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STARLIKENESS CONDITIONS FOR AN INTEGRAL OPERATOR

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ABSTRACT. Let for fixed $n \in \mathbb{N}$, Σ_n denotes the class of function of the following form

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k,$$

which are analytic in the punctured open unit disk $\Delta^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$. In the present paper we defined and studied an operator in

$$F(z) = \left[\frac{c+1-\mu}{z^{c+1}} \int_0^z \left(\frac{f(t)}{t}\right)^{\mu} t^{c+\mu} dt\right]^{\frac{1}{\mu}}, \qquad \text{for } \ f \in \Sigma_n \ \text{ and } \ c+1-\mu > 0.$$

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1. INTRODUCTION

Let $\mathcal{H}(\Delta) = \mathcal{H}$ denote the class of analytic functions in Δ, where $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$. For a fixed positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f(z) \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

with $\mathcal{H}_0 = \mathcal{H}[0, 1]$. Let $\mathcal{A}_n$ be the class of analytic functions defined on the unit disc with the normalized conditions $f(0) = 0 = f'(0) - 1$, that is $f \in \mathcal{A}_n$ has the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (z \in \Delta \text{ and } n \in \mathbb{N}). \tag{1.1}$$

Let $\mathcal{A}_1 = \mathcal{A}$ and let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in Δ.

221-09

A function $f \in \mathcal{A}$ is said to be in $\mathcal{S}^*$ iff $f(\Delta)$ is a starlike domain with respect to the origin. Let for $0 \le \alpha < 1$,

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in \Delta \right\}$$

be the class of all starlike functions of order α. So $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. We denote $\mathcal{S}_n^*(\alpha) \equiv \mathcal{S}^*(\alpha) \bigcap \mathcal{A}_n$ for $n \in \mathbb{N}$.

A function $f \in \mathcal{A}$ is said to be in $\mathcal{C}$ iff $f(\Delta)$ is a convex domain. Let for $0 \le \alpha < 1$,

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f(z)} \right) > \alpha, z \in \Delta \right\}$$

be the class of convex functions of order α. So $\mathcal{C}(0) \equiv \mathcal{C}$.

Let for fixed $n \in \mathbb{N}$, Σ_n denote the class of meromorphic functions of the following form

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k, \tag{1.2}$$

which are analytic in the punctured open unit disk $\Delta^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \Delta - \{0\}$. Let $\Sigma_0 = \Sigma$.

A function $f \in \Sigma$ is said to be meromorphically starlike of order α in Δ^* if it satisfies the condition

$$-\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad (0 \le \alpha < 1; z \in \Delta^*).$$

We denote by $\Sigma^*(\alpha)$, the subclass of Σ consisting of all meromorphically starlike functions of order α in Δ^* and $\Sigma_n^*(\alpha) \equiv \Sigma^*(\alpha) \bigcap \Sigma_n$ for $n \in \mathbb{N}$.

We say that $f(z)$ is subordinate to $g(z)$ and $f \prec g$ in Δ or $f(z) \prec g(z)$ $(z \in \Delta)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z)),\ z \in \Delta$. Furthermore, if the function g is univalent in Δ, $f(z) \prec g(z)$ $(z \in \Delta) \Leftrightarrow f(0) = g(0)$ and$f(\Delta) \subset g(\Delta)$.

In the present paper, for $f(z) \in \Sigma_n$, we define and study a generalized operator $I[f]$

$$I[f] = F(z) = \left[\frac{c+1-\mu}{z^{c+1}} \int_0^z \left(\frac{f(t)}{t} \right)^{\mu} t^{c+\mu} dt \right]^{\frac{1}{\mu}}, \quad (c+1-\mu > 0,\ z \in \Delta^*), \tag{1.3}$$

which is similar to the Alexander transform when $c = \mu = 1$ and is similar to Bernardi transformation when $\mu = 1$ and $c > 0$.

2. Main Results

For our main results we need the following lemmas.

Lemma 2.1 (Goluzin [5]). *If $f \in \mathcal{A}_n \bigcap S^*$, then*

$$\operatorname{Re} \left[\frac{f(z)}{z} \right]^{\frac{n}{2}} > \frac{1}{2}.$$

This inequality is sharp with extremal function $f(z) = \frac{z}{(1-z^n)^{\frac{2}{n}}}$.

