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SOME PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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ABSTRACT. The object of the present paper is to derive some properties of analytic functions defined by the Carlson - Shaffer linear operator $L(a,c)f(z)$.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.3}$$

Define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \tag{1.4}$$

$$(a \in \mathbb{R};\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\ldots\},\ z \in \mathcal{U}),$$

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where $(\lambda)_n$ is the Pochhammer symbol given in terms of Gamma functions,

$$\begin{aligned}(\lambda)_n &:= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1, & n=0, \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n \in \mathbb{N}:\{1,2,\dots\}. \end{cases}\end{aligned}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] introduced a linear operator $L(a,c): \mathcal{A} \to \mathcal{A}$ by

$$L(a,c)f(z) := \phi(a,c;z) * f(z), \tag{1.5}$$

or, equivalently, by

$$L(a,c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \qquad (z \in \mathcal{U}). \tag{1.6}$$

It follows from (1.6) that

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z), \tag{1.7}$$

and

$$\begin{aligned} L(1,1)f(z) &= f(z), \quad L(2,1)f(z) = zf'(z), \\ L(3,1)f(z) &= zf'(z) + \frac{1}{2}z^2 f''(z). \end{aligned}$$

Many properties of analytic functions defined by the Carlson-Shaffer linear operator were studied by (among others) Owa and Srivastava [10], Ding [5], Kim and Lee [6], Ravichandran *et al.* ([8, 9]), Shanmugam *et al.* [7] and Frasin ([2, 3]).

In this paper we shall derive some properties of analytic functions defined by the linear operator $L(a,c)f(z)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1 ([4]). *Let $\Phi(u,v)$ be a complex valued function,*

$$\Phi : D \to \mathbb{C}, \qquad (D \subset \mathbb{C}^2; \mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\Phi(u,v)$ satisfies:

(i) $\Phi(u,v)$ *is continuous in* D;
(ii) $(1,0) \in D$ *and* $\mathrm{Re}(\Phi(1,0)) > 0$;
(iii) $\mathrm{Re}(\Phi(iu_2, v_1)) \le 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \le -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in \mathcal{U}$. If $\mathrm{Re}(\Phi(p(z), zp'(z))) > 0 \ (z \in \mathcal{U})$, then $\mathrm{Re}(p(z)) > 0 \ (z \in \mathcal{U})$.

2. MAIN RESULTS

Theorem 2.1. *Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \ge 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that*

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] + a\alpha \right\} > \gamma \tag{2.1}$$

for some $\gamma(\gamma < (\alpha-\beta)(a+1))$ and $2(\alpha-\beta) + \mathrm{Re}(\beta) \ne 0$, then

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \right\} > \frac{2\gamma - 2a(\alpha-\beta) + \mathrm{Re}(\beta)}{2(\alpha-\beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.2}$$

Proof. Define the function $p(z)$ by

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} := \delta + (1-\delta)p(z), \tag{2.3}$$

where

$$\delta = \frac{2\gamma - 2a(\alpha-\beta) + \operatorname{Re}(\beta)}{2(\alpha-\beta)+\operatorname{Re}(\beta)} < 1. \tag{2.4}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is regular in $\mathcal{U}$. It follows from (2.3) that

$$\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)} = \frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)} \tag{2.5}$$

by making use of the familiar identity (1.7) in (2.5), we get

$$\frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} = 1 + \frac{a}{\delta+(1-\delta)p(z)} - \frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)} \tag{2.6}$$

or, equivalently,

$$\begin{aligned}\frac{L(a,c)f(z)}{L(a+1,c)f(z)}&\left[\alpha - \beta\frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)}\right] + a\alpha \\ &= (\delta+a)(\alpha-\beta) + (1-\delta)(\alpha-\beta)p(z) + \beta(1-\delta)zp'(z).\end{aligned} \tag{2.7}$$

Therefore, we have

$$\begin{aligned}\operatorname{Re}&\left\{\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\left[\alpha - \beta\frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)}\right] + a\alpha - \gamma\right\} \\ &= \operatorname{Re}\left\{(\delta+a)(\alpha-\beta) - \gamma + (1-\delta)(\alpha-\beta)p(z) + \beta(1-\delta)zp'(z)\right\} > 0.\end{aligned} \tag{2.8}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = (\delta+a)(\alpha-\beta) - \gamma + (1-\delta)(\alpha-\beta)u + \beta(1-\delta)v \tag{2.9}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1,0) \in D$ and $\operatorname{Re}(\Phi(1,0)) = (\alpha-\beta)(a+1) - \gamma > 0$;

