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MAXIMIZATION FOR INNER PRODUCTS UNDER QUASI-MONOTONE CONSTRAINTS

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Abstract

This paper studies optimization for inner products of real vectors assuming monotonicity properties for the entries in one of the vectors. Resulting inequalities have been useful recently in bounding reciprocals of power series with rapidly decaying coefficients and in proving that all symmetric Toeplitz matrices generated by monotone convex sequences have off-diagonal decay preserved through triangular decompositions. An example of an application of the theory to global optimization for inner products is also provided.

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Contents

1. Introduction

This paper studies inequalities for inner products of real vectors assuming monotonicity and boundedness properties for the entries in one of the vectors. In particular, for $r \in (0,1]$, we consider inner products $\boldsymbol{p} \cdot \boldsymbol{q}$, for vectors $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, satisfying $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$, $p_i \in [A, B]$ for $1 \le i \le n$, and one of the following properties

1. (r-quasi-monotonicity) $p_{i+1} \ge r p_i$ for $1 \le i \le n-1$.
2. (r-geometric monotonicity) $p_{i+1} \ge \frac{1}{r} p_i$ for $1 \le i \le n-1$.
3. (monotonicity) $p_{i+1} \ge p_i$ for $1 \le i \le n-1$.

For discussion of various classes of sequences of monotone type, see for instance, Kijima [12], and Leindler [15, 14].

Our method involves, for each of the three cases mentioned, obtaining *finite* sets $\mathcal{P}_n = \mathcal{P}_n(A, B, r)$ such that

$$\min\{\boldsymbol{v} \cdot \boldsymbol{q} : \boldsymbol{v} \in \mathcal{P}_n\} \le \boldsymbol{p} \cdot \boldsymbol{q} \le \max\{\boldsymbol{v} \cdot \boldsymbol{q} : \boldsymbol{v} \in \mathcal{P}_n\},$$

for all $\boldsymbol{p}$ satisfying the respective monotonicity assumption, above.

The paper proceeds as follows. In Section 2, we consider obtaining the sets $\mathcal{P}_n$ corresponding to Property (1), above. An application to linear recurrences, which has been useful in the recent literature is also given. In Section 3, we consider the case of r-geometric monotonicity. The paper includes examples which provide an application of the theory to global optimization for inner products, for a specific vector $\boldsymbol{q}$.

2. The Case of r-quasi-monotonicity

In this section we consider the assumption of r-*quasi-monotonicity* of the entries in $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ (as defined in (1), above), i.e.

$$p_{i+1} \geq rp_i \tag{2.1}$$

for $1 \leq i \leq n-1$. The motivation for consideration of such a condition arose in a probability related context of investigating a monotone sequence $\{q_i\}$ with a geometric bound, i.e.

$$q_i \leq Ar^i$$

where $A > 0$ and $r < 1$ (see [2]). In this case the sequence $\{\phi_i\}$ defined by $\phi_i = \frac{q_i}{r^i}$, satisfies $0 \leq \phi_i \leq A$, and

$$\phi_i = \frac{q_i}{r^i} \geq \frac{q_{i+1}}{r^i} = \phi_{i+1}r.$$

For a given vector $\boldsymbol{t} = (t_0, t_1, t_2, \dots, t_k)$ satisfying $t_0 \geq 0$, $t_i \geq 1$ for $1 \leq i \leq k$ and

$$\sum_i t_i = N, \tag{2.2}$$

define the vector $\boldsymbol{v_t}$ via

$$\boldsymbol{v_t} \overset{def}{=} A\big(\overbrace{0, 0, \dots, 0}^{t_0}; r^0, r^1, \dots, r^{t_1-1}; \\ r^0, r^1, \dots, r^{t_2-1}; r^0, r^1, \dots, r^{t_k-1}\big). \tag{2.3}$$

In addition, define the set of vectors

$$\mathcal{P}_N = \mathcal{P}_N(A, 0, r) = \{\boldsymbol{v_t} : \boldsymbol{t} \text{ satisfies } (2.2)\ \}. \tag{2.4}$$

We have the following result regarding inner products.

Theorem 2.1. *Suppose that* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *are* n*-vectors where* $\boldsymbol{p}$ *satisfies (2.1), for* $1 \le i \le n-1$ *and* $0 \le p_i \le A$ *for* $1 \le i \le n$*. We have,*

$$\min\{\boldsymbol{w} \cdot \boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n\} \le \boldsymbol{p} \cdot \boldsymbol{q} \le \max\{\boldsymbol{w} \cdot \boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n\} \tag{2.5}$$

where $\boldsymbol{p} \cdot \boldsymbol{q}$ *denotes the standard dot product* $\sum_{i=1}^{n} p_i q_i$*.*

The value in Theorem 2.1 lies in the fact that for any given n, $\mathcal{P}_n$ is a finite set.

