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ON AN INEQUALITY OF OSTROWSKI TYPE

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ABSTRACT. We prove an inequality of Ostrowski type for p-norm, generalizing a result of Dragomir [1].

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1. INTRODUCTION

For a differentiable function $f\colon [a,b] \to \mathbb{R}$, $a \cdot b > 0$, Dragomir has in [1] proved, using Pompeiu's mean value theorem [3], the following Ostrowski type inequality:

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right| \le D(x) \cdot \|f - \iota f'\|_\infty , \tag{1.1}$$

where $\iota(t) = t$, $t \in [a,b]$, and

$$D(x) = \frac{b-a}{|x|} \left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right) . \tag{1.2}$$

We are going to prove a general estimate with the p-norm, $1 \le p \le \infty$, which will for $p = \infty$ give the Dragomir result.

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2. THE MAIN RESULT

Theorem 2.1. *Let the function* $f\colon [a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\le p,q\le\infty$, *and all* $x\in[a,b]$, *the following inequality holds:*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\le PU(x,p)\cdot\|f-\iota\,f'\|_p \tag{2.1}$$

where $\iota(t)=t$, $t\in[a,b]$, *and*

$$PU(x,p)=(b-a)^{\frac{1}{p}-1}\cdot\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\right]. \tag{2.2}$$

Note that in cases $(p,q)=(1,\infty)$, $(\infty,1)$, and $(2,2)$ the constant $PU(x,p)$ has to be taken as the limit as $p\to1$, ∞, and 2, respectively.

Proof. Define $F\colon[1/b,1/a]\to\mathbb{R}$ by $F(t):=t\,f(\frac{1}{t})$. The function F is continuous and is differentiable on $(1/b,1/a)$, and for all $x_1,x_2\in[1/b,1/a]$ we have

$$F(x_1)-F(x_2)=\int_{x_2}^{x_1}F'(t)\,\mathrm{d}t$$

$$=\int_{x_2}^{x_1} f\left(\frac{1}{t}\right)-\frac{1}{t}f'\left(\frac{1}{t}\right)\,\mathrm{d}t\qquad\left(\left[u:=\frac{1}{t}\right]\right) \tag{2.3}$$

$$=-\int_{1/x_2}^{1/x_1}\left(f(u)-uf'(u)\right)\,\frac{1}{u^2}\,\mathrm{d}u\,. \tag{2.4}$$

Denote $x_1=:1/x$ and $x_2=:1/t$. Then for all $x,t\in[a,b]$ from (2.4) we get

$$\frac{1}{x}\,f(x)-\frac{1}{t}\,f(t)=\int_x^t\left(f(u)-uf'(u)\right)\,\frac{1}{u^2}\,\mathrm{d}u,$$

i.e.,

$$tf(x)-xf(t)=x\,t\int_x^t\left(f(u)-uf'(u)\right)\,\frac{1}{u^2}\,\mathrm{d}u\,. \tag{2.5}$$

Integrating on t and dividing by x, we obtain

$$\frac{b^2-a^2}{2}\cdot\frac{f(x)}{x}-\int_a^b f(t)\,\mathrm{d}t=\int_a^b t\left(\int_x^t\left(f(u)-uf'(u)\right)\,\frac{1}{u^2}\,\mathrm{d}u\right)\mathrm{d}t\,,$$

and therefore

$$\left|\frac{b^2-a^2}{2}\cdot\frac{f(x)}{x}-\int_a^b f(t)\,\mathrm{d}t\right| \tag{2.6}$$

$$\le\int_a^b\left|\int_x^t\left|\left(f(u)-uf'(u)\right)\,\frac{t}{u^2}\right|\mathrm{d}u\right|\mathrm{d}t$$

$$=\int_a^x\left|\int_t^x\left|\left(f(u)-uf'(u)\right)\,\frac{t}{u^2}\right|\mathrm{d}u\right|\,\mathrm{d}t+\int_x^b\left|\int_x^t\left|\left(f(u)-uf'(u)\right)\,\frac{t}{u^2}\right|\,\mathrm{d}u\right|\mathrm{d}t\,.$$

