

GENERALIZATIONS OF SOME NEW ČEBYŠEV TYPE INEQUALITIES

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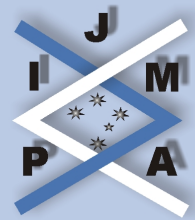
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Abstract: We provide generalizations of some recently published Čebyšev type inequalities.

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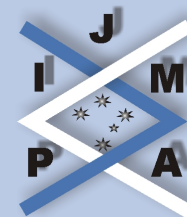
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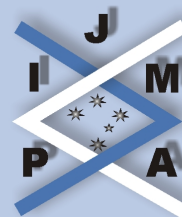
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1. Introduction

In a recent paper [1], B.G. Pachpatte proved the following Čebyšev type inequalities:

Theorem 1.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions on $[a, b]$ with $f', g' \in L_2[a, b]$, then,*

$$(1.1) \quad |P(F, G, f, g)| \leq \frac{(b-a)^2}{12} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \\ \times \left[\frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}},$$

$$(1.2) \quad |P(A, B, f, g)| \leq \frac{(b-a)^2}{12} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \\ \times \left[\frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}},$$

where

$$(1.3) \quad P(\alpha, \beta, f, g) = \alpha\beta - \frac{1}{b-a} \left(\alpha \int_a^b g(t) dt + \beta \int_a^b f(t) dt \right) \\ + \left(\frac{1}{b-a} \int_a^b f(t) dt \right) \left(\frac{1}{b-a} \int_a^b g(t) dt \right),$$

$$(1.4) \quad [f; a, b] = \frac{f(b) - f(a)}{b - a},$$

$$F = \frac{f(a) + f(b)}{2}, \quad G = \frac{g(a) + g(b)}{2}, \quad A = f\left(\frac{a+b}{2}\right), \quad B = g\left(\frac{a+b}{2}\right),$$

and

$$\|f\|_2 := \left[\int_a^b f^2(t) dt \right]^{\frac{1}{2}}.$$

Theorem 1.2. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be differentiable functions so that f', g' are absolutely continuous on $[a, b]$, then,

$$(1.5) \quad |P(\overline{F}, \overline{G}, f, g)| \leq \frac{(b-a)^4}{144} \|f'' - [f'; a, b]\|_\infty \|g'' - [g'; a, b]\|_\infty,$$

where

$$\overline{F} = \frac{f(a) + f(b)}{2} - \frac{(b-a)^2}{12} [f'; a, b],$$

$$\overline{G} = \frac{g(a) + g(b)}{2} - \frac{(b-a)^2}{12} [g'; a, b],$$

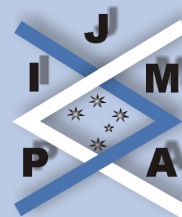
$P(\alpha, \beta, f, g)$ and $[f; a, b]$ are as defined in (1.3) and (1.4), and

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)| < \infty.$$

In [2], B.G. Pachpatte presented an additional Čebyšev type inequality given in Theorem 1.3 below.

Theorem 1.3. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$, then we have,

$$(1.6) \quad |P(C, D, f, g)| \leq \frac{1}{(b-a)^2} M^{\frac{2}{q}} \|f'\|_p \|g'\|_p,$$



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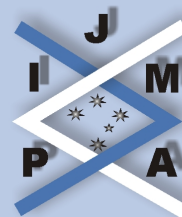
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where $P(\alpha, \beta, f, g)$ is as defined in (1.3),

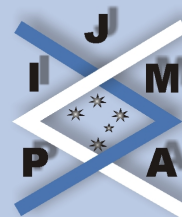
$$C = \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{a+b}{2} \right) \right],$$
$$D = \frac{1}{3} \left[\frac{g(a) + g(b)}{2} + 2g \left(\frac{a+b}{2} \right) \right],$$

$$(1.7) \quad M = \frac{(2^{q+1} + 1)(b-a)^{q+1}}{3(q+1)6^q}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

In this paper, we provide some generalizations of the above three theorems.