For a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, we will use the notation $\boldsymbol{p}^{i,j}$ to indicate the vector consisting of the i^{th} through j^{th} entries in $\boldsymbol{p}$, i.e.

$$\boldsymbol{p}^{i,j} = (p_i, p_{i+1}, \ldots, p_j) \tag{2.6}$$

Proof of Theorem 2.1. First, suppose $\boldsymbol{p} \cdot \boldsymbol{q} > 0$, and note that the lower bound in (2.5), for such vectors, follows from the fact that $\boldsymbol{v_t} = \boldsymbol{0}$ for $\boldsymbol{t} = (n, 0, \ldots, 0)$. We will obtain a sequence of vectors $\{\widetilde{\boldsymbol{p}}_i\}_{i=1}^{n+1}$, satisfying

$$0 \le \boldsymbol{p} \cdot \boldsymbol{q} = \widetilde{\boldsymbol{p}}_{n+1} \cdot \boldsymbol{q} \le \widetilde{\boldsymbol{p}}_n \cdot \boldsymbol{q} \le \cdots \le \widetilde{\boldsymbol{p}}_1 \cdot \boldsymbol{q},$$

such that $\widetilde{\boldsymbol{p}}_1 \in \mathcal{P}_n$.

In particular, consider the vectors $\widetilde{\boldsymbol{p}}_i = (\widetilde{\boldsymbol{p}}_i(1), \widetilde{\boldsymbol{p}}_i(2), \ldots, \widetilde{\boldsymbol{p}}_i(n)) \in \mathbb{R}^n$, $i = 1, \ldots, n+1$ defined recursively according to the following scheme.

1. $\widetilde{\boldsymbol{p}}_{n+1} = \boldsymbol{p}$.
2. For $1 \le i \le n$, set

$$S_i = \{s : i+1 \le s \le n \quad \text{and} \quad \widetilde{\boldsymbol{p}}_{i+1}(s) = A\}, \quad \text{and}$$

$$v_i = \min\left(S_i \bigcup \{n+1\}\right).$$

3. For $1 \le i \le n$, define $\widetilde{\boldsymbol{p}}_i$ (a function of $\widetilde{\boldsymbol{p}}_{i+1}$) via

$$\widetilde{\boldsymbol{p}}_i = \Big(\widetilde{\boldsymbol{p}}_{i+1}(1), \widetilde{\boldsymbol{p}}_{i+1}(2), \dots, \widetilde{\boldsymbol{p}}_{i+1}(i-1), c_i\widetilde{\boldsymbol{p}}_{i+1}(i), c_i\widetilde{\boldsymbol{p}}_{i+1}(i+1),$$
$$\dots, c_i\widetilde{\boldsymbol{p}}_{i+1}(v_i - 1), A, \widetilde{\boldsymbol{p}}_{i+1}(v_i+1), \dots, \widetilde{\boldsymbol{p}}_{i+1}(n)\Big)$$

$$= (\boldsymbol{w}^1_{i+1}; c_i\boldsymbol{w}^2_{i+1}; \boldsymbol{w}^3_{i+1}), \tag{2.7}$$

say, where c_i is given by

$$c_i = \begin{cases} \dfrac{A}{p_i}, & \text{if } \boldsymbol{w}^2_{i+1} \cdot \boldsymbol{q}^{i,v_i-1} > 0 \\ \dfrac{rp_{i-1}}{p_i}, & \text{if } \boldsymbol{w}^2_{i+1} \cdot \boldsymbol{q}^{i,v_i-1} \le 0 \text{ and } i > 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.8}$$

Note that $\widetilde{\boldsymbol{p}}_{i+1} = (\boldsymbol{w}^1_{i+1}, \boldsymbol{w}^2_{i+1}, \boldsymbol{w}^3_{i+1})$.