First, consider the case $1 < p, q < \infty$. Applying Hölder's inequality, the sum in the last line of (2.5) is

$$\begin{aligned}
(2.7)\qquad &\le \left(\int_a^x \left(\int_t^x |f(u) - uf'(u)|^p \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{p}} \cdot \left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_x^b \left(\int_x^t |f(u) - uf'(u)|^p \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{p}} \cdot \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left(\int_a^b \left(\int_a^b |f(u) - uf'(u)|^p \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The first factor in (2.7) equals

$$(2.8)\qquad \left(\int_a^b \left(\int_a^b |f(u) - uf'(u)|^p \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \, \|f - \iota\, f'\|_p \,,$$

and for the second factor, for $p, q \neq 2$, we get

$$\begin{aligned}
(2.9)\quad &\left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{q}} \\
&= \left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Putting (2.8) and (2.9) into (2.7) and dividing (2.6) and (2.7) by $(b-a)$ gives the required inequality (2.1) in the case $1 < p, q < \infty$, $p, q \neq 2$.

For $p = q = 2$, instead of (2.9) we obtain

$$\begin{aligned}
(2.10)\quad &\left(\int_a^x \left(\int_t^x \frac{t^2}{u^4} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_x^b \left(\int_x^t \frac{t^2}{u^4} \,\mathrm{d}u\right) \mathrm{d}t\right)^{\frac{1}{2}} \\
&= \frac{1}{3}\left(\left(\ln\left(\frac{x}{a}\right)^3 + \frac{a^3}{x^3} - 1\right)^{\frac{1}{2}} + \left(\ln\left(\frac{x}{b}\right)^3 + \frac{b^3}{x^3} - 1\right)^{\frac{1}{2}}\right)
\end{aligned}$$

which is easily shown to be equal to the limit of the right hand side in (2.9) for $q \to 2$, i. e.

$$(2.11)\quad \lim_{p\to 2} PU(x,p) = \frac{1}{3(b-a)^{\frac{1}{2}}}\left(\left(\ln\left(\frac{x}{a}\right)^3 + \frac{a^3}{x^3} - 1\right)^{\frac{1}{2}} + \left(\ln\left(\frac{x}{b}\right)^3 + \frac{b^3}{x^3} - 1\right)^{\frac{1}{2}}\right).$$

Now consider the case $p = \infty$, $q = 1$. The last line in (2.6) is

$$\begin{aligned}
(2.12)\qquad &\le \sup_{a\le u\le b} |f(u) - u\, f'(u)| \left(\int_a^x \left(\int_t^x \frac{t}{u^2} \,\mathrm{d}u\right) \mathrm{d}t + \int_x^b \left(\int_x^t \frac{t}{u^2} \,\mathrm{d}u\right) \mathrm{d}t\right) \\
&= \|f - \iota\, f'\|_\infty \cdot \left(\frac{a^2 + b^2}{2x} + x - a - b\right).
\end{aligned}$$

Putting (2.12) into (2.6) and dividing by $(b-a)$ gives

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\le\frac{1}{b-a}\left(\frac{a^2+b^2}{2x}+x-a-b\right)\cdot\|f-\iota\,f'\|_\infty$$

which reproves the Dragomir's result [1].

It is easy to see that

$$\frac{1}{b-a}\left(\frac{a^2+b^2}{2x}+x-a-b\right)=\lim_{p\to\infty}PU(x,p)$$

proving (2.1) in the case $p=\infty$, $q=1$.

Finally consider the case $p=1$, $q=\infty$. In this case the last line of (2.6) is

$$\begin{aligned}(2.13)\qquad &\le\int_a^x\left|\int_t^x|f(u)-uf'(u)|\max_{\substack{t\le u\le x\\ a\le t\le x}}\frac{t}{u^2}\,\mathrm{d}u\right|\mathrm{d}t\\ &\quad+\int_x^b\left|\int_x^t|f(u)-uf'(u)|\max_{\substack{x\le u\le t\\ x\le t\le b}}\frac{t}{u^2}\,\mathrm{d}u\right|\mathrm{d}t\\ &\le\int_a^b\int_a^b|f(u)-u\,f'(u)|\,\mathrm{d}u\,\mathrm{d}t\cdot\left(\frac{1}{a}+\frac{b}{x^2}\right)\\ &=(b-a)\left(\frac{1}{a}+\frac{b}{x^2}\right)\cdot\|f-\iota\,f'\|_1\,.\end{aligned}$$