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2. Statement of Results

We use the following notation to simplify the detail of presentation. For suitable functions $f, g : [a, b] \rightarrow \mathbb{R}$ and real number $\theta \in [0, 1]$ we set,

$$\Gamma_{\theta} = \frac{\theta}{2}[f(a) + f(b)] + (1 - \theta)f\left(\frac{a + b}{2}\right),$$

$$\Delta_{\theta} = \frac{\theta}{2}[g(a) + g(b)] + (1 - \theta)g\left(\frac{a + b}{2}\right),$$

$$\bar{\Gamma}_{\theta} = \Gamma_{\theta} + \frac{(1 - 3\theta)(b - a)^2}{24}[f', a, b],$$

$$\bar{\Delta}_{\theta} = \Delta_{\theta} + \frac{(1 - 3\theta)(b - a)^2}{24}[g', a, b],$$

where $[f; a, b]$ is as defined in (1.4).

We also use $P(\alpha, \beta, f, g)$ as defined in (1.3), where α and β are real constants.

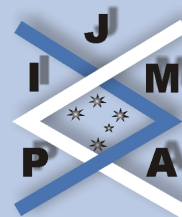
The results are stated as Theorems 2.1, 2.2 and 2.3.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold, then for any $\theta \in [0, 1]$,*

$$(2.1) \quad |P(\Gamma_{\theta}, \Delta_{\theta}, f, g)| \leq \frac{(b - a)^2}{12}[\theta^3 + (1 - \theta)^3] \\ \times \left[\frac{1}{b - a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \left[\frac{1}{b - a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}.$$

Theorem 2.2. *Let the assumptions of Theorem 1.2 hold, then for any $\theta \in [0, 1]$,*

$$(2.2) \quad |P(\bar{\Gamma}_{\theta}, \bar{\Delta}_{\theta}, f, g)| \leq (b - a)^4 I^2(\theta) \|f'' - [f'; a, b]\|_{\infty} \|g'' - [g'; a, b]\|_{\infty},$$



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where

$$(2.3) \quad I(\theta) = \begin{cases} \frac{\theta^3}{3} - \frac{\theta}{8} + \frac{1}{24}, & 0 \leq \theta \leq \frac{1}{2}, \\ \frac{1}{8}(\theta - \frac{1}{3}), & \frac{1}{2} < \theta \leq 1. \end{cases}$$

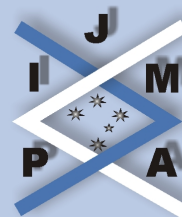
Theorem 2.3. *Let the assumptions of Theorem 1.3 hold, then for any $\theta \in [0, 1]$,*

$$(2.4) \quad |P(\Gamma_\theta, \Delta_\theta, f, g)| \leq \frac{1}{(b-a)^2} M_\theta^{\frac{2}{q}} \|f'\|_p \|g'\|_p,$$

where

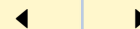
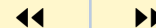
$$(2.5) \quad M_\theta = \frac{\theta^{q+1} + (1-\theta)^{q+1}}{(q+1)2^q} (b-a)^{q+1},$$

and $\frac{1}{p} + \frac{1}{q} = 1$.



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3. Proof of Theorem 2.1

Define the function,

$$(3.1) \quad K(\theta, t) = \begin{cases} t - (a + \theta \frac{b-a}{2}), & t \in [a, \frac{a+b}{2}], \\ t - (b - \theta \frac{b-a}{2}), & t \in (\frac{a+b}{2}, b], \end{cases}$$

and we obtain the following identities:

$$(3.2) \quad \Gamma_{\theta} - \frac{1}{b-a} \int_a^b f(t) dt = O(f; a, b; \theta),$$

$$(3.3) \quad \Delta_{\theta} - \frac{1}{b-a} \int_a^b g(t) dt = O(g; a, b; \theta),$$

where

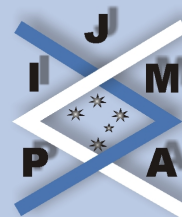
$$O(f; a, b; \theta) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(k(\theta, t) - k(\theta, s)) dt ds.$$

Multiplying the left sides and right sides of (3.2) and (3.3) we get,

$$(3.4) \quad P(\Gamma_{\theta}, \Delta_{\theta}, f, g) = O(f; a, b; \theta)O(g; a, b; \theta).$$

From (3.4),

$$(3.5) \quad |P(\Gamma_{\theta}, \Delta_{\theta}, f, g)| = |O(f; a, b; \theta)||O(g; a, b; \theta)|.$$



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Using the Cauchy-Schwarz inequality for double integrals,

$$(3.6) \quad |O(f; a, b; \theta)| \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f'(t) - f'(s)| |k(\theta, t) - k(\theta, s)| dt ds \\ \leq \left[\frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 dt ds \right]^{\frac{1}{2}} \\ \times \left[\frac{1}{2(b-a)^2} \int_a^b \int_a^b (k(\theta, t) - k(\theta, s))^2 dt ds \right]^{\frac{1}{2}}.$$