It is not difficult to verify by induction that $\boldsymbol{w}^j_{i+1}$, $j = 1, 2, 3$, are of the form

$$\boldsymbol{w}^1_{i+1} = \widetilde{\boldsymbol{p}}^{1,i-1}_{i+1} = (p_1, p_2, \dots, p_{i-1}) \tag{2.9}$$

$$\boldsymbol{w}^2_{i+1} = \widetilde{\boldsymbol{p}}^{i,v_i-1}_{i+1} = (p_i, rp_i, r^2p_i, \dots, r^{v_i-i-1}p_i) \tag{2.10}$$

$$\boldsymbol{w}^3_{i+1} = \widetilde{\boldsymbol{p}}^{v_i,n}_{i+1} \in \mathcal{P}_{n-v_i+1}, \tag{2.11}$$

We have that (2.7) and (2.8) imply

$$\widetilde{\boldsymbol{p}}_i \cdot \boldsymbol{q} - \widetilde{\boldsymbol{p}}_{i+1} \cdot \boldsymbol{q} = (c_i - 1)(\boldsymbol{w}_{i+1}^2 \cdot \boldsymbol{q}^{n-i,v_i-1}) \geq 0,$$

and, for $1 \leq i \leq n+1$,

$$\begin{aligned}\widetilde{\boldsymbol{p}}_i \in \big\{&(p_1, p_2, \dots, p_{i-1}, rp_{i-1}, r^2p_{i-1}, r^3p_{i-1}, \dots, r^{v_i-i}p_{i-1}; \boldsymbol{w}_{i+1}^3),\\ &(p_1, p_2, \dots, p_{i-1}, A, rA, r^2A, \dots, r^{v_i-i-1}A; \boldsymbol{w}_{i+1}^3)\big\}.\end{aligned} \tag{2.12}$$

Thus $v_{i-1} \in \{v_i, i\}$, and in particular, for $i = 2$, we have

$$\begin{aligned}\widetilde{\boldsymbol{p}}_2 \in \big\{&(p_1, rp_1, r^2p_1, r^3p_1, \dots, r^{v_2-2}p_1; \boldsymbol{w}_3^3),\\ &(p_1, A, rA, r^2A, \dots, r^{v_2-3}A; \boldsymbol{w}_3^3)\big\}.\end{aligned} \tag{2.13}$$

The vector $\widetilde{\boldsymbol{p}}_1$ then satisfies

$$\begin{aligned}\widetilde{\boldsymbol{p}}_1 \in \big\{&(0, 0, \dots, 0; \boldsymbol{w}_3^3), (A, rA, r^2A, r^3A, \dots, r^{v_2-2}A; \boldsymbol{w}_3^3),\\ &(A, A, rA, r^2A, \dots, r^{v_2-3}A; \boldsymbol{w}_3^3),\\ &(0, A, rA, r^2A, \dots, r^{v_2-3}A; \boldsymbol{w}_3^3)\big\} \subset \mathcal{P}_n\end{aligned} \tag{2.14}$$

and the theorem is proven in this case. The proof follows similarly, if $\boldsymbol{p} \cdot \boldsymbol{q} \leq 0$, and the proof of the theorem is complete. □

The following example provides an application of Theorem 2.1 to global optimization for inner products, for a specific vector $\boldsymbol{q}$.

Example 2.1. *Consider the vector* $q \in \mathbb{R}^{15}$ *given by*

$$\boldsymbol{q} = \big(0.4361725, 0.6454718, 2.0226176, \\ -4.1395363, 0.9749134, 4.3806500, -4.0035597, \\ 0.6773984, -3.7420053, -2.7051776, 3.8209032, \\ 0.6327872, 1.4719490, 1.2277661, 4.1026365\big). \tag{2.15}$$

The entries in q are depicted in Figure 1. Now, consider optimizing $\boldsymbol{p} \cdot \boldsymbol{q}$ over all $\boldsymbol{p} = (p_1, p_2, \ldots, p_{15}) \in \mathbb{R}^{15}$, satisfying $0 \le p_i \le 1$ and (2.1) for some $0 < r \le 1$. Theorem 2.1 implies that we need only compute and compare inner products with $\boldsymbol{q}$ over the finite set $\mathcal{P}_{15}(1, 0, r)$ as given in (2.4).

The results of the computations for $r \in \{.1, .3, .7, .9\}$, are given in Figure 2.

It is possible to apply Theorem 2.1 in sequence to obtain bounds for linear recurrences, as is shown by the following theorem.