(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \le -(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2,v_1)) &= \operatorname{Re}\{(\delta+a)(\alpha-\beta) - \gamma + \beta(1-\delta)v_1\} \\ &\le (\delta+a)(\alpha-\beta) - \gamma - \frac{(1-\delta)(1+u_2^2)\operatorname{Re}(\beta)}{2} \\ &= -\frac{(1-\delta)u_2^2\operatorname{Re}(\beta)}{2} \le 0.\end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1.1. Thus we have $\operatorname{Re}(p(z)) > 0\,(z \in \mathcal{U})$, that is

$$\operatorname{Re}\left\{\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\right\} > \frac{2\gamma - 2a(\alpha-\beta)+\operatorname{Re}(\beta)}{2(\alpha-\beta)+\operatorname{Re}(\beta)}.$$

□

Letting $\beta = -\bar{\alpha}$ in Theorem 2.1, we have

Corollary 2.2. *Let* $\alpha \in \mathbb{C}$, $(\mathrm{Re}(\alpha) < 0)$, *and suppose that*

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left[\alpha + \bar{\alpha} \frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] + a\alpha \right\} > \gamma \tag{2.10}$$

for some γ $(\gamma < 2(a+1)\,\mathrm{Re}(\alpha))$, *then*

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \right\} > \frac{2\gamma - (4a+1)\,\mathrm{Re}(\alpha)}{3\,\mathrm{Re}(\alpha)} \qquad (z \in \mathcal{U}). \tag{2.11}$$

Letting $a = c = 1$ in Theorem 2.1, we have

Corollary 2.3. *Let* $\alpha, \beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, *and suppose that*

$$\mathrm{Re}\left\{ \frac{f(z)}{zf'(z)} \left[\alpha - 2\beta \left(1 + \frac{zf''(z)}{2f'(z)} \right) \right] + \alpha \right\} > \gamma \tag{2.12}$$

for some $\gamma(\gamma < 2(\alpha - \beta))$ *and* $2(\alpha-\beta) + \mathrm{Re}(\beta) \neq 0$, *then*

$$\mathrm{Re}\left\{ \frac{f(z)}{zf'(z)} \right\} > \frac{2\gamma - 2(\alpha-\beta) + \mathrm{Re}(\beta)}{2(\alpha-\beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.13}$$

Theorem 2.4. *Let* $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, *and suppose that*

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] + a\alpha \right\} < \gamma \tag{2.14}$$

for some $\gamma(\gamma > (\alpha-\beta)(a+1))$ *and* $2(\alpha-\beta) + \mathrm{Re}(\beta) \neq 0$, *then*

$$\mathrm{Re}\left\{ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \right\} < \frac{2\gamma - 2a(\alpha-\beta) + \mathrm{Re}(\beta)}{2(\alpha-\beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.15}$$

Proof. Define the function $p(z)$ by

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} := \delta + (1-\delta)p(z), \tag{2.16}$$

where

$$\delta = \frac{2\gamma - 2a(\alpha-\beta) + \mathrm{Re}(\beta)}{2(\alpha-\beta) + \mathrm{Re}(\beta)} > 1. \tag{2.17}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is regular in $\mathcal{U}$. It follows from (2.16) that

$$\begin{aligned} \mathrm{Re}\left\{ \gamma - \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] - a\alpha \right\} \\ = \mathrm{Re}\left\{ \gamma - (\delta + a)(\alpha-\beta) - (1-\delta)(\alpha-\beta)p(z) - \beta(1-\delta)zp'(z) \right\} > 0. \end{aligned} \tag{2.18}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = \gamma - (\delta + a)(\alpha-\beta) - (1-\delta)(\alpha-\beta)u - \beta(1-\delta)v \tag{2.19}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1,0) \in D$ and $\mathrm{Re}(\Phi(1,0)) = \gamma - (\alpha-\beta)(a+1) > 0$;