Appending (2.13) to (2.6) and dividing by $(b-a)$ gives

$$(2.14)\qquad\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\le\left(\frac{1}{a}+\frac{b}{x^2}\right)\cdot\|f-\iota\,f'\|_1\,.$$

It is not too difficult to show that

$$(2.15)\qquad\lim_{p\to 1}PU(x,p)=\frac{1}{a}+\frac{b}{x^2}\,,$$

so (2.14) proves formula (2.1) for $p=1$, $q=\infty$, proving the theorem. □

Remark 2.2. We have considered only the positive case, $0<a<b$. In the case $a<b<0$, a similar but more cumbersome formula holds, where most a's, b's and x's have to be replaced by their absolute values.

3. THE WEIGHTED CASE

In the weighted case we have the following result:

Theorem 3.1. *Let the function* $f\colon[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$*, and let* $w\colon[a,b]\to\mathbb{R}$ *be a nonnegative integrable function. Then for* $\frac{1}{p}+\frac{1}{q}=1$*, with* $1\le p,q\le\infty$*, and all* $x\in[a,b]$*, the following inequality holds:*

$$\begin{aligned}(3.1)\qquad &\left|\frac{f(x)}{x}\int_a^b t\,w(t)\,\mathrm{d}t-\int_a^b f(t)\,w(t)\,\mathrm{d}t\right|\\ &\le\|f-\iota f'\|_p\cdot\frac{(b-a)^{\frac{1}{p}}}{(1-2q)^{\frac{1}{q}}}\cdot\left[\left(x^{1-2q}\int_a^x t^q\,(w(t))^q\,\mathrm{d}t-\int_a^x t^{1-q}\,(w(t))^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\int_x^b t^{1-q}\,(w(t))^q\,\mathrm{d}t-x^{1-2q}\int_x^b t^q\,(w(t))^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. Multiplying (2.5) by $w(t)/x$ and integrating on t, we get

$$\frac{f(x)}{x}\int_a^b t\,w(t)\,\mathrm{d}t-\int_a^b f(t)\,w(t)\,\mathrm{d}t=\int_a^b t\,w(t)\left(\int_x^t (f(u)-u\,f'(u))\,\frac{1}{u^2}\,\mathrm{d}u\right)\,\mathrm{d}t\,,$$

and as in the proof of Theorem 2.1 we have

$$\begin{aligned}
&\left|\frac{f(x)}{x}\int_a^b t\,w(t)\,\mathrm{d}t-\int_a^b f(t)\,w(t)\,\mathrm{d}t\right|\\
&\le \int_a^x \left|\int_t^x \left|(f(u)-uf'(u))\,\frac{t\,w(t)}{u^2}\right|\,\mathrm{d}u\right|\,\mathrm{d}t+\int_x^b \left|\int_x^t \left|(f(u)-uf'(u))\,\frac{t\,w(t)}{u^2}\right|\,\mathrm{d}u\right|\,\mathrm{d}t\\
&\le \left(\int_a^x\left(\int_t^x |f(u)-u\,f'(u)|^p\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{p}}\cdot\left(\int_a^x\left(\int_t^x \frac{t^q\ (w(t))^q}{u^{2q}}\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\qquad+\left(\int_x^b\left(\int_x^t |f(u)-u\,f'(u)|^p\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{p}}\cdot\left(\int_x^b\left(\int_x^t \frac{t^q\ (w(t))^q}{u^{2q}}\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le \left(\int_a^b\left(\int_a^b |f(u)-u\,f'(u)|^p\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\qquad\cdot\left[\left(\int_a^x\left(\int_t^x \frac{t^q\ (w(t))^q}{u^{2q}}\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_x^b\left(\int_x^t \frac{t^q\ (w(t))^q}{u^{2q}}\,\mathrm{d}u\right)\,\mathrm{d}t\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which gives (3.1). □

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