Theorem 2.2. *Suppose that* $\{b_i\}$ *and* $\{\alpha_{i,j}\}$ *satisfy*

$$b_n = \sum_{k=0}^{n-1} (-\alpha_{n,k}) b_k, \qquad n \ge 1, \tag{2.16}$$

where $b_0 = 1$ *and*

$$\alpha_{n,k} \in [0, A], \tag{2.17}$$

for $0 \le k \le n-1$ *and* $n \ge 1$, *and*

$$r\alpha_{n,k} \le \alpha_{n,k+1}. \tag{2.18}$$

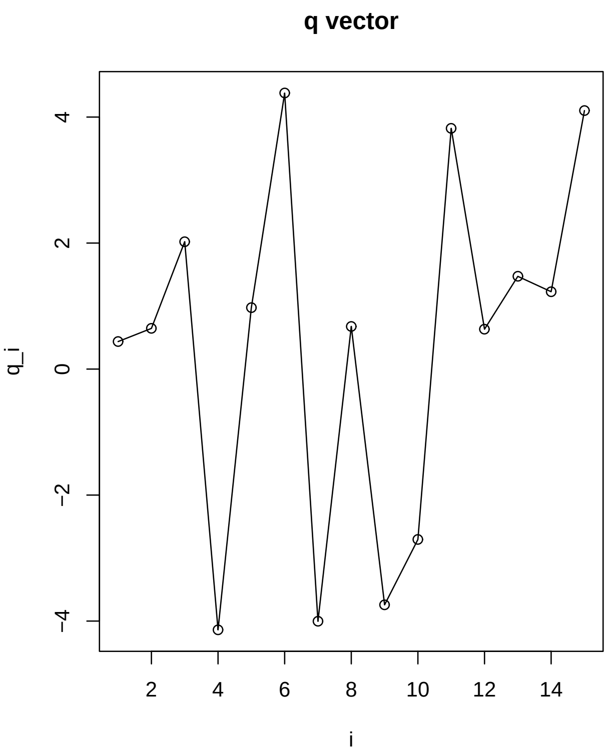

Figure 1: The vector q in (2.15).

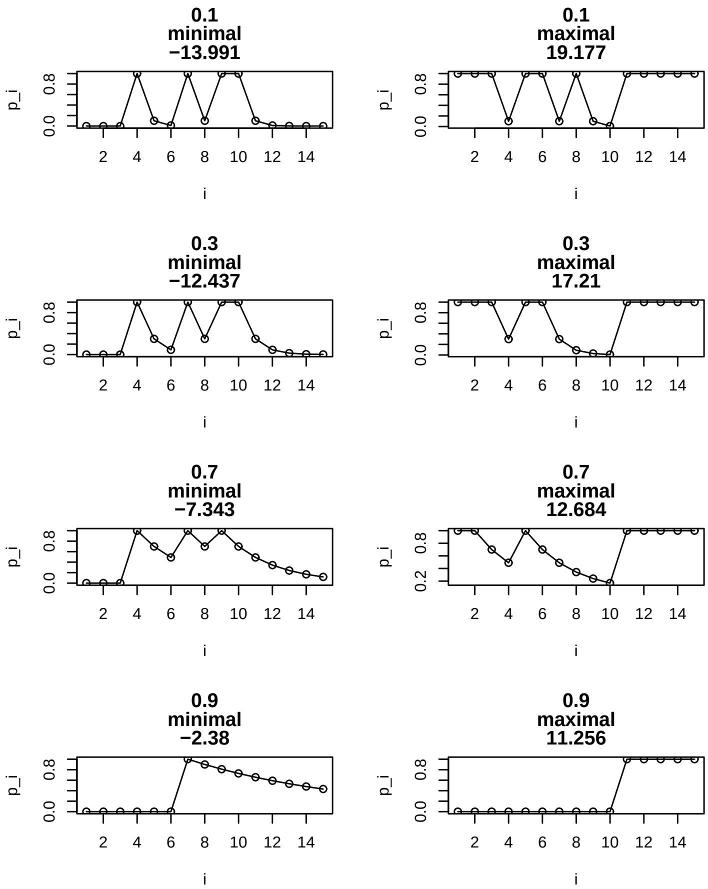

Figure 2: Maximal and minimal values for inner products under the constraint in (2.1)

Then, there exist $\{b_i'\}$ and $\{\alpha_{i,j}'\}$ such that

$$|b_n| \leq |b_n'|$$

and

$$b_n' = \sum_{k=0}^{n-1} (-\alpha_{n,k}') b_k', \qquad n \geq 1, \tag{2.19}$$

with each vector

$$\boldsymbol{\alpha}_i' = (\alpha_{i,0}', \alpha_{i,1}', \ldots, \alpha_{i,i-1}') \in \mathcal{P}_i,$$

for $1 \leq i \leq n$, where $\mathcal{P}_i$ is as in (2.4).