(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2$,

$$\begin{aligned} \mathrm{Re}(\Phi(iu_2, v_1)) &= \mathrm{Re}\{\gamma - (\delta+a)(\alpha-\beta) - \beta(1-\delta)v_1\} \\ &\leq \gamma - (\delta+a)(\alpha-\beta) + \frac{(1-\delta)(1+u_2^2)\,\mathrm{Re}(\beta)}{2} \\ &= \frac{(1-\delta)u_2^2\,\mathrm{Re}(\beta)}{2} \leq 0, \end{aligned}$$

applying Lemma 1.1, we have $\mathrm{Re}(p(z)) > 0\,(z \in \mathcal{U})$, that is

$$\mathrm{Re}\left\{\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\right\} < \frac{2\gamma - 2a(\alpha-\beta) + \mathrm{Re}(\beta)}{2(\alpha-\beta) + \mathrm{Re}(\beta)}.$$

□

Theorem 2.5. *Let* $a > -1$*,* $\mu > 0$*,* $\alpha \in \mathbb{C}$*,* $\beta \in \mathbb{C}$*,* $(\mathrm{Re}(\beta) \geq 0)$*,* $(\alpha+\beta) \in \mathbb{R}$*, and suppose that*

$$\mathrm{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\left[\alpha + \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\} > \gamma \tag{2.20}$$

for some $\gamma(\gamma < \alpha+\beta)$ *and* $2\mu(\alpha+\beta)(a+1) + \mathrm{Re}(\beta) \neq 0$*, then*

$$\mathrm{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\right\} > \frac{2\mu\gamma(a+1) + \mathrm{Re}(\beta)}{2\mu(\alpha+\beta)(a+1) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.21}$$

Proof. Define the function $p(z)$ by

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} := \delta + (1-\delta)p(z), \tag{2.22}$$

where

$$\delta = \frac{2\mu\gamma(a+1) + \mathrm{Re}(\beta)}{2\mu(\alpha+\beta)(a+1) + \mathrm{Re}(\beta)} > 1.$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is regular in $\mathcal{U}$. Also, by a simple computation and by making use of the familiar identity (1.7) we find from (2.22) that

$$\frac{\beta L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{\beta(1-\delta)zp'(z)}{\mu(a+1)(\delta + (1-\delta)p(z))} + \beta, \tag{2.23}$$

and by using (2.22) and (2.23), we get

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\left(\frac{\beta L(a+2,c)f(z)}{L(a+1,c)f(z)} + \alpha\right) = \frac{\beta(1-\delta)zp'(z)}{\mu(a+1)} + (\alpha+\beta)(\delta + (1-\delta)p(z)). \tag{2.24}$$

Therefore, we have

$$\begin{aligned}\mathrm{Re}&\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\left(\frac{\beta L(a+2,c)f(z)}{L(a+1,c)f(z)} + \alpha\right) - \gamma\right\} \\ &= \mathrm{Re}\left\{(\alpha+\beta)\delta - \gamma + (1-\delta)(\alpha+\beta)p(z) + \frac{\beta(1-\delta)}{\mu(a+1)}zp'(z)\right\} > 0.\end{aligned} \tag{2.25}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = (\alpha+\beta)\delta - \gamma + (1-\delta)(\alpha+\beta)u + \frac{\beta(1-\delta)}{\mu(a+1)}v \tag{2.26}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then
(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1,0) \in D$ and $\mathrm{Re}(\Phi(1,0)) = (\alpha+\beta) - \gamma > 0$;

(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2, v_1)) &= \operatorname{Re}\{(\alpha+\beta)\delta - \gamma + \frac{\beta(1-\delta)}{\mu(a+1)}v_1\} \\ &\leq (\alpha+\beta)\delta - \gamma - \frac{(1-\delta)(1+u_2^2)\operatorname{Re}(\beta)}{2\mu(a+1)} \\ &= -\frac{(1-\delta)u_2^2 \operatorname{Re}(\beta)}{2\mu(a+1)} \leq 0.\end{aligned}$$

Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1.1. Thus we have $\operatorname{Re}(p(z)) > 0\,(z \in \mathcal{U})$, that is,

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\right\} > \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha+\beta)(a+1) + \operatorname{Re}(\beta)} \qquad (z \in \mathcal{U}).$$

