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A NEW REFINEMENT OF THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish a new refinement of the Hermite-Hadamard inequality for convex functions.

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1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the following inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality [5].

In recent years there have been many extensions, generalizations and similar results of the inequality (1.1).

In [2], Dragomir established the following theorem which is a refinement of the left side of (1.1).

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)dx.$$

In [6] Yang and Hong established the following theorem which is a refinement of the right side of inequality (1.1).

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Theorem 1.2. *If $f:[a,b]\to\mathbb{R}$ is a convex function, and F is defined by*

$$F(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx,$$

then F is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx=F(0)\le F(t)\le F(1)=\frac{f(a)+f(b)}{2}.$$

In this paper we establish a refinement of the both sides of inequality (1.1). For this we first define two sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned}x_n&=\frac{1}{2^n}\sum_{i=1}^{2^n}f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)\\&=\frac{1}{2^n}\sum_{i=1}^{2^n}f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right),\end{aligned}\tag{1.2}$$

$$\begin{aligned}y_n&=\frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\left[f\left(\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b\right)+f\left(\left(1-\frac{i-1}{2^n}\right)a+\frac{i-1}{2^n}b\right)\right]\\&=\frac{1}{2^{n+1}}\left[f(a)+f(b)+2\sum_{i=1}^{2^n-1}f\left(\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b\right)\right]\end{aligned}\tag{1.3}$$

and we prove the following

$$\begin{aligned}f\left(\frac{a+b}{2}\right)=x_0&\le\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\&=x_1\le\cdots\le x_n\le\cdots\\&\le\frac{1}{b-a}\int_a^b f(x)dx\le\cdots\le y_n\le\cdots\le y_1\\&=\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\\&\le y_0=\frac{f(a)+f(b)}{2},\end{aligned}$$

which is a new refinement of the Hermite-Hadamard inequality (1.1). For a similar discussion, see [1] or the monograph online [7, p. 19 – 22].

2. A Refinement Result

In this section, using the terminologies of the introduction, we refine the Hermite-Hadamard inequality via the sequences $\{x_n\}$ and $\{y_n\}$.

Theorem 2.1. *Let f be a convex function on $[a,b]$. Then we have*

$$f\left(\frac{a+b}{2}\right)\le x_n\le\frac{1}{b-a}\int_a^b f(x)dx\le y_n\le\frac{f(a)+f(b)}{2}.$$

Proof. By the right side of Hermite-Hadamard inequality (1.1) we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{b-a}\sum_{i=1}^{2^n}\int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}} f(x)dx \\
&\le \frac{1}{b-a}\sum_{i=1}^{2^n}\left(a+i\frac{b-a}{2^n}-a-(i-1)\frac{b-a}{2^n}\right)\frac{f\left(a+i\frac{b-a}{2^n}\right)+f\left(a+(i-1)\frac{b-a}{2^n}\right)}{2} \\
&= \frac{1}{2^{n+1}}\left[\sum_{i=1}^{2^n} f\left[\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b\right]+f\left[\left(1-\frac{i-1}{2^n}a\right)+\frac{i-1}{2^n}b\right]\right] \\
&= y_n.
\end{aligned}$$

By the convexity of f we obtain

$$\begin{aligned}
y_n &\le \frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\left[\left(1-\frac{i}{2^n}\right)f(a)+\frac{i}{2^n}f(b)+\left(1-\frac{i-1}{2^n}\right)f(a)+\frac{i-1}{2^n}f(b)\right] \\
&= \frac{1}{2^{n+1}}\left[f(a)\sum_{i=1}^{2^n}\left(2-\frac{i}{2^{n-1}}+\frac{1}{2^n}\right)+f(b)\sum_{i=1}^{2^n}\left(\frac{i}{2^{n-1}}-\frac{1}{2^n}\right)\right] \\
&= \frac{1}{2^{n+1}}\left[f(a)\left(2^{n+1}-\frac{1}{2^{n-1}}\frac{2^n(2^n+1)}{2}+\frac{2^n}{2^n}\right)+f(b)\left(\frac{1}{2^{n-1}}\cdot\frac{2^n(2^n+1)}{2}-\frac{2^n}{2^n}\right)\right] \\
&= \frac{1}{2^{n+1}}[f(a)(2^{n+1}-2^n)+f(b)(2^n)] = \frac{f(a)+f(b)}{2},
\end{aligned}$$