In fact, there exists a set $\{\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \ldots, \boldsymbol{\alpha}_n'\}$, with $\boldsymbol{\alpha}_i' \in \mathcal{P}_i$, such that b_i' is as large as possible (with its inherent sign) given $b_0, b_1', b_2', \ldots, b_{i-1}'$.

Remark 1. *While Theorem 2.2 looks relatively simple, it has proven indispensable recently in two quite unrelated interesting contexts. The theorem was crucial, in proving a recent optimal explicit form of Kendall's Renewal Theorem (see Berenhaut, Allen and Fraser [2]) stemming from bounds on reciprocals of power series with rapidly decaying coefficients. In a quite unrelated context, a simpler version of Theorem 2.2 was also employed in Berenhaut and Bandyopadhyay [3] in proving that all symmetric Toeplitz matrices generated by monotone convex sequences have off-diagonal decay preserved through triangular decompositions.*

Proof of Theorem 2.2. The proof, here, involves applying Theorem 2.1 to successively “scale" the rows of the coefficient matrix

$$[-\alpha_{i,j}] = \begin{bmatrix} -\alpha_{1,0} & 0 & \cdots & 0 \\ -\alpha_{2,0} & -\alpha_{2,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -\alpha_{n,0} & -\alpha_{n,1} & \cdots & -\alpha_{n,n-1} \end{bmatrix}.$$

while not decreasing the value of $|b_n|$ at any step.

First, define the sequences

$$\bar{\boldsymbol{\alpha}}_i = (\alpha_{i,0}, \dots, \alpha_{i,i-1}) \qquad \text{and}$$
$$\boldsymbol{b}^{k,j} = (b_k, \dots, b_j),$$

for $0 \le k \le j \le n-1$ and $1 \le i \le n$.

Suppose that $b_n > 0$. Expanding via (2.16), b_n can be written as

$$b_n = C_1^0 b_0 + C_1^1 b_1, \tag{2.20}$$

where C_1^0 and C_1^1 are constants, which depend on $\{\alpha_{i,j}\}$. If $C_1^1 > 0$, then select $\bar{\boldsymbol{\alpha}}_1' = (\alpha_{1,0}') \in \mathcal{P}_1$ so that $-\bar{\boldsymbol{\alpha}}_1' \cdot \boldsymbol{b}^{0,0}$ is maximal, via Theorem 2.1. Similarly, if $C_1^1 < 0$, select $\bar{\boldsymbol{\alpha}}_1' = (\alpha_{1,0}') \in \mathcal{P}_1$ so that $-\bar{\boldsymbol{\alpha}}_1' \cdot \boldsymbol{b}^{0,0}$ is minimal. In either case, replacing $\alpha_{1,0}$ by $\alpha_{1,0}'$ in (2.16) will result in a larger (or equal) value for $C_1^1 b_1$, and in turn, referring to (2.20), a larger (or equal) value of $|b_n|$.

Now, suppose that the first through $(k-1)^{th}$ rows of the α matrix are of the form described in the theorem (i.e. resulting in maximal b_i values for $1 \le i \le$

$k-1$ with respect to the preceeding b_j, $0 \leq j \leq i-1$), and express b_n in the form

$$b_n = C_k^0 b_0 + C_k^1 b_1 + \cdots + C_k^k b_k, \tag{2.21}$$

via (2.16). If $C_k^k \geq 0$, then select $\bar{\boldsymbol{\alpha}}'_k \in \mathcal{P}_k$ so that $-\bar{\boldsymbol{\alpha}}'_k \cdot \boldsymbol{b}^{0,k-1}$ is maximal, via Theorem 2.1. Similarly, if $C_k^k < 0$, select $\bar{\boldsymbol{\alpha}}'_k \in \mathcal{P}_k$ so that $-\bar{\boldsymbol{\alpha}}'_k \cdot \boldsymbol{b}^{0,0}$ is minimal. In either case, referring to (2.21), replacing the values in $\bar{\boldsymbol{\alpha}}_k$ by those in $\bar{\boldsymbol{\alpha}}'_k$ in (2.16) will not decrease the value of $|b_n|$. The result follows by induction for this case. The case $b_n < 0$ is similar and the theorem is proven. □

For further results along these lines in the case $r = 1$ and $B = 0$, see [4].