□

Letting $\alpha = \bar{\beta}$ in Theorem 2.5, we have

Corollary 2.6. *Let* $a > -1$, $\mu > 0$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) > 0)$*, and suppose that*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\left[\alpha + \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\} > \gamma \tag{2.27}$$

for some $\gamma(\gamma < 2\operatorname{Re}(\beta))$*, then*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\right\} > \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{(4\mu(a+1)+1)\operatorname{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.28}$$

Letting $a = c = 1$ in Theorem 2.5, we have,

Corollary 2.7. *Let* $\mu > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha+\beta) \in \mathbb{R}$*, and suppose that*

$$\operatorname{Re}\left\{(f'(z))^{\mu}\left[\alpha + \beta\left(1 + \frac{zf''(z)}{2f'(z)}\right)\right]\right\} > \gamma \tag{2.29}$$

for some $\gamma(\gamma < \alpha+\beta)$ *and* $4\mu(\alpha+\beta) + \operatorname{Re}(\beta) \neq 0$*, then*

$$\operatorname{Re}\left\{(f'(z))^{\mu}\right\} > \frac{4\mu\gamma + \operatorname{Re}(\beta)}{4\mu(\alpha+\beta) + \operatorname{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.30}$$

Employing the same manner as in the proofs of Theorems 2.4 and 2.5, we have:

Theorem 2.8. *Let* $a > -1$, $\mu > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha+\beta) \in \mathbb{R}$*, and suppose that*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\left[\alpha + \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\} < \gamma \tag{2.31}$$

for some $\gamma(\gamma > \alpha+\beta)$ *and* $2\mu(\alpha+\beta)(a+1) + \operatorname{Re}(\beta) \neq 0$*, then*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\right\} < \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha+\beta)(a+1) + \operatorname{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.32}$$

Theorem 2.9. *Let* $\lambda > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \geq 0)$, $(\alpha + \beta) \in \mathbb{R}$, *and suppose that*

$$\mathrm{Re}\left\{\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \times \left[\lambda\beta(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\lambda\beta\frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \lambda\alpha\right]\right\} > \gamma \tag{2.33}$$

for some $\gamma(\gamma < \lambda(\alpha+\beta))$ *and* $2\lambda(\alpha+\beta) + \mathrm{Re}(\beta) \neq 0$, *then*

$$\mathrm{Re}\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} > \frac{2\gamma + \mathrm{Re}(\beta)}{2\lambda(\alpha+\beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.34}$$

Proof. Define the function $p(z)$ by

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} := \delta + (1-\delta)p(z), \tag{2.35}$$

where

$$\delta = \frac{2\gamma + \mathrm{Re}(\beta)}{2\lambda(\alpha+\beta) + \mathrm{Re}(\beta)} < 1. \tag{2.36}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is regular in $\mathcal{U}$. Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.35) that

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda}\left[\lambda\beta(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\lambda\beta\frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \lambda\alpha\right] = \lambda(\alpha+\beta)(\delta + (1-\delta)p(z)) + \beta(1-\delta)zp'(z) \tag{2.37}$$

Therefore, we have

$$\begin{aligned}\mathrm{Re}\Bigg\{&\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \times \left[\lambda\beta(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\lambda\beta\frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \lambda\alpha\right] - \gamma\Bigg\} \\ &= \mathrm{Re}\left\{\lambda(\alpha+\beta)(\delta + (1-\delta)p(z)) + \beta(1-\delta)zp'(z) - \gamma\right\} > 0.\end{aligned} \tag{2.38}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = \lambda\delta(\alpha+\beta) - \gamma + \lambda(1-\delta)(\alpha+\beta)u + \beta(1-\delta)v \tag{2.39}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1,0) \in D$ and $\mathrm{Re}(\Phi(1,0)) = \lambda(\alpha+\beta) - \gamma > 0$;