By simple computation,

$$(3.7) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 dt ds \\ = \frac{1}{b-a} \int_a^b (f'(t))^2 dt - \left(\frac{1}{b-a} \int_a^b f'(t) dt \right)^2,$$

and

$$(3.8) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (k(\theta, t) - K(\theta, s))^2 dt ds = \frac{(b-a)^2}{12} [\theta^3 + (1-\theta)^3].$$

Using (3.7), (3.8) in (3.6),

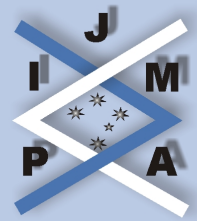
$$(3.9) \quad |O(f; a, b; \theta)| \leq \frac{b-a}{2\sqrt{3}} [\theta^3 + (1-\theta)^3]^{\frac{1}{2}} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}}.$$

Similarly,

$$(3.10) \quad |O(g; a, b; \theta)| \leq \frac{b-a}{2\sqrt{3}} [\theta^3 + (1-\theta)^3]^{\frac{1}{2}} \left[\frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}.$$

Using (3.9) and (3.10) in (3.5), (2.1) follows.

Remark 1. If $\theta = 1$ and $\theta = 0$ in (2.1), the inequalities (1.1) and (1.2) are recaptured. Thus Theorem 2.1 may be regarded as a generalization of Theorem 1.1.



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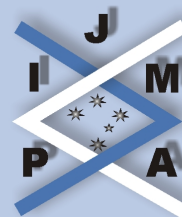
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4. Proof of Theorem 2.2

Define the function

$$L(\theta, t) = \begin{cases} \frac{1}{2}(t-a)[t - (1-\theta)a - \theta b], & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(t-b)[t - \theta a - (1-\theta)b], & t \in (\frac{a+b}{2}, b]. \end{cases}$$

It is not difficult to find the following identities:

$$(4.1) \quad \frac{1}{b-a} \int_a^b f(t) dt - \bar{\Gamma}_\theta = Q(f', f''; a, b),$$

$$(4.2) \quad \frac{1}{b-a} \int_a^b g(t) dt - \bar{\Delta}_\theta = Q(g', g''; a, b),$$

where

$$Q(f', f''; a, b) = \frac{1}{b-a} \int_a^b L(\theta, t) \{f''(t) - [f'; a, b]\} dt.$$

Multiplying the left sides and right sides of (4.1) and (4.2), we get,

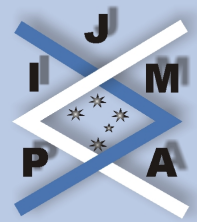
$$(4.3) \quad P(\bar{\Gamma}_\theta, \bar{\Delta}_\theta, f, g) = Q(f', f''; a, b)Q(g', g''; a, b).$$

From (4.3),

$$(4.4) \quad |P(\bar{\Gamma}_\theta, \bar{\Delta}_\theta, f, g)| = |Q(f', f''; a, b)||Q(g', g''; a, b)|.$$

By simple computation, we have,

$$(4.5) \quad |Q(f', f''; a, b)| \leq \frac{1}{b-a} \int_a^b |L(\theta, t)| |f''(t) - [f'; a, b]| dt \\ \leq \frac{1}{b-a} \|f''(t) - [f'; a, b]\|_\infty \int_a^b |L(\theta, t)| dt,$$



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and similarly,

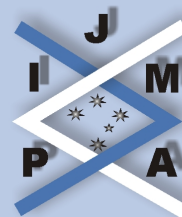
$$(4.6) \quad |Q(f', f''; a, b)| \leq \frac{1}{b-a} \|f''(t) - [f'; a, b]\|_\infty \int_a^b |L(\theta, t)| dt,$$

where

$$(4.7) \quad \int_a^b |L(\theta, t)| dt = (b-a)^3 \times \begin{cases} \frac{\theta^3}{3} - \frac{\theta}{8} + \frac{1}{24}, & 0 \leq \theta \leq \frac{1}{2}, \\ \frac{1}{8}(\theta - \frac{1}{3}), & \frac{1}{2} < \theta \leq 1. \end{cases}$$

Consequently, the inequalities (2.2) and (2.3) follow from (4.4) – (4.7).

