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UNIFORMLY STARLIKE AND UNIFORMLY CONVEX FUNCTIONS PERTAINING TO SPECIAL FUNCTIONS

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ABSTRACT. The main object of this paper is to derive the sufficient conditions for the function $z\{{}_p\psi_q(z)\}$ to be in the classes of uniformly starlike and uniformly convex functions. Similar results using integral operator are also obtained.

Key words and phrases: Analytic functions, Univalent functions, Starlike functions, Convex functions, Integral operator, Fox-Wright function.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$.

Also let S denote the subclass of A consisting of all functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1.2}$$

A function $f \in A$ is said to be starlike of order α, $0 \leq \alpha < 1$, if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, $z \in \Delta$. Also f of the form (1.1) is uniformly starlike, whenever $\left(\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\right) \geq 0$, $(z,\xi) \in$

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$\Delta \times \Delta$. This class of all uniformly starlike functions is denoted by UST [4] (see also [5], [10] and [14]).

The function f of the form (1.1) is uniformly convex in Δ whenever $\mathrm{Re}\left(1 + (z-\xi)\frac{f''(z)}{f'(z)}\right) \geq 0$, $(z,\xi) \in \Delta \times \Delta$. This class of all uniformly convex functions is denoted by UCV [3] (also refer [2], [6], [9] and [13]). Further it is said to be in the class $UCV(\alpha)$, $\alpha \geq 0$ if $\mathrm{Re}\left(1 + \frac{zf'(z)}{f(z)}\right) \geq \alpha \left|\frac{zf''(z)}{f'(z)}\right|$.

A function f of the form (1.2) is said to be in the class $USTN(\alpha)$, $0 \leq \alpha \leq 1$, if $\mathrm{Re}\left(\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\right) \geq \alpha$, $(z,\xi) \in \Delta \times \Delta$.

In the present paper, we shall use analogues of the lemmas in [8] and [7] respectively in the following form.

Lemma 1.1. *A function f of the form (1.1) is in the class $UST(\alpha)$, if*

$$\sum_{n=2}^{\infty}[(3-\alpha)n-2]\,|a_n| \leq (1-\alpha)M_1,$$

where $M_1 > 0$ is a suitable constant. In particular, $f \in UST$ whenever

$$\sum_{n=2}^{\infty}(3n-2)\,|a_n| \leq M_1.$$

Lemma 1.2. *A sufficient condition for a function f of the form (1.1) to be in the class $UCV(\alpha)$ is that $\sum_{n=2}^{\infty} n[(\alpha+1)n-\alpha]\;a_n \leq M_2$, where $M_2 > 0$ is a suitable constant. In particular, $f \in UCV$ whenever $\sum_{n=2}^{\infty} n^2 a_n \leq M_2$.*

The Fox-Wright function [12, p. 50, equation 1.5] appearing in the present paper is defined by

$$(1.3)\qquad {}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(a_j,\alpha_j)_{1,p};\\(b_j,\beta_j)_{1,q};\end{array}\; z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)z^n}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!},$$

where α_j $(j = 1,\ldots,p)$ and β_j $(j = 1,\ldots,q)$ are real and positive and $1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j$.

2. MAIN RESULTS

Theorem 2.1. *If*

$$\sum_{j=1}^{q}|b_j| > \sum_{j=1}^{p}|a_j| + 1,\;\; a_j > 0 \qquad \text{and} \qquad 1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j,$$

then a sufficient condition for the function $z\{{}_p\psi_q(z)\}$ to be in the class $UST(\alpha)$, $0 \leq \alpha < 1$, is

$$(2.1)\quad \left(\frac{3-\alpha}{1-\alpha}\right){}_p\psi_q\left[\begin{array}{c}(|a_j+\alpha_j|,\alpha_j)_{1,p};\\(|b_j+\beta_j|,\beta_j)_{1,q};\end{array}\;1\right] + {}_p\psi_q\left[\begin{array}{c}(|a_j|,\alpha_j)_{1,p};\\(|b_j|,\beta_j)_{1,q};\end{array}\;1\right] \leq M_1 + \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.$$