(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2$,

$$\begin{aligned}\mathrm{Re}(\Phi(iu_2, v_1)) &= \mathrm{Re}\{\lambda\delta(\alpha+\beta) - \gamma + \beta(1-\delta)v_1\} \\ &\leq \lambda\delta(\alpha+\beta) - \gamma - \frac{(1-\delta)(1+u_2^2)}{2}\mathrm{Re}(\beta) \\ &= -\frac{(1-\delta)u_2^2\,\mathrm{Re}(\beta)}{2} \leq 0.\end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1.1. Thus we have

$$\operatorname{Re}\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} > \frac{2\gamma+\operatorname{Re}(\beta)}{2\lambda(\alpha+\beta)+\operatorname{Re}(\beta)} \qquad (z\in\mathcal{U}).$$

□

Letting $\alpha=\bar{\beta}$ in Theorem 2.9, we have:

Corollary 2.10. *Let* $\lambda>0$, $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$, $(\operatorname{Re}(\beta)>0)$, $(\alpha+\beta)\in\mathbb{R}$, *and suppose that*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \times\left[\lambda\beta(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}-a\lambda\beta\frac{L(a+1,c)f(z)}{L(a,c)f(z)}+\lambda\bar{\beta}\right]\right\}>\gamma \tag{2.40}$$

for some $\gamma(\gamma<\lambda(\alpha+\beta))$, *then*

$$\operatorname{Re}\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} > \frac{2\gamma+\operatorname{Re}(\beta)}{(4\lambda+1)\operatorname{Re}(\beta)} \qquad (z\in\mathcal{U}). \tag{2.41}$$

Letting $a=c=\lambda=1$ in Theorem 2.9, we have:

Corollary 2.11. *Let* $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$, $(\operatorname{Re}(\beta)\geq 0)$, $(\alpha+\beta)\in\mathbb{R}$, *and suppose that:*

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)\left[(2\beta+\alpha)+\beta z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right]\right\}>\gamma \tag{2.42}$$

for some $\gamma(\gamma<\alpha+\beta)$ *and* $2(\alpha+\beta)+\operatorname{Re}(\beta)\neq 0$, *then* $f(z)$ *is starlike of order* δ, *where* $\delta=\frac{2\gamma+\operatorname{Re}(\beta)}{2(\alpha+\beta)+\operatorname{Re}(\beta)}$.

Employing the same manner as in the proofs of Theorems 2.4 and 2.9, we have:

Theorem 2.12. *Let* $\lambda>0$, $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$, $(\operatorname{Re}(\beta)\geq 0)$, $(\alpha+\beta)\in\mathbb{R}$, *and suppose that:*

$$\operatorname{Re}\left\{\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \times\left[\lambda\beta(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}-a\lambda\beta\frac{L(a+1,c)f(z)}{L(a,c)f(z)}+\lambda\alpha\right]\right\}<\gamma \tag{2.43}$$

for some $\gamma(\gamma>\lambda(\alpha+\beta))$ *and* $2\lambda(\alpha+\beta)+\operatorname{Re}(\beta)\neq 0$, *then*

$$\operatorname{Re}\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} < \frac{2\gamma+\operatorname{Re}(\beta)}{2\lambda(\alpha+\beta)+\operatorname{Re}(\beta)} \qquad (z\in\mathcal{U}). \tag{2.44}$$

Theorem 2.13. *Let* $a>-1$, $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$, $(\operatorname{Re}(\beta)\geq 0)$, $(\alpha-\beta)\in\mathbb{R}$, *and suppose that*

$$\operatorname{Re}\left\{\frac{z}{L(a+1,c)f(z)}\left[(a+1)\alpha-\beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\}>\gamma \tag{2.45}$$

for some $\gamma(\gamma<(\alpha-\beta)(a+1))$, *then*

$$\operatorname{Re}\left\{\frac{z}{L(a+1,c)f(z)}\right\} > \frac{2\gamma+\operatorname{Re}(\beta)}{2(a+1)(\alpha-\beta)+\operatorname{Re}(\beta)} \qquad (z\in\mathcal{U}). \tag{2.46}$$

Proof. Define the function $p(z)$ by

$$\frac{z}{L(a+1,c)f(z)} := \delta + (1-\delta)p(z), \tag{2.47}$$

where

$$\delta = \frac{2\gamma + \operatorname{Re}(\beta)}{2(a+1)(\alpha-\beta) + \operatorname{Re}(\beta)} > 1. \tag{2.48}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is regular in $\mathcal{U}$. Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.47) that

$$\begin{aligned}\frac{z}{L(a+1,c)f(z)} &\left[(a+1)\alpha - \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right] \\ &= (a+1)(\alpha-\beta)(\delta + (1-\delta)p(z)) + \beta(1-\delta)zp'(z).\end{aligned} \tag{2.49}$$