Note that, recurrences with varying or random coefficients have been studied by many previous authors. For a partial survey of such literature see Viswanath [22] and [23], Viswanath and Trefethen [24], Embree and Trefethen [10], Wright and Trefethen [26], Mallik [16], Popenda [18], Kittapa [13], Odlyzko [17], Berenhaut and Goedhart [6, 7], Berenhaut and Morton [9], Berenhaut and Foley [5], and Stević [19, 20, 21] (and the references therein). For a comprehensive treatment of difference equations and inequalities, c.f. Agarwal [1].

We now turn to consideration of the remaining cases of r-geometric decay and monotonicity mentioned in the introduction.

3. The Case of r-geometric Monotonicity

In this section we consider the assumption of r-geometric monotonicity of the entries in $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, i.e.

$$p_{i+1} \geq \frac{1}{r} p_i$$

for $1 \leq i \leq n-1$.

First, for a given integer $0 \leq t \leq n$, define the vector $\boldsymbol{v}_t$ via $v_0 = \mathbf{0}$, and

$$\boldsymbol{v}_t \stackrel{def}{=} (\overbrace{0, 0, \ldots, 0}^{n-t}, Ar^{t-1}, Ar^{t-2}, \ldots, Ar, A).$$

In addition, define the set of vectors

$$\mathcal{P}_n^2 = \mathcal{P}_n^2(A, 0, r) = \{\boldsymbol{v_t} : \ \ 0 \leq t \leq n\}. \tag{3.1}$$

Here, we have the following theorem.

Theorem 3.1. *Suppose that $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ are n-vectors where $\boldsymbol{p}$ satisfies*

$$p_{i+1} \geq \frac{1}{r} p_i \tag{3.2}$$

for $1 \leq i \leq n-1$, and $0 \leq p_i \leq A$ for $1 \leq i \leq n$. We have,

$$\min\{\boldsymbol{w} \cdot \boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n\} \leq \boldsymbol{p} \cdot \boldsymbol{q} \leq \max\{\boldsymbol{w} \cdot \boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n\}.$$

Proof. First, suppose $\boldsymbol{p}\cdot\boldsymbol{q}>0$, and note that the lower bound in (2.5) follows from the fact that $\boldsymbol{v}_t=\boldsymbol{0}$ for $t=0$. As in the proof of Theorem 2.1, we will, again, obtain a sequence of vectors $\{\widetilde{\boldsymbol{p}}_i\}_{i=1}^{n+1}$, satisfying

$$0\le \boldsymbol{p}\cdot\boldsymbol{q}=\widetilde{\boldsymbol{p}}_{n+1}\cdot\boldsymbol{q}\le\widetilde{\boldsymbol{p}}_n\cdot\boldsymbol{q}\le\cdots\le\widetilde{\boldsymbol{p}}_1\cdot\boldsymbol{q},$$

such that $\widetilde{\boldsymbol{p}}_1\in\mathcal{P}_n^2$.

In particular, consider the vectors $\widetilde{\boldsymbol{p}}_i=(\widetilde{\boldsymbol{p}}_i(1),\widetilde{\boldsymbol{p}}_i(2),\dots,\widetilde{\boldsymbol{p}}_i(n))\in\mathbb{R}^n$, $i=1,2,\dots,n+1$ defined recursively according to the following scheme.

1. $\widetilde{\boldsymbol{p}}_{n+1}=\boldsymbol{p}$.
2. For $1\le i\le n$, set $S_i=\{s: i+1\le s\le n \text{ and } \widetilde{\boldsymbol{p}}_{i+1}(s)=Ar^{n-s}\}$, and $v_i=\min(S_i\bigcup\{n+1\})$.
3. For $1\le i\le n$, set

$$\begin{aligned}\widetilde{\boldsymbol{p}}_i&=\Big(\widetilde{\boldsymbol{p}}_{i+1}(1),\widetilde{\boldsymbol{p}}_{i+1}(2),\dots,\widetilde{\boldsymbol{p}}_{i+1}(i-1),c_i\widetilde{\boldsymbol{p}}_{i+1}(i),c_i\widetilde{\boldsymbol{p}}_{i+1}(i+1),\\&\qquad\dots,c_i\widetilde{\boldsymbol{p}}_{i+1}(v_i-1),\widetilde{\boldsymbol{p}}_{i+1}(v_i),\widetilde{\boldsymbol{p}}_{i+1}(v_i+1),\dots,\widetilde{\boldsymbol{p}}_{i+1}(n)\Big)\\&=(\boldsymbol{w}_{i+1}^1;c_i\boldsymbol{w}_{i+1}^2;\boldsymbol{w}_{i+1}^3),\end{aligned}\tag{3.3}$$

where c_i is given by

$$c_i=\begin{cases}\dfrac{Ar^{n-i}}{p_i}, & \text{if } \boldsymbol{w}_{i+1}^2\cdot\boldsymbol{q}^{i,v_i-1}>0\\[2ex] \dfrac{\frac{1}{r}p_{i-1}}{p_i}, & \text{if } \boldsymbol{w}_{i+1}^2\cdot\boldsymbol{q}^{i,v_i-1}\le 0 \text{ and } i>1\\[2ex] 0, & \text{otherwise}\end{cases}.\tag{3.4}$$