Lemma 2.2 ([9]). *Let u and v denote complex variables, $u = \alpha + i\rho$, $v = \sigma + i\delta$ and let $\Psi(u, v)$ be a complex valued function that satisfies the following conditions:*

(i) $\Psi(u, v)$ *is continuous in a domain* $\Omega \subset \mathbb{C}^2$*;*
(ii) $(1, 0) \in \Omega$ *and* $\operatorname{Re}(\Psi(1, 0)) > 0$*;*
(iii) $\operatorname{Re}(\Psi(i\rho, \sigma)) \le 0$ *whenever* $(i\rho, \sigma) \in \Omega$*,* $\sigma \le -\frac{1+\rho^2}{2}$ *and* ρ*,* σ *are real.*

If $p(z) \in \mathcal{H}[a,n]$ *is a function that is analytic in* Δ*, such that* $(p(z), zp'(z)) \in \Omega$ *and* $\operatorname{Re}(\Psi(p(z), zp'(z))) > 0$ *hold for all* $z \in \Delta$*, then* $\operatorname{Re} p(z) > 0$*, when* $z \in \Delta$*.*

Lemma 2.3 ([9, p. 34], [8])**.** *Let* $p \in \mathcal{H}[a,n]$

(i) *If* $\Psi \in \Psi_n[\Omega,\ M,\ a]$*, then*

$$\Psi(p(z), zp'^2 p''(z); z) \in \Omega \Rightarrow |p(z)| < M.$$

(ii) *If* $\Psi \in \Psi_n[M,\ a]$*, then*

$$|\Psi(p(z), zp'^2 p''(z); z)| < M \Rightarrow |p(z)| < M.$$

Lemma 2.4 ([6])**.** *Let* $h(z)$ *be an analytic and convex univalent function in* Δ*, with* $h(0) = a$*,* $c \neq 0$ *and* $\operatorname{Re} c \geq 0$*. If* $p \in \mathcal{H}[a,n]$ *and*

$$p(z) + \frac{zp'(z)}{c} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) \prec \frac{c}{nz^{\frac{c}{n}}} \int_0^z t^{\frac{c}{n}-1} f(t)dt, \quad z \in \Delta.$$

The function q *is convex and the best dominant.*

Theorem 2.5. *Let* $c > 0$ *and* $0 < \mu < 1$*. If* $f \in \Sigma_n^*(\alpha)$ *for* $0 < \alpha < 1$*, then* $\mathrm{I}(f) = F(z) \in \Sigma_n^*(\beta)$*, where*

$$\begin{aligned} \beta &= \beta(\alpha, c, \mu) \\ &= \frac{1}{4\mu}\Big[2c + 2\alpha\mu + n + 2 \\ &\qquad - \sqrt{[4(c-\alpha\mu)]^2 + (n+2)(n+2+4c+4\mu\alpha) - 16\alpha - 8\mu n}\Big]. \end{aligned} \tag{2.1}$$

Proof. Here we have the conditions

$$0 < \alpha < 1,\ 0 < \mu < 1 \quad \text{and} \quad c > 0, \tag{2.2}$$

which will imply that $\beta < 1$.

Let $f(z) \in \Sigma_n^*(\alpha)$. We first show that $F(z)$ defined by (1.3) will become nonzero for $z \in \Delta^*$. Again since $f \in \Sigma_n^*(\alpha)$, we have $f(z) \neq 0$, for $z \in \Delta^*$.

Let $g(z) = \frac{1}{(f(z))^\mu}$, then a simple computation shows that $g(z) \in S_n^*(\alpha\mu)$.

If we define

$$I_g = \left[\frac{g(z)}{z}\right]^{\left\{\frac{1}{1-\alpha\mu}\right\}},$$

then $I(g) \in S_n^*$ and by Goluzin's subordination result (by Lemma 2.1), we obtain

$$\left[\frac{I_g}{z}\right]^{\frac{n}{2}} \prec \frac{1}{1+z}.$$

From the relation between I_g, g and f we get that

$$\frac{g(z)}{z} \prec (1+z)^{\frac{2}{n}(\alpha\mu-1)},$$

which implies

$$z(f(z))^\mu \prec (1+z)^{\frac{2}{n}(1-\alpha\mu)}$$

and since $0 < \alpha\mu < 1$, we have $z(f(z))^\mu \prec (1+z)^{\frac{2}{n}}$. Combining this with