Therefore, we have

$$\begin{aligned}\operatorname{Re}&\left\{\frac{z}{L(a+1,c)f(z)}\left[(a+1)\alpha - \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right] - \gamma\right\} \\ &= \operatorname{Re}\{(a+1)(\alpha-\beta)(\delta + (1-\delta)p(z)) + \beta(1-\delta)zp'(z) - \gamma\} > 0.\end{aligned} \tag{2.50}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = \delta(a+1)(\alpha-\beta) - \gamma + (a+1)(\alpha-\beta)(1-\delta)u + \beta(1-\delta)v \tag{2.51}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then
(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1,0) \in D$ and $\operatorname{Re}(\Phi(1,0)) = (\alpha-\beta)(a+1) - \gamma > 0$;
(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \le -(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2, v_1)) &= \operatorname{Re}\{\delta(a+1)(\alpha-\beta) - \gamma + \beta(1-\delta)v_1\} \\ &\le \delta(a+1)(\alpha-\beta) - \gamma - \frac{(1-\delta)(1+u_2^2)}{2}\operatorname{Re}(\beta) \\ &= -\frac{(1-\delta)u_2^2\operatorname{Re}(\beta)}{2} \le 0.\end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1.1. Thus we have

$$\operatorname{Re}\left\{\frac{z}{L(a+1,c)f(z)}\right\} > \frac{2\gamma + \operatorname{Re}(\beta)}{2(a+1)(\alpha-\beta) + \operatorname{Re}(\beta)} \qquad (z \in \mathcal{U}).$$

□

Letting $\beta = -\bar{\alpha}$ in Theorem 2.13, we have:

Corollary 2.14. *Let* $a > -1, a \neq 0, \alpha \in \mathbb{C}, (\operatorname{Re}(\alpha) < 0)$, *and suppose that*

$$\operatorname{Re}\left\{\frac{z}{L(a+1,c)f(z)}\left[(a+1)\alpha + \bar{\alpha}\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\} > \gamma \tag{2.52}$$

for some $\gamma(\gamma < 2(a+1)\operatorname{Re}(\alpha))$, *then*

$$\operatorname{Re}\left\{\frac{z}{L(a+1,c)f(z)}\right\} > \frac{2\gamma - \operatorname{Re}(\alpha)}{(4a+3)\operatorname{Re}(\alpha)} \qquad (z \in \mathcal{U}). \tag{2.53}$$

Letting $a = c = 1$ in Theorem 2.13, we have:

Corollary 2.15. *Let* $a > -1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, *and suppose that*

$$\mathrm{Re}\left\{\frac{1}{f'(z)}\left[2\alpha - \beta - \frac{zf''(z)}{2f'(z)}\beta\right]\right\} > \gamma \tag{2.54}$$

for some $\gamma(\gamma < 2(\alpha - \beta))$ *and* $4(\alpha - \beta) + \mathrm{Re}(\beta) \neq 0$, *then*

$$\mathrm{Re}\left\{\frac{1}{f'(z)}\right\} > \frac{2\gamma + \mathrm{Re}(\beta)}{4(\alpha - \beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.55}$$

Employing the same manner as in the proofs of Theorem 2.4 and 2.13, we have:

Theorem 2.16. *Let* $a > -1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\mathrm{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, *and suppose that*

$$\mathrm{Re}\left\{\frac{z}{L(a+1,c)f(z)}\left[(a+1)\alpha - \beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right]\right\} < \gamma \tag{2.56}$$

for some $\gamma(\gamma > (\alpha - \beta)(a+1))$ *and* $2(a+1)(\alpha - \beta) + \mathrm{Re}(\beta) \neq 0$, *then*

$$\mathrm{Re}\left\{\frac{z}{L(a+1,c)f(z)}\right\} < \frac{2\gamma + \mathrm{Re}(\beta)}{2(a+1)(\alpha - \beta) + \mathrm{Re}(\beta)} \qquad (z \in \mathcal{U}). \tag{2.57}$$

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