It is not difficult to verify by induction that $\boldsymbol{w}_{i+1}^{j}$, $j = 1, 2, 3$, are of the form

$$\boldsymbol{w}_{i+1}^{1} = \widetilde{\boldsymbol{p}}_{i+1}^{1,i-1} = (p_1, p_2, \dots, p_{i-1}) \tag{3.5}$$

$$\boldsymbol{w}_{i+1}^{2} = \widetilde{\boldsymbol{p}}_{i+1}^{1,v_i-1} = \left(p_i, \frac{1}{r}p_i, \frac{1}{r^2}p_i, \dots, \frac{1}{r^{v_i-i-1}}p_i\right) \tag{3.6}$$

$$\boldsymbol{w}_{i+1}^{3} = \widetilde{\boldsymbol{p}}_{i+1}^{v_i,n} = (Ar^{n-v_i}, Ar^{n-v_i-1} \cdots, Ar, A) \in \mathcal{P}_{n-v_i+1}. \tag{3.7}$$

Now, note that from (3.2), and the bound $p_n \leq A$, we have that

$$p_i \leq Ar^{n-i},$$

for $1 \leq i \leq n$, and $p_{i-1}/r \leq p_i$ for $2 \leq i \leq n$. Hence, (3.3) and (3.4) imply that

$$\widetilde{\boldsymbol{p}}_i \cdot \boldsymbol{q} - \widetilde{\boldsymbol{p}}_{i-1} \cdot \boldsymbol{q} = (c_i - 1)(\boldsymbol{w}_{i+1}^{2} \cdot \boldsymbol{q}^{i,v_i-1}) \geq 0,$$

and that,

$$\begin{aligned} \widetilde{\boldsymbol{p}}_i \in \Bigg\{ \Bigg(& p_1, p_2, \dots, p_{i-2}, p_{i-1}, \frac{1}{r}p_{i-1}, \frac{1}{r^2}p_{i-1}, \dots, \frac{1}{r^{v_i-i-1}}p_{i-1}, \\ & Ar^{n-v_i}, Ar^{n-(v_i+1)}, \dots, Ar, A \Bigg), \Big(p_1, p_2, \dots, p_{i-1}, Ar^{n-i}, \\ & Ar^{n-(i+1)}, \dots, Ar^{n-(v_i-1)}, Ar^{n-v_i}, Ar^{n-(v_i+1)}, \dots, Ar, A\Big) \Bigg\}. \end{aligned} \tag{3.8}$$

Thus $v_{i-1} \in \{v_i, i\}$, and for $i = 2$, we have

$$\widetilde{\boldsymbol{p}}_2 \in \Bigg\{ \Bigg(p_1, \frac{1}{r}p_1, \frac{1}{r^2}p_1, \dots, \frac{1}{r^{v_2-i-1}}p_{i-1}, Ar^{n-v_2}, Ar^{n-(v_2+1)}, \tag{3.9}$$

$$\ldots, Ar, A\Big), \big(p_1, Ar^{n-2}, Ar^{n-3}, \ldots, Ar^2, Ar, A\big) \Big\}.$$

The vector $\widetilde{\boldsymbol{p}}_1$ then satisfies

$$\begin{aligned}(3.10)\quad \widetilde{\boldsymbol{p}}_1 \in \Big\{ &\big(0, 0, \ldots, 0, Ar^{n-v_2}, Ar^{n-(v_2+1)}, \ldots, Ar, A\big)\,, \\ &(Ar^{n-1}, Ar^{n-2}, Ar^{n-3}, \ldots, Ar^2, Ar, A), \\ &\qquad \big(0, Ar^{n-2}, Ar^{n-3}, \ldots, Ar^2, Ar, A\big) \Big\} \subset \mathcal{P}_n^2,\end{aligned}$$

and the theorem is proven in this case. The proof follows similarly, if $\boldsymbol{p} \cdot \boldsymbol{q} \leq 0$, and the proof of the theorem is complete. □