$$\min_{|z|=1} \operatorname{Re}(1+z)^{\frac{2}{n}} = 0,$$

we deduce that

$$\operatorname{Re}[z(f(z))^\mu] > 0. \tag{2.3}$$

By differentiating (1.3), we obtain

$$(c+1)(F(z))^\mu + z\frac{d}{dz}(F(z))^\mu = (c+1-\mu)(f(z))^\mu. \tag{2.4}$$

If we let

$$\frac{P(z)}{z} = (F(z))^\mu, \tag{2.5}$$

then (2.4) becomes

$$P(z) + \frac{1}{c}zP'(z) = \frac{c+1-\mu}{c}z(f(z))^\mu.$$

Hence from (2.3) we have

$$\operatorname{Re}\Psi(P(z), zP'(z)) = \operatorname{Re}\left[P(z) + \frac{zP'(z)}{c}\right], \tag{2.6}$$

where $\Psi(r,s) = r + \frac{s}{c}$. To show that $\operatorname{Re}P(z) > 0$, condition (iii) of Lemma 2.2 must be satisfied. Since $c > 0$, (2.6) implies that

$$\operatorname{Re}\Psi(i\rho,\sigma) = \operatorname{Re}\left(i\rho + \frac{\sigma}{c}\right) \leq -\frac{n(1+\rho^2)}{2c} \leq 0,$$

when $\sigma \leq -\frac{n(1+\rho^2)}{2}$, for all $\rho \in \mathbb{R}$. Hence from (2.6) we deduce that $\operatorname{Re}P(z) > 0$, which implies that $F(z) \neq 0$ for $z \in \Delta^*$.

We next determine β such that $F \in \Sigma_n^*(\beta)$. Let us define $p(z) \in \mathcal{H}[1,n]$ by

$$-\frac{zF'(z)}{F(z)} = (1-\beta)p(z) + \beta. \tag{2.7}$$

By applying (part iii) of Lemma 2.2 again with different Ψ we finish the proof of the theorem. Since $f \in \Sigma_n^*(\alpha)$, by differentiating (2.4) we easily get

$$\operatorname{Re}\Psi(p(z), zp'(z)) > 0,$$

where

$$\Psi(r,s) = (1-\beta)r + \beta + \frac{(1-\beta)\sigma}{c+1-\mu\beta-\mu(1-\beta)p(z)} - \alpha.$$

For $\beta \leq \beta(\alpha,c,\mu)$, where $\beta(\alpha,c,\mu)$ is given by (2.1), a simple calculation shows that the admissibility condition (iii) of Lemma 2.2 is satisfied. Hence by Lemma 2.2, we get $\operatorname{Re}p(z) > 0$. Using this result in (2.7) together with $\beta < 1$ shows that $F(z) \in \Sigma_n^*(\beta)$. $\square$

Theorem 2.6. *Let $0 < c+1-\mu < 1$. If, for $0 < \alpha < 1$, $f \in \Sigma^*(\alpha)$, then $\mathrm{I}(f) \in \Sigma^*(\beta)$, where*

$$\beta = \beta(\alpha,\mu,c) = \frac{1}{2\mu}\left[2c + 2\alpha\mu + 3 - \sqrt{[2(c-\alpha\mu)]^2 + 3(3+4c) - 4\mu(2+\alpha)}\right]. \tag{2.8}$$

The proof is very similar to that of Theorem 2.5.

In the special case when the meromorphic function given in (1.2) has a coefficient $a_0 = 0$, it is possible to obtain a stronger result than (2.8).

Theorem 2.7. *Let* $c > 0$, $0 < \mu < 1$, $0 < \alpha < 1$, $f \in \Sigma_1^*(\alpha)$, *then* $\mathrm{I}(f) \in \Sigma_1^*(\beta)$, *where*

$$\beta = \beta(\alpha, \mu, c) = \frac{1}{2\mu}\left[c + \alpha\mu + 1 - \sqrt{(c - \alpha\mu)^2 + 4(c + 1 - \mu)}\right]. \tag{2.9}$$

The proof is similar to that of Theorem 2.5.