so

$$\frac{1}{b-a}\int_a^b f(x)dx \le y_n \le \frac{f(a)+f(b)}{2}.$$

On the other hand, by the left side of inequality (1.1) we have

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\sum_{i=1}^{2^n}\int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}} f(x)dx \ge \frac{1}{b-a}\sum_{i=1}^{2^n}\frac{b-a}{2^n},$$

$$f\left(\frac{a+i\frac{b-a}{2^n}+a+(i-1)\frac{b-a}{2^n}}{2}\right) = \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right) = x_n.$$

By the convexity of f and Jensen's inequality we obtain

$$\begin{aligned}
x_n &= \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right) \\
&\ge f\left[\frac{1}{2^n}\sum_{i=1}^{2^n}\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)\right] \\
&= f\left[\frac{1}{2^n}\left(2^n a+\frac{b-a}{2^n}\cdot\frac{2^n(2^n+1)}{2}-\frac{b-a}{2^{n+1}}2^n\right)\right] \\
&= f\left(a+\frac{b-a}{2}\right) = f\left(\frac{a+b}{2}\right).
\end{aligned}$$

□

Theorem 2.2. *Let f be a convex function on $[a,b]$, then $\{x_n\}$ is increasing, $\{y_n\}$ is decreasing and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{1}{b-a}\int_a^b f(x)dx.$$

Proof. We have

$$\begin{aligned}
x_n &= \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(a + i\frac{b-a}{2^n} - \frac{b-a}{2^{n+1}}\right) \\
&= \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(\frac{(2^{n+1}-2i+1)a+(2i-1)b}{2^{n+1}}\right) \\
&= \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(\frac{1}{2}\cdot\frac{(2^{n+3}-8i+4)a+(8i-4)b}{2^{n+2}}\right) \\
&= \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(\frac{1}{2}\cdot\frac{(2^{n+2}+3-4i)a+(4i-3)b+(2^{n+2}+1-4i)a+(4i-1)b}{2^{n+2}}\right) \\
&\le \frac{1}{2^{n+1}}\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2}+3-4i)a+(4i-3)b}{2^{n+2}}\right) \\
&\qquad + \frac{1}{2^{n+1}}\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2}+1-4i)a+(4i-1)b}{2^{n+2}}\right)
\end{aligned}$$

set $A=\{1,3,\dots,2^{n+1}-1\}$ and $B=\{2,4,\dots,2^{n+1}\}$, thus we obtain

$$\begin{aligned}
\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2}+3-4i)a+(4i-3)b}{2^{n+2}}\right) &= \sum_{A} f\left(\frac{(2^{n+2}+1-2i)a+(2i-1)b}{2^{n+2}}\right) \\
\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2}+1-4i)a+(4i-1)b}{2^{n+2}}\right) &= \sum_{B} f\left(\frac{(2^{n+2}+1-2i)a+(2i-1)b}{2^{n+2}}\right),
\end{aligned}$$

which implies that

$$x_n \le \frac{1}{2^{n+1}}\left[\sum_{A\cup B} f\left(\frac{(2^{n+2}+1-2i)a+(2i-1)b}{2^{n+2}}\right)\right] = x_{n+1},$$

so $\{x_n\}$ is increasing. On the other hand we have

$$\begin{aligned}
y_{n+1} &= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^{n+1}-1} f\left[\left(1-\frac{i}{2^{n+1}}\right)a + \frac{i}{2^{n+1}}b\right]\right] \\
&= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^{n+1}-1} f\left(\frac{(2^{n+1}-i)a+ib}{2^{n+1}}\right)\right].
\end{aligned}$$