Remark 2. If $\theta = 1$ in (2.2) with (2.3), the inequality (1.5) is recaptured. Thus Theorem 2.2 may be regarded as a generalization of Theorem 1.2.

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5. Proof of Theorem 2.3

From (3.1), we can also find the following identities:

$$(5.1) \quad \Gamma_{\theta} - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b K(\theta, t) f'(t) dt,$$

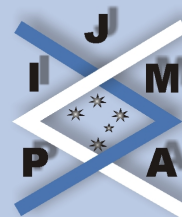
$$(5.2) \quad \Delta_{\theta} - \frac{1}{b-a} \int_a^b g(t) dt = \frac{1}{b-a} \int_a^b K(\theta, t) g'(t) dt.$$

Multiplying the left sides and right sides of (5.1) and (5.2) we get,

$$(5.3) \quad P(\Gamma_{\theta}, \Delta_{\theta}, f, g) = \frac{1}{(b-a)^2} \left(\int_a^b k(\theta, t) f'(t) dt \right) \left(\int_a^b k(\theta, t) g'(t) dt \right).$$

From (5.3) and using the properties of modulus and Hölder's integral inequality, we have,

$$(5.4) \quad \begin{aligned} & |P(\Gamma_{\theta}, \Delta_{\theta}, f, g)| \\ & \leq \frac{1}{(b-a)^2} \left(\int_a^b |k(\theta, t)| |f'(t)| dt \right) \left(\int_a^b |k(\theta, t)| |g'(t)| dt \right) \\ & \leq \frac{1}{(b-a)^2} \left[\left(\int_a^b |k(\theta, t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b |f'|^p dt \right)^{\frac{1}{p}} \right] \\ & \quad \times \left[\left(\int_a^b |k(\theta, t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b |g'|^p dt \right)^{\frac{1}{p}} \right] \\ & = \frac{1}{(b-a)^2} \left(\int_a^b |k(\theta, t)|^q dt \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p. \end{aligned}$$

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A simple computation gives,

$$\begin{aligned}(5.5) \quad & \int_a^b |k(\theta, t)|^q dt \\ &= \int_a^{\frac{a+b}{2}} \left| t - \left(a + \theta \frac{b-a}{2} \right) \right|^q dt + \int_{\frac{a+b}{2}}^b \left| t - \left(b - \theta \frac{b-a}{2} \right) \right|^q dt \\ &= \int_a^{a+\theta \frac{b-a}{2}} \left(a + \theta \frac{b-a}{2} - t \right)^q dt + \int_{a+\theta \frac{b-a}{2}}^{\frac{a+b}{2}} \left(t - a - \theta \frac{b-a}{2} \right)^q dt \\ &\quad + \int_{\frac{a+b}{2}}^{b-\theta \frac{b-a}{2}} \left(b - \theta \frac{b-a}{2} - t \right)^q dt + \int_{b-\theta \frac{b-a}{2}}^b \left(t - b + \theta \frac{b-a}{2} \right)^q dt \\ &= \frac{2}{q+1} \left[\left(\frac{\theta}{2} \right)^{q+1} (b-a)^{q+1} + \left(\frac{1-\theta}{2} \right)^{q+1} (b-a)^{q+1} \right] \\ &= \frac{\theta^{q+1} + (1-\theta)^{q+1}}{(q+1)2^q} (b-a)^{q+1} = M_\theta.\end{aligned}$$

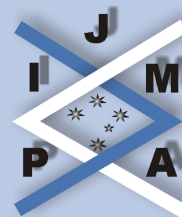
Consequently, the inequality (2.4) with (2.5) follow from (5.4) and (5.5).

Remark 3. If we take $\theta = \frac{1}{3}$ in (2.4) with (2.5), we recapture the inequality (1.6) with (1.7). Thus Theorem 2.3 may be regarded as a generalization of Theorem 1.3.

Remark 4. If we take $p = 2$ in Theorem 2.3, and replace $f(t)$ and $g(t)$ by $f(t) - [f; a, b]t$ and $g(t) - [g; a, b]t$ in (2.4), respectively, then inequality (2.1) is recaptured.

References

- [1] B.G. PACHPATTE, New Čebyšev type inequalities via trapezoidal-like rules, *J. Inequal. Pure and Appl. Math.*, **7**(1) (2006), Art. 31. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=637>].
- [2] B.G. PACHPATTE, On Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces, *J. Inequal. Pure and Appl. Math.*, **7**(2) (2006), Art. 58. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=675>].



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