}_{i,x}|^3 \le \ 1.039\beta_x + 1.819\alpha_x.$$

From these facts, (2.6) and Theorem 1.2, we have

$$\begin{aligned}
&\left|P\left(\bar{S}_x \le \frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right| \\
&\le \frac{(31.935)\sum_{i=1}^n E|\bar{Y}_{i,x}|^3}{1+\left(\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)^3} \le \frac{(31.935)(2.29)\sum_{i=1}^n E|\bar{Y}_{i,x}|^3}{\left(1+\frac{x-ES_x}{\sqrt{\operatorname{Var} S_x}}\right)^3} \\
&\le \frac{73.131(1.039\beta_x + 1.818\alpha_x)}{(0.99687+x)^3} \le \frac{(1.0008)^3(75.983\beta_x + 132.952\alpha_x)}{(1+x)^3} \\
&\le \frac{76.17\beta_x}{(1+x)^3} + \frac{133.27\alpha_x}{(1+x)^3} \le \frac{76.17\beta_x}{(1+x)^3} + \frac{33.318\alpha_x}{(1+x)^2},
\end{aligned}$$

where we use the fact that

$$\frac{1+x}{0.99687+x} \le 1.0008 \quad \text{for all} \quad 3 \le x < 7.98$$

in the fourth inequality.
Case $x \ge 7.98$.
We can prove the result of this case by using the same argument as the case $3 \le x < 7.98$. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. It suffices to consider only $x \ge 0$ as we can simply apply the results to $-W_n$ when $x < 0$.
Case 1. $0 \le x < 1.3$.
Note that for $x \ge 0$,

$$EX_i^2 I(|X_i| \ge 1) + E|X_i|^3 I(|X_i| < 1) \le EX_i^2 I(|X_i| \ge 1+x) + E|X_i|^3 I(|X_i| < 1+x)$$

and for $0 \le x \le 1.3$, $(1+x)^3 \le 12.167$.
From these facts and (1.3), we have

$$\begin{aligned}
&|P(W_n \le x) - \Phi(x)| \\
&\quad \le 4.1 \sum_{i=1}^{n} \left\{ EX_i^2 I(|X_i| \ge 1) + E|X_i|^3 I(|X_i| < 1) \right\} \\
&\quad \le 4.1 \sum_{i=1}^{n} \left\{ EX_i^2 I(|X_i| \ge 1+x) + E|X_i|^3 I(|X_i| < 1+x) \right\} \\
&\quad \le \frac{4.1(12.167)}{(1+x)^3} \sum_{i=1}^{n} \left\{ EX_i^2 I(|X_i| \ge 1+x) + E|X_i|^3 I(|X_i| < 1+x) \right\}
\end{aligned}$$

$$\leq 49.89 \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \right\}.$$

Before proving another case, we need the equation

$$|P(W_n \leq x) - \Phi(x)| \leq \frac{4.931\alpha_x}{(1+x)^2} + \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| \tag{2.7}$$

for $\alpha_x \leq 0.11$ and $x \geq 1.3$.
By (2.9) of ([14]), it suffices to show that for $\alpha_x \leq 0.11$ and $x \geq 1.3$,

$$|P(S_x \leq x) - \Phi(x)| \leq \frac{3.319\alpha_x}{(1+x)^2} + \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right|. \tag{2.8}$$

By Proposition 2.1(1) and Proposition 2.1(2), we have

$$\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}} \geq x - ES_x \geq x - \frac{\alpha_x}{(1+x)},$$

which implies

$$\min\left\{x, \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right\} \geq x - \frac{\alpha_x}{1+x}.$$

From this and the fact that

$$\Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{\frac{-t^2}{2}} dt \leq \frac{1}{\sqrt{2\pi} e^{\frac{a^2}{2}}} \int_a^b 1 dt = \frac{(b-a)}{\sqrt{2\pi} e^{\frac{a^2}{2}}}$$

for $0 < a < b$, we have

$$\begin{aligned}
&|P(S_x \leq x) - \Phi(x)| \\
&\leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| + \left| \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi(x) \right| \\
&\leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| \\
&\quad + \frac{1}{\sqrt{2\pi} e^{\frac{1}{2}\left[\min\left(x, \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right]^2}} \left| \frac{x}{\sqrt{\operatorname{Var} S_x}} - x - \frac{ES_x}{\sqrt{\operatorname{Var} S_x}} \right| \\
&\leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) \right| \\
&\quad + \frac{1}{\sqrt{2\pi} e^{\frac{1}{2}\left(x - \frac{\alpha_x}{1+x}\right)^2}} \left| \frac{x}{\sqrt{\operatorname{Var} S_x}} - x - \frac{ES_x}{\sqrt{\operatorname{Var} S_x}} \right|.
\end{aligned} \tag{2.9}$$