Corollary 2.8. *Let* $n \geq 1$, $c + n + 1 > 0$ *and* $g(z) \in \mathcal{H}[0,\ n]$. *If* $|\,((g(z))^{\mu})'\,| \leq \lambda$ *and*

$$\mathcal{F}(z) = \left[\frac{1}{z^{c+1}}\int_0^z (g(t))^{\mu}\, t^c dt\right]^{\frac{1}{\mu}}, \tag{2.10}$$

then

$$|\,((\mathcal{F}(z))^{\mu})'\,| \leq \frac{\lambda}{c + n + 1}.$$

Proof. From (2.10) we deduce $(c+1)(\mathcal{F}(z))^{\mu} + z\,((\mathcal{F}(z))^{\mu})' = g^{\mu}(z)$. If we set $z\,((\mathcal{F}(z))^{\mu})' = P(z)$, then $P \in \mathcal{H}[0,\ n]$ and

$$(c + 1)P(z) + zP'(z) = z(g^{\mu}(z))' \prec \lambda z.$$

From part(i) of Lemma 2.3, it follows that this differential subordination has the best dominant

$$P(z) \prec Q(z) = \frac{\lambda z}{c + n + 1}.$$

Hence we have

$$|\,((\mathcal{F}(z))^{\mu})'\,| \leq \frac{\lambda}{c + n + 1}.$$

□

Corollary 2.9. *Let* $c + n + 1 > 0$ *and* $f \in \Sigma_n$ *be given as*

$$f(z) = \frac{1}{z} + g(z),$$

where $n \geq 1$ *and* $g(z) \in \mathcal{H}[0,\ n]$. *Let* $\mathcal{F}$ *be defined by*

$$\mathcal{F}(z) \equiv \frac{1}{z} + G(z) = \frac{1}{z} + \left[\frac{1}{z^{c+1}}\int_0^z (g(t))^{\mu}\, t^c dt\right]^{\frac{1}{\mu}}. \tag{2.11}$$

Then

$$|\,((g(z))^{\mu})'\,| \leq \frac{n(c + n + 1)}{\sqrt{n^2 + 1}}.$$

Proof. From Corollary 2.8 we obtain

$$|\,((G(z))^{\mu})'\,| \leq \frac{n}{\sqrt{n^2 + 1}},$$

since from (2.11), we have

$$|z^2\,((\mathcal{F}(z))^{\mu})' + 1| = |G'(z)|.$$

Hence from [2], we conclude that $\mathcal{F} \in \Sigma_n^*$. □

Corollary 2.10. *Let* n *be a fixed positive integer and* $c > 0$. *Let* q *be a convex function in* Δ, *with* $q(0) = 1$ *and let* h *be defined by*

$$h(z) = q(z) + \frac{n + 1}{c} zq'(z). \tag{2.12}$$

If $f \in \Sigma_n$ *and* $F(z)$ *is given by* (1.3), *then*

$$-\frac{c + 1 - \mu}{c} z^2\,((f(z))^{\mu})' \prec h(z) \Rightarrow -z^2\,((F(z))^{\mu})' \prec q(z),$$

and this result is sharp.

Proof. From the definition of $h(z)$, it is a convex function. If we obtain

$$p(z) = -z^2(F^\mu(z))',$$

then $p \in \mathcal{H}[1,\ n+1]$ and from (2.3), we get

$$p(z) + \frac{1}{c}zp'(z) = -\frac{c+1-\mu}{c}z^2\left((f(z))^\mu\right)' \prec h(z).$$

The conclusion of the corollary follows by Lemma 2.4. □

Corollary 2.11. *Let* $n \geq 1$ *and* $c > 0$. *Let* $f \in \Sigma_n$ *and let* $F(z)$ *given by* (1.3). *If* $\lambda > 0$, *then*

$$|z^2((f(z))^\mu)' + 1| < \lambda \Rightarrow |z^2((F(z))^\mu)' + 1| < \frac{\lambda c}{c+n+1}.$$

In particular,

$$|z^2((f(z))^\mu)' + 1| < \frac{c+n+1}{c} \Rightarrow |z^2((F(z))^\mu)' + 1| < 1.$$

Hence $(F(z))^\mu$ *is univalent.*

Proof. If we take

$$q(z) = 1 + \frac{\lambda cz}{c+n+1},$$

then (2.12) becomes

$$h(z) = 1 + \lambda z.$$

The conclusion of the corollary follows by Corollary 2.10. □

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