Proof. Since

$$z\{{}_p\psi_q(z)\} = \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}z + \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n-1)]z^n}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n-1)](n-1)!}$$

so by virtue of Lemma 1.1, we need only to show that

$$(2.2)\qquad \sum_{n=2}^{\infty}[(3-\alpha)n-2]\left|\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n-1)](n-1)!}\right| \leq (1-\alpha)M_1.$$

Now, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(3-\alpha)n-2]\left|\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n-1)](n-1)!}\right| \\
&\qquad\qquad = \sum_{n=0}^{\infty}[(3-\alpha)(n+2)-2]\left|\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n+1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n+1)](n+1)!}\right| \\
&\qquad\qquad = (3-\alpha)\sum_{n=0}^{\infty}\left|\frac{\prod_{j=1}^{p}\Gamma[(a_j+\alpha_j)+n\alpha_j]}{\prod_{j=1}^{q}\Gamma[(b_j+\beta_j)+n\beta_j]n!}\right| \\
&\qquad\qquad\qquad + (1-\alpha)\left[\sum_{n=0}^{\infty}\left|\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\right|\frac{1}{n!}-\frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\right] \\
&\qquad\qquad = (3-\alpha)\ {}_p\psi_q\left[\begin{array}{l}(|a_j+\alpha_j|,\alpha_j)_{1,p};\\ (|b_j+\beta_j|,\beta_j)_{1,q};\end{array}1\right] \\
&\qquad\qquad\qquad + (1-\alpha)\ {}_p\psi_q\left[\begin{array}{l}(|a_j|,\alpha_j)_{1,p};\\ (|b_j|,\beta_j)_{1,q};\end{array}1\right]-(1-\alpha)\frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j} \\
&\qquad\qquad \le (1-\alpha)M_1
\end{aligned}$$

which in view of Lemma 1.1 gives the desired result. □

Theorem 2.2. *If*

$$\sum_{j=1}^{q}b_j>\sum_{j=1}^{p}a_j+1,\ \ a_j>0 \qquad \textit{and} \qquad 1+\sum_{j=1}^{q}\beta_j>\sum_{j=1}^{p}\alpha_j,$$

then a sufficient condition for the function $z\{{}_p\psi_q(z)\}$ *to be in the class* $USTN(\alpha)$, $0\le\alpha<1$, *is:*

$$\left(\frac{3-\alpha}{1-\alpha}\right)\ {}_p\psi_q\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j)_{1,p};\\ (b_j+\beta_j,\beta_j)_{1,q};\end{array}1\right]+\ {}_p\psi_q\left[\begin{array}{l}(a_j,\alpha_j)_{1,p};\\ (b_j,\beta_j)_{1,q};\end{array}1\right]\le M_1+\frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.$$

Proof. The proof of Theorem 2.2 is a direct consequence of Theorem 2.1. □

Theorem 2.3. *If*

$$\sum_{j=1}^{q}b_j>\sum_{j=1}^{p}a_j+2, a_j>0 \qquad \textit{and} \qquad 1+\sum_{j=1}^{q}\beta_j>\sum_{j=1}^{p}\alpha_j,$$

then a sufficient condition for the function $z\{{}_p\psi_q(z)\}$ *to be in the class* $UCV(\alpha)$, $0\le\alpha<1$, *is*

$$\begin{aligned}
(2.3)\quad &(1+\alpha)\ {}_p\psi_q\left[\begin{array}{l}(a_j+2\alpha_j,\alpha_j)_{1,p};\\ (b_j+2\beta_j,\beta_j)_{1,q};\end{array}1\right] \\
&\qquad + (2\alpha+3)\ {}_p\psi_q\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j)_{1,p};\\ (b_j+\beta_j,\beta_j)_{1,q};\end{array}1\right]+\ {}_p\psi_q(1)\le M_2+\frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.
\end{aligned}$$

Proof. By virtue of Lemma 1.2, it suffices to prove that

$$(2.4)\qquad \sum_{n=2}^{\infty}n[(\alpha+1)n-\alpha]\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n-1)](n-1)!}\le M_2.$$

Now, we have

$$(2.5)\quad \sum_{n=2}^{\infty} n[(\alpha+1)n-\alpha]\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(n-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(n-1)](n-1)!}$$
$$= (1+\alpha)\sum_{n=1}^{\infty}(n+1)^2\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma[(b_j+\beta_j n)n!} - \alpha\sum_{n=1}^{\infty}(n+1)\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!}.$$