Note that for $x \geq 1.3$

$$e^{\frac{x^2}{2}} \geq 0.933(1+x), \qquad e^{\frac{x^2}{2}} \geq 0.193(1+x)^3$$

and

$$e^{\frac{1}{2}(x - \frac{\alpha_x}{1+x})^2} \geq e^{\frac{x^2}{2} - (\frac{x}{1+x})\alpha_x} \geq 0.89 e^{\frac{x^2}{2}}.$$

From these facts, Proposition 2.1(1) , Proposition 2.1(3) and $\alpha_x \leq 0.11$, we have

$$\frac{1}{\sqrt{2\pi} e^{\frac{1}{2}\left(x - \frac{\alpha_x}{1+x}\right)^2}} \left| \frac{x}{\sqrt{\operatorname{Var} S_x}} - x - \frac{ES_x}{\sqrt{\operatorname{Var} S_x}} \right|$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{2\pi}\left(0.89e^{\frac{x^2}{2}}\right)}\left|\frac{x}{\sqrt{\operatorname{Var} S_x}} - x\right| + \frac{1}{\sqrt{2\pi}\left(0.89e^{\frac{x^2}{2}}\right)}\left|\frac{ES_x}{\sqrt{\operatorname{Var} S_x}}\right| \\
&\leq \frac{1.452\alpha_x x}{\sqrt{2\pi}(0.89)(0.193)(1+x)^3} + \frac{\alpha_x}{(1+x)}\frac{(1+1.452\alpha_x)}{\sqrt{2\pi}(0.89)(0.933)(1+x)} \\
&\leq \frac{3.373\alpha_x x}{(1+x)^3} + \frac{0.558\alpha_x}{(1+x)^2} \leq \frac{3.931\alpha_x}{(1+x)^2}.
\end{aligned}$$

From this fact, (2.8) and (2.9), we have (2.7)
Case 2. $1.3 \leq x < 2$.
By the fact that $|P(W_n \leq x) - \Phi(x)| \leq 0.55$ ([3, pp. 246]), we can assume $\frac{\alpha_x}{(1+x)^2} \leq 0.011$, i.e. $\alpha_x \leq 0.099$.
From this fact, (2.7) and Proposition 2.2(1), we have

$$\begin{aligned}
|P(W_n \leq x) - \Phi(x)| &\leq \frac{4.931\alpha_x}{(1+x)^2} + \left|P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\operatorname{Var} S_x}}\right)\right| \\
&\leq \frac{4.931\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} + \frac{54.513\alpha_x}{(1+x)^2} \\
&= \frac{59.444\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} \leq 59.444\delta_x.
\end{aligned}$$

Case 3. $2 \leq x \leq 14$.
Subcase 3.1. $(1+x)^2\alpha_x \geq \frac{1}{5}$.
Using the same argument of subcase 1.1 in Theorem 1.2 of ([14]) and the facts that

$$\frac{1+x}{x} = 1 + \frac{1}{x} \leq 1.5 \text{ and } e^{\frac{x^2}{2}} \geq 0.92x^3 \text{ for } 2 \leq x \leq 14, \tag{2.10}$$

we can show that

$$|P(W_n \leq x) - \Phi(x)| \leq 37.408\delta_x.$$

Subcase 3.2. $(1+x)^2\alpha_x < \frac{1}{5}$.
Note that for $x \geq 2$, we have

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq 0.023 \leq 0.11.$$

By (2.7) and Proposition 2.2(2), we obtain the required bounds.
Case 4. $x > 14$.
Follows the argument of case 3 on replacing the inequalities

$$e^{\frac{x^2}{2}} \geq 60x^3 \text{ and } \quad \frac{1+x}{x} = 1 + \frac{1}{x} \leq 1.071$$

in (2.10). □

Proof of Corollary 1.4. If $0 \leq x < 1.3$.
We used the same argument as case 1 of Theorem 1.3 and the fact that $(1 + \frac{x}{4})^3 \leq 2.327$ to get $C = 9.54$.
Suppose that $x \geq 1.3$. By the fact that

$$\delta_x \leq \left(\frac{1+\frac{x}{4}}{1+x}\right)^2 \delta_{\frac{x}{4}},$$

we have

$$\delta_x \le \begin{cases} 0.332\delta_{\frac{x}{4}} & \text{if} \quad 1.3 \le x < 2, \\ 0.250\delta_{\frac{x}{4}} & \text{if} \quad 2 \le x < 3, \\ 0.192\delta_{\frac{x}{4}} & \text{if} \quad 3 \le x < 7.98, \\ 0.112\delta_{\frac{x}{4}} & \text{if} \quad 7.98 \le x < 14, \\ 0.090\delta_{\frac{x}{4}} & \text{if} \quad x \ge 14. \end{cases}$$

Then Corollary 1.4 follows from this fact and Theorem 1.3.

□

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