Setting $C = \{2, 4, 6, \dots, 2^{n+1} - 2\}$, we obtain

$$\begin{aligned}
y_{n+1} &= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i \in C} f\left(\frac{(2^{n+1} - i)a + ib}{2^{n+1}} \right) + 2 \sum_{i \in A} f\left(\frac{(2^{n+1} - i)a + ib}{2^{n+1}} \right) \right] \\
&= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^{n+1} - 2i)a + 2ib}{2^{n+1}} \right) \right. \\
&\qquad \left. + 2 \sum_{i=1}^{2^n} f\left(\frac{(2^{n+1} - 2i + 1)a + (2i - 1)b}{2^{n+1}} \right) \right] \\
&= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
&\qquad \left. + 2 \sum_{i=1}^{2^n} f\left(\frac{1}{2} \cdot \frac{(2^n - i)a + ib + (2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
&\leq \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
&\qquad \left. + \sum_{i=1}^{2^n} f\left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n} f\left(\frac{(2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
&= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
&\qquad \left. + f(b) + f(a) + \sum_{i=2}^{2^n} f\left(\frac{(2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
&= \frac{1}{2^{n+2}} \left[2f(a) + 2f(b) + 3 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) \right] \\
&= \frac{1}{2^{n+1}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n - 1} f\left(\frac{(2^n - i)a + ib}{2^n} \right) \right] = y_n,
\end{aligned}$$

so $\{y_n\}$ is decreasing.

For the proof of the last assertions, since f is continuous on $[a, b]$, we use the following well known equality:

$$\lim_{n \to \infty} \frac{b - a}{n} \sum_{i=1}^{n} f\left(a + i \frac{b - a}{n} \right) = \int_a^b f(x)dx.$$

So we obtain

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \frac{1}{b - a} \int_a^b f(x)dx.$$

□

Remark 1. Let f be a convex function on $[a,b]$. In conclusion, we can state that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) = x_0 &\le \frac{1}{2} f\left[\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &= x_1 \le \cdots \le x_n \le \cdots \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \le \cdots \le y_n \le \cdots \le y_1 \\ &= \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] \\ &\le y_0 = \frac{f(a)+f(b)}{2}. \end{aligned}$$

3. Applications for Special Means

Recall the following means

a) The arithmetic mean

$$A(a,b) = \frac{a+b}{2} \quad (a,b>0);$$

b) The geometric neam

$$G(a,b) = \sqrt{ab} \quad (a,b>0);$$

c) The harmonic mean

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} \quad (a,b>0);$$

d) The logarithmic mean

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & b \neq a; \\ a & b = 0; \end{cases} \qquad (a,b>0).$$

We define the two new means by the following:

e) The $n-$harmonic mean

$$H_n(a,b) = 2^{n+1}\left[\frac{1}{a} + 2\sum_{i=1}^{2^n-1} \frac{1}{\left(1-\frac{i}{2^n}\right)a + \frac{i}{2^n}b} + \frac{1}{b}\right]^{-1}$$
$$(n = 0,1,2,\dots, \quad a,b>0)$$

f) The $n-$arithmetic mean

$$A_n(a,b) = 2^n\left[\sum_{i=1}^{2^n} \frac{1}{\left(1-\frac{i}{2^n}+\frac{1}{2^{n+1}}\right)a + \left(\frac{i}{2^n}-\frac{1}{2^{n+1}}\right)b}\right]^{-1}$$
$$(n = 0,1,2,\dots; a,b>0).$$

It is clear that $H_0(a,b) = H(a,b)$ and $A_0(a,b) = A(a,b)$. By the above terminology we have the following simple proposition:

Proposition 3.1. *Let* $0 < a < b < \infty$. *Then we have*

$$H(a,b) \le H_n(a,b) \le L(a,b) \le A_n(a,b) \le A(a,b),$$
$$\lim_{n\to\infty} H_n(a,b) = \lim_{n\to\infty} A_n(a,b) = L(a,b).$$

Proof. Let $f : [a,b] \to (0,\infty)$, $f(x) = \frac{1}{x}$ and use Remark 1. We omit the details. □

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