Using $(n+1)^2 = n(n+1)+(n+1)$, (2.5) may be expressed as

$$(2.6)\quad (1+\alpha)\sum_{n=1}^{\infty}(n+1)\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)(n-1)!} + \sum_{n=1}^{\infty}(n+1)\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!}$$
$$= (1+\alpha)\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)(n-2)!} + (2\alpha+3)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j+\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j+\beta_j)+\beta_j n]n!}$$
$$+ \sum_{n=1}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!}$$
$$= (1+\alpha)\ {}_p\psi_q\left[\begin{array}{l}(a_j+2\alpha_j,\alpha_j)_{1,p};\\ (b_j+2\beta_j,\beta_j)_{1,q};\end{array} 1\right] + (2\alpha+3)\ {}_p\psi_q\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j)_{1,p};\\ (b_j+\beta_j,\beta_j)_{1,q};\end{array} 1\right]$$
$$+\ {}_p\psi_q(1) - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j},$$

which is bounded above by M_2 if and only if (2.3) holds. Hence the theorem is proved. □

3. AN INTEGRAL OPERATOR

In this section we obtain sufficient conditions for the function

$${}_p\phi_q\left[\begin{array}{l}(a_j,\alpha_j)_{1,p};\\ (b_j,\beta_j)_{1,q};\end{array} z\right] = \int_0^z {}_p\psi_q(x)dx$$

to be in the classes UST and UCV.

Theorem 3.1. *If*

$$\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j,\ \ a_j>0 \qquad \textit{and} \qquad 1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j,$$

then a sufficient condition for the function ${}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx$ *to be in the class* UST *is*

$$(3.1)\quad 3\ {}_p\psi_q(1) - 2\ {}_p\psi_q\left[\begin{array}{l}(a_j-\alpha_j,\alpha_j)_{1,p};\\ (b_j-\beta_j,\beta_j)_{1,q};\end{array} 1\right] + 2\frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} \le M_1 + \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.$$

Proof. Since

$$(3.2)\quad {}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx = \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}z + \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]}\frac{z^n}{n!},$$

we have

$$\begin{aligned}
(3.3)\qquad &\sum_{n=2}^{\infty}(3n-2)\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]n!}\\
&= 3\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!} - 2\left[\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]n!}\right.\\
&\qquad\left.-\frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\right]\\
&= 3\ {}_p\psi_q(1) - 2\ {}_p\psi_q\left[\begin{array}{l}(a_j-\alpha_j,\alpha_j)_{1,p};\\ (b_j-\beta_j,\beta_j)_{1,q};\end{array} 1\right] + 2\frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.
\end{aligned}$$

In view of Lemma 1.1, (3.3) leads to the result (3.1). □

Theorem 3.2. *If*

$$\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j,\, a_j > 0 \qquad \textit{and} \qquad 1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j,$$

then a sufficient condition for the function ${}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx$ *to be in the class* UCV *is*

$$(3.4)\qquad {}_p\psi_q\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j)_{1,p};\\ (b_j+\beta_j,\beta_j)_{1,q};\end{array} 1\right] + {}_p\psi_q(1) \le M_2 + \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}.$$

Proof. Since ${}_p\phi_q(z)$ has the form (3.2), then

$$\begin{aligned}
(3.5)\qquad &\sum_{n=2}^{\infty} n^2\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]n!}\\
&= \sum_{n=1}^{\infty}(n+1)\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!}\\
&= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j+\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j+\beta_j)+\beta_j n]n!} + \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\\
&= {}_p\psi_q\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j)_{1,p};\\ (b_j+\beta_j,\beta_j)_{1,q};\end{array} 1\right] + {}_p\psi_q(1) - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j},
\end{aligned}$$

which in view of Lemma 1.2 gives the desired result (3.4). □

4. PARTICULAR CASES

4.1. By setting $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 1; \beta_1 = \beta_2 = \cdots = \beta_q = 1$ and

$$M_1 = M_2 = M_3 = \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j},$$

Theorems 2.1, 2.3, 3.1 and 3.2 reduce to the results recently obtained by Shanmugam, Ramachandran, Sivasubramanian and Gangadharan [11].

4.2. By specifying the parameters suitably, the results of this paper readily yield the results due to Dixit and Verma [1].

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