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SOME RESULTS ASSOCIATED WITH FRACTIONAL CALCULUS OPERATORS INVOLVING APPELL HYPERGEOMETRIC FUNCTION

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ABSTRACT. A class of fractional derivative operators (with the Appell hypergeometric function in the kernel) is used here to define a new subclass of analytic functions and a coefficient bound inequality is established for this class of functions. Also, an inclusion theorem for a class of fractional integral operators involving the Hardy space of analytic functions is proved. The concluding remarks briefly mentions the relevances of the main results and possibilities of further work by using these new classes of fractional calculus operators.

Key words and phrases: Analytic functions, Hardy space, Fractional derivatives and fractional integrals, Appell hypergeometric function, Inclusion relation.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\Delta_n^{(\alpha,\alpha',\beta,\beta',\gamma)}(\sigma)$ the subclass of functions in $\mathcal{A}(n)$ which also satisfy the inequality:

$$\operatorname{Re}\left\{\chi_1(\alpha,\alpha',\beta,\beta',\gamma) z^{\alpha+\alpha'+\gamma-1} D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\} > \sigma \qquad (z \in \mathbb{U}), \tag{1.2}$$

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where $D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}$ is the generalized fractional derivative operator (defined below), and (for convenience)

$$\begin{aligned}(1.3)\quad &\chi_m(\alpha,\alpha',\beta,\beta',\gamma)\\ &= \frac{\Gamma(1+m+\beta')\Gamma(1+m-\alpha-\alpha'-\gamma)\Gamma(1+m-\alpha'-\beta-\gamma)}{\Gamma(1+m)\Gamma(1+m-\alpha'+\beta')\Gamma(1+m-\alpha-\alpha'-\beta-\gamma)} \quad (m\in\mathbb{N}),\end{aligned}$$

provided that

$$\begin{aligned}(1.4)\qquad & 0\le\sigma<1; 0\le\gamma<1;\\ &\gamma< \min(-\alpha-\alpha', -\alpha'-\beta, -\alpha-\alpha'-\beta)+m+1;\\ &\beta'> \max(0,\alpha')-m-1.\end{aligned}$$

Following [8], a function $f(z)$ is said to be in the class $\mathcal{V}_n(\theta_k)$ if $f(z)\in\mathcal{A}(n)$ satisfies the condition that

$$\arg(a_k)=\theta_k \quad (k\ge n+1; n\in\mathbb{N})$$

and if there exists a real number ρ such that

$$(1.5)\qquad \theta_k+(k-1)\rho\equiv\pi(\mathrm{mod}2\pi) \quad (k\ge n+1; n\in\mathbb{N}),$$

then we say that $f(z)$ is in the class $\mathcal{V}_n(\theta_k;\rho)$. Suppose $\mathcal{V}_n=\cup\mathcal{V}_n(\theta_k;\rho)$ over all possible sequences θ_k with ρ satisfying (1.5), then we denote by $\nabla_n^{(\alpha,\alpha',\beta,\beta',\gamma)}(\sigma)$ the subclass of $\mathcal{V}_n$ which consists of functions $f(z)$ belonging to the class $\Delta_n^{(\alpha,\alpha',\beta,\beta',\gamma)}(\sigma)$.

We present here the following family of fractional integral (and derivative) operators which involve the familiar Appell hypergeometric function F_3 (see also Kiryakova [4] and Saigo and Maeda [9]).

Definition 1.1. Let $\gamma>0$ and $\alpha,\alpha',\beta,\beta'\in\mathbb{R}$. Then the fractional integral operator $I_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}$ of a function $f(z)$ is defined by

$$\begin{aligned}(1.6)\quad &I_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}f(z)\\ &=\frac{z^{-\alpha}}{\Gamma(\gamma)}\int_0^z (z-\zeta)^{\gamma-1}\zeta^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{\zeta}{z},1-\frac{z}{\zeta}\right)f(\zeta)d\zeta \quad (\gamma>0),\end{aligned}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, and it is understood that $(z-\zeta)^{\gamma-1}$ denotes the principal value for $0\leqq\arg(z-t)<2\pi$. The function F_3 occurring in the kernel of (1.6) is the familiar Appell hypergeometric function of third type (also known as Horn's F_3 - function; see, for example, [10]) defined by

$$(1.7)\qquad F_3(\alpha,\alpha',\beta,\beta';\gamma;z,\xi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m}{m!}\frac{\xi^n}{n!} \quad (|z|<1,|\xi|<1),$$

which is related to the Gaussian hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ by the following relationship:

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;z)&=F_3(\alpha,\alpha',\beta,\beta';\gamma;z,0)\\ &=F_3(\alpha,0,\beta,\beta';\gamma;z,\xi)=F_3(\alpha,\alpha',\beta,0;\gamma;z,\xi).\end{aligned}$$

Definition 1.2. The fractional derivative operator $D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}$ of a function $f(z)$ is defined by

$$(1.8)\qquad D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}f(z)=\frac{d^n}{dz^n}I_{0,z}^{(\alpha,\alpha',\beta-n,\beta',n-\gