

REDHEFFER TYPE INEQUALITY FOR BESSEL FUNCTIONS

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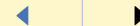
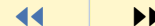
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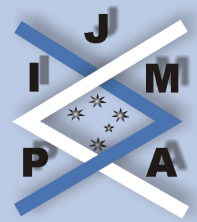
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Abstract:

In this short note, by using mathematical induction and infinite product representations of the functions $\mathcal{J}_p : \mathbb{R} \rightarrow (-\infty, 1]$ and $\mathcal{I}_p : \mathbb{R} \rightarrow [1, \infty)$, defined by

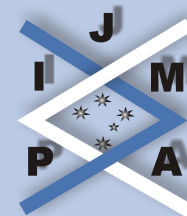
$$\mathcal{J}_p(x) = 2^p \Gamma(p+1) x^{-p} J_p(x) \quad \text{and} \quad \mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x),$$

an extension of Redheffer's inequality for the function \mathcal{J}_p and a Redheffer-type inequality for the function \mathcal{I}_p are established. Here J_p and I_p , denotes the Bessel function and modified Bessel function, while Γ stands for the Euler gamma function. At the end of this work a lower bound for the Γ function is deduced, using Euler's infinite product formula. Our main motivation to write this note is the publication of C.P. Chen, J.W. Zhao and F. Qi [2], which we wish to complement.

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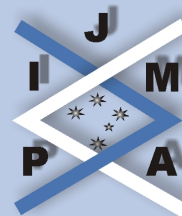
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1. Introduction and Preliminaries

The following inequality

$$(1.1) \quad \frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all } x \in \mathbb{R}$$

is known in literature as Redheffer's inequality [4, 5]. Motivated by this inequality recently C.P. Chen, J.W. Zhao and F. Qi [2] (see also the survey article of F. Qi [3]) using mathematical induction and infinite product representation of $\cos x$, $\sinh x$ and $\cosh x$ established the following Redheffer-type inequalities:

$$(1.2) \quad \cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad \text{and} \quad \cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad \text{for all } |x| \leq \frac{\pi}{2}.$$

Moreover, the authors found the hyperbolic analogue of inequality (1.1), by showing that

$$(1.3) \quad \frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2} \quad \text{for all } |x| < \pi.$$

As we mentioned above, the proofs of inequalities (1.2) and (1.3) by C.P. Chen, J. W. Zhao and F. Qi are based on the following representations [1, p. 75 and 85] of $\cos x$, $\sinh x$ and $\cosh x$:

$$(1.4) \quad \cos x = \prod_{n \geq 1} \left[1 - \frac{4x^2}{(2n-1)^2 \pi^2} \right], \quad \cosh x = \prod_{n \geq 1} \left[1 + \frac{4x^2}{(2n-1)^2 \pi^2} \right],$$

and

$$(1.5) \quad \frac{\sinh x}{x} = \prod_{n \geq 1} \left(1 + \frac{x^2}{n^2 \pi^2} \right)$$

respectively. In this paper our aim is to show that the idea of using mathematical induction and infinite product representation is also fruitful for Bessel functions as well as for the Γ function.



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2. An Extension of Redheffer's Inequality and its Hyperbolic Analogue

Our first main result reads as follows.

Theorem 2.1. *Let us consider the functions $\mathcal{J}_p : \mathbb{R} \rightarrow (-\infty, 1]$ and $\mathcal{I}_p : \mathbb{R} \rightarrow [1, \infty)$, defined by the relations*

$$\mathcal{J}_p(x) = 2^p \Gamma(p+1) x^{-p} J_p(x) \quad \text{and} \quad \mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x),$$

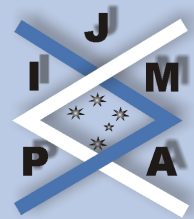
where J_p and I_p are the well-known Bessel function, and modified Bessel function respectively. Furthermore suppose that $p > -1$ and let $j_{p,n}$ be the n -th positive zero of the Bessel function J_p . If $\Delta_p(n) := j_{p,n+1}^2 - j_{p,1} j_{p,n} - j_{p,n} j_{p,n+1} \geq 0$, where $n = 1, 2, \dots$, then the following inequalities hold

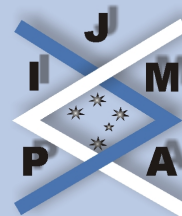
$$(2.1) \quad \mathcal{J}_p(x) \geq \frac{j_{p,1}^2 - x^2}{j_{p,1}^2 + x^2}, \quad \text{for all } |x| \leq \alpha_p := \min_{n \geq 1, p > -1} \left\{ j_{p,1}, \sqrt{\Delta_p(n)} \right\},$$

$$(2.2) \quad \mathcal{I}_p(x) \leq \frac{j_{p,1}^2 + x^2}{j_{p,1}^2 - x^2}, \quad \text{for all } |x| < j_{p,1}.$$

Remark 1. For later use it is worth mentioning that in particular for $p = -1/2$ and $p = 1/2$ respectively the functions \mathcal{J}_p and \mathcal{I}_p reduce to some elementary functions [1, p. 438 and 443], such as

$$(2.3) \quad \begin{aligned} \mathcal{J}_{-1/2}(x) &= \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \\ \mathcal{J}_{1/2}(x) &= \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \end{aligned}$$





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with their hyperbolic analogs

$$(2.4) \quad \begin{aligned} \mathcal{I}_{-1/2}(x) &= \sqrt{\pi/2} \cdot x^{1/2} I_{-1/2}(x) = \cosh x, \\ \mathcal{I}_{1/2}(x) &= \sqrt{\pi/2} \cdot x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}. \end{aligned}$$

Recall that $\mathcal{J}_{-1/2}$ has the infinite product representation (1.4) and [1, p. 75]

$$\mathcal{J}_{1/2}(x) = \frac{\sin x}{x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2 \pi^2} \right).$$

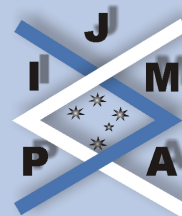
Thus using the relations (2.6) and (2.3) it is clear that $j_{-1/2,n} = (2n - 1)\pi/2$ and $j_{1/2,n} = n\pi$ for all $n = 1, 2, \dots$. Consequently for all $n = 1, 2, \dots$ we have $\sqrt{\Delta_{1/2}(n)} = j_{1/2,1} = \pi$ and

$$\sqrt{\Delta_{-1/2}(n)} = \frac{\pi}{2} \sqrt{2n + 3} \geq \frac{\pi}{2} \sqrt{5} > \frac{\pi}{2} = j_{-1/2,1},$$

which imply that $\alpha_{-1/2} = \pi/2$ and $\alpha_{1/2} = \pi$. So in view of (2.3), if we take in (2.1) $p = -1/2$ and $p = 1/2$ respectively, then we reobtain the first inequality from (1.2) and Redheffer's inequality (1.1) respectively, with the intervals of validity $[-\pi/2, \pi/2]$ and $[-\pi, \pi]$, respectively. The situation is similar to inequality (2.2), namely if we choose in (2.2) $p = -1/2$ and $p = 1/2$ respectively, then by using (2.4), we reobtain the second inequality from (1.2) and inequality (1.3), with the same intervals of validity, i.e. $[-\pi/2, \pi/2]$ and $[-\pi, \pi]$, respectively.

Proof of Theorem 2.1. First observe that to prove (2.1) it is enough to show that

$$(2.5) \quad \mathcal{J}_p(xj_{p,1}) \geq \frac{1 - x^2}{1 + x^2}$$



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holds for all $|x| \leq \alpha_p/j_{p,1}$. It is known that for the Bessel function of the first kind J_p the following infinite product formula [6, p. 498]

$$(2.6) \quad \mathcal{J}_p(x) = 2^p \Gamma(p+1) x^{-p} J_p(x) = \prod_{n \geq 1} \left(1 - \frac{x^2}{j_{p,n}^2} \right)$$

is valid for arbitrary x and $p \neq -1, -2, \dots$. From this we deduce

$$(2.7) \quad \mathcal{J}_p(x j_{p,1}) = \frac{1-x^2}{1+x^2} \left[(1+x^2) \lim_{n \rightarrow \infty} Q_{p,n} \right],$$

$$\text{where } Q_{p,n} := \prod_{k=2}^n \left(1 - \frac{x^2 j_{p,1}^2}{j_{p,k}^2} \right).$$

In what follows we want to prove by mathematical induction that

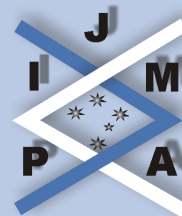
$$(2.8) \quad (1+x^2) Q_{p,n} \geq 1 + \frac{x^2 j_{p,1}}{j_{p,n}}$$

holds for all $p > -1$, $n \geq 2$ and $|x| \leq \alpha_p/j_{p,1}$. For $n = 2$ clearly by assumptions we have

$$(1+x^2) Q_{p,2} - \left(1 + \frac{x^2 j_{p,1}}{j_{p,2}} \right) = \frac{x^2}{j_{p,2}^2} [\Delta_p(1) - j_{p,1}^2 x^2] \geq 0.$$

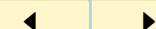
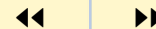
Now suppose that (2.8) holds for some $m \geq 2$. From the definition of $Q_{p,m}$, we easily get

$$Q_{p,m+1} = Q_{p,m} \cdot \left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2} \right), \text{ for all } m = 2, 3, 4, \dots,$$



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thus

$$\begin{aligned}(1+x^2)Q_{p,m+1} - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right) &= (1+x^2)Q_{p,m} \left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right) \\ &\geq \left(1 + \frac{x^2 j_{p,1}}{j_{p,m}}\right) \left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right) \\ &= \frac{x^2 j_{p,1}}{j_{p,m} j_{p,m+1}^2} [\Delta_p(m) - j_{p,1}^2 x^2] \geq 0,\end{aligned}$$

and hence by induction (2.8) follows. Here we used the fact that from the hypothesis we obtain $|x| \leq \sqrt{\Delta_p(m)}/j_{p,1} \leq j_{p,m+1}/j_{p,1}$. On the other hand from the MacMahon expansion [6, p. 506],

$$j_{p,n} = (n + p/2 - 1/4)\pi + \mathcal{O}(n^{-1}), \quad n \rightarrow \infty,$$

we have $j_{p,n} \rightarrow \infty$, as n tends to infinity. Finally from (2.8) we obtain

$$\lim_{n \rightarrow \infty} (1+x^2)Q_{p,n} \geq \lim_{n \rightarrow \infty} \left(1 + \frac{x^2 j_{p,1}}{j_{p,n}}\right) = 1,$$

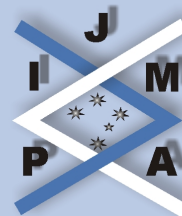
which in view of (2.7) implies (2.5). This completes the proof of (2.1).

Proceeding similarly as in the proof of (2.1) now we prove (2.2). It suffices to show that

$$(2.9) \quad \mathcal{I}_p(x j_{p,1}) \leq \frac{1+x^2}{1-x^2}$$

holds for all $|x| < 1$. Analogously, using the factorisation (2.6), it is known that for the modified Bessel function of the first kind I_p the following infinite product formula

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x) = \prod_{n \geq 1} \left(1 + \frac{x^2}{j_{p,n}^2}\right)$$



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is also valid for arbitrary x and $p \neq -1, -2, \dots$. From this we get that

$$(2.10) \quad \mathcal{I}_p(xj_{p,1}) = \frac{1+x^2}{1-x^2} \left[(1-x^2) \lim_{n \rightarrow \infty} R_{p,n} \right],$$

$$\text{where } R_{p,n} := \prod_{k=2}^n \left(1 + \frac{x^2 j_{p,1}^2}{j_{p,k}^2} \right).$$

In what follows we want to show by mathematical induction that

$$(2.11) \quad (1-x^2)R_{p,n} \leq 1 - \frac{x^2 j_{p,1}}{j_{p,n}}$$

holds for all $p > -1$, $n \geq 2$ and $|x| < 1$. For $n = 2$, clearly we have

$$(1-x^2)R_{p,2} - \left(1 - \frac{x^2 j_{p,1}}{j_{p,2}} \right) = \frac{x^2}{j_{p,2}^2} [-\Delta_p(1) - j_{p,1}^2 x^2] \leq 0.$$

Now suppose that (2.11) holds for some $m \geq 2$. From the definition of $R_{p,m}$, we easily get

$$R_{p,m+1} = R_{p,m} \cdot \left(1 + \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2} \right), \text{ for all } m = 2, 3, 4, \dots,$$

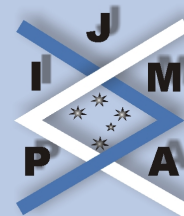
thus

$$\begin{aligned} (1-x^2)R_{p,m+1} - \left(1 - \frac{x^2 j_{p,1}}{j_{p,m+1}} \right) &= (1-x^2)R_{p,m} \left(1 + \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2} \right) - \left(1 - \frac{x^2 j_{p,1}}{j_{p,m+1}} \right) \\ &\leq \left(1 - \frac{x^2 j_{p,1}}{j_{p,m}} \right) \left(1 + \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2} \right) - \left(1 - \frac{x^2 j_{p,1}}{j_{p,m+1}} \right) \\ &= \frac{x^2 j_{p,1}}{j_{p,m} j_{p,m+1}^2} [-\Delta_p(m) - j_{p,1}^2 x^2] \leq 0, \end{aligned}$$

and hence by induction, (2.11) follows. Finally using again the fact that $j_{p,n} \rightarrow \infty$, as n tends to infinity, from (2.11) we obtain

$$\lim_{n \rightarrow \infty} (1 - x^2)R_{p,n} \leq \lim_{n \rightarrow \infty} \left(1 - \frac{x^2 j_{p,1}}{j_{p,n}}\right) = 1,$$

which in view of (2.10) implies (2.9). Thus the proof is complete. ■



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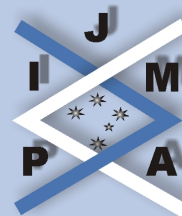
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3. A Lower Bound for the Γ Function

In an effort to popularize the method that C.P. Chen, J. W. Zhao and F. Qi used in the previous section, we illustrate it below by giving a lower bound for the Γ function.