Now, for a given integer $0 \leq t \leq n$, define the vector $\boldsymbol{v}_t$ via $v_0 = \mathbf{0}$, and

$$\boldsymbol{v}_t \stackrel{def}{=} \big(\overbrace{B, B, \ldots, B}^{n-t}, \overbrace{A, A, \ldots, A}^{t}\big).$$

In addition, define the set of vectors

$$(3.11)\qquad \mathcal{P}_n^3 = \mathcal{P}_n^3(A, B, 1) = \{\boldsymbol{v}_t : \ \ 0 \leq t \leq n\}.$$

For the case $r = 1$ in either (2.1) or (3.2), we can similarly prove the following result. For $B = 0$ the theorem follows directly from either Theorem 2.1 or Theorem 3.1 (see also Lemma 2.2 in [4])). For $0 < B < A$, the proof is similar to that for Theorems 2.1 and 3.1, and will be omitted.

Theorem 3.2 (Monotonicity). *Suppose that* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *are* n*-vectors where* $\boldsymbol{p}$ *satisfies*

$$p_{i+1} \geq p_i$$

for $1 \le i \le n-1$, *and* $0 \le B \le p_i \le A$ *for* $1 \le i \le n$. *We have,*

$$\min\{\boldsymbol{w}\cdot\boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n^3\} \le \boldsymbol{p}\cdot\boldsymbol{q} \le \max\{\boldsymbol{w}\cdot\boldsymbol{q} : \boldsymbol{w} \in \mathcal{P}_n^3\}.$$

We conclude with a return to global optimization for inner products for the vector $\boldsymbol{q}$ as given in Example 2.1.

Example 2.1 (revisited). Consider the vector $q \in \mathbb{R}^{15}$ as given in (2.15).

The entries in q are depicted in Figure 1. Now, consider optimizing $\boldsymbol{p}\cdot\boldsymbol{q}$ over all $\boldsymbol{p} = (p_1, p_2, \dots, p_{15}) \in \mathbb{R}^{15}$, satisfying $0 \le p_i \le 1$ and (3.2) for some $0 < r \le 1$. Theorem 3.2 implies that we need only check over the finite set $\mathcal{P}_{15}^2(1,0,r)$ as given in (3.11). The results of the computations for $r \in \{.1, .3, .7, .9\}$, are given in Figure 3.

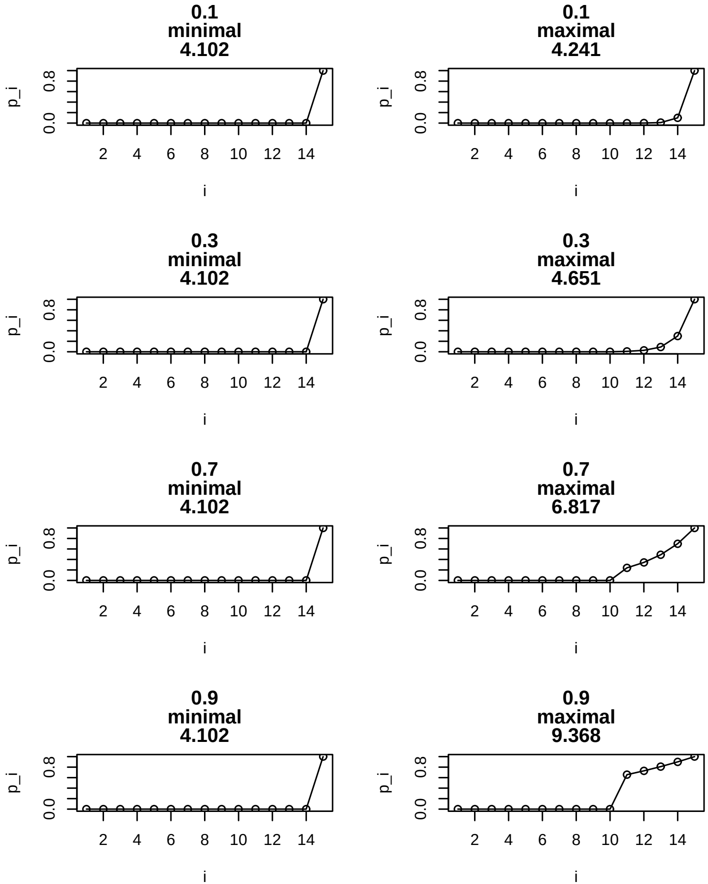

Figure 3: Maximal and minimal values for inner products under the constraint in (3.2).

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