Theorem 3.1. *If $x \in (0, 1]$, then*

$$(3.1) \quad \Gamma(x) \geq \frac{1-x}{1+x} \cdot \frac{e^{(1-\gamma)x}}{x},$$

where $\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right) = 0.5772156649 \dots$ is the Euler constant.

Proof. From the well-known Euler infinite product formula [1, p. 255] for the Γ function,

$$\frac{1}{xe^{\gamma x}\Gamma(x)} = \prod_{n \geq 1} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}},$$

we have

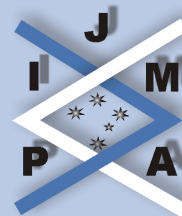
$$(3.2) \quad \frac{e^{(1-\gamma)x}}{x\Gamma(x)} = \frac{1+x}{1-x} \left[(1-x) \lim_{n \rightarrow \infty} S_n \right],$$

$$\text{where } S_n = \prod_{k=2}^n \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}}, \quad n = 2, 3, \dots$$

Observe that to prove (3.1), it is enough to show that for all $n = 2, 3, \dots$

$$(3.3) \quad (1-x)S_n < \left(1 - \frac{x}{n}\right)$$

holds, hence from this we get $\lim_{n \rightarrow \infty} (1-x)S_n \leq 1$, and consequently from (3.2) the inequality (3.1) follows. To prove (3.3) we use mathematical induction again. For



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$n = 2$ we easily get for $x \in (0, 1)$ that

$$(1-x)S_2 < \left(1 - \frac{x}{2}\right) \iff e^{x/2} > \frac{(1-x)\left(1 + \frac{x}{2}\right)}{1 - \frac{x}{2}},$$

which clearly holds because

$$e^{x/2} - \frac{(1-x)\left(1 + \frac{x}{2}\right)}{1 - \frac{x}{2}} = \sum_{k \geq 1} \left(x + \frac{1}{k!}\right) \frac{x^k}{2^k} > 0.$$

Now suppose that (3.3) holds for some $m \geq 2$. Then from (3.2) and (3.3) we obtain that

$$\begin{aligned} (1-x)S_{m+1} - \left(1 - \frac{x}{m+1}\right) &= (1-x)S_m \left(1 + \frac{x}{m+1}\right) e^{-\frac{x}{m+1}} - \left(1 - \frac{x}{m+1}\right) \\ &< \left(1 - \frac{x}{m}\right) \left(1 + \frac{x}{m+1}\right) e^{-\frac{x}{m+1}} - \left(1 - \frac{x}{m+1}\right) \end{aligned}$$

and this is negative if and only if

$$e^{\frac{x}{m+1}} - \frac{\left(1 - \frac{x}{m}\right) \left(1 + \frac{x}{m+1}\right)}{\left(1 - \frac{x}{m+1}\right)} = \sum_{k \geq 1} \left(\frac{x}{m} + \frac{1}{m} - 1 + \frac{1}{k!}\right) \frac{x^k}{(m+1)^k} > 0.$$

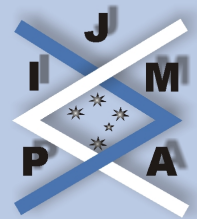
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Remark 2. Numerical experiments in Maple6 show that the lower bound from (3.1) is far from being the best possible one. For example for $x = 0.5$ we have that $\Gamma(0.5) = \sqrt{\pi} = 1.772453851\dots$, while the right hand side of (3.1) is just $0.8235978287\dots$. Similarly for $x = 0.25$ we have $\Gamma(0.25) = 3.6256099082\dots$, while the right hand

side of (3.1) is just 2.667561665... In fact graphics in Maple6 suggest that the function $f : (-1, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \Gamma(x) - \frac{1-x}{1+x} \cdot \frac{e^{(1-\gamma)x}}{x}$$

is convex and satisfies the inequality $f(x) \geq 1$, for all $x \in (-1, 0]$ or $x \in [1, \infty)$. Moreover $f(x) \in (0.94, 1]$, for all $x \in [0, 1]$.



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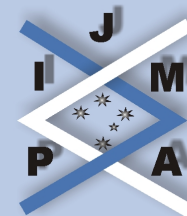
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Redheffer Type Inequality
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Árpád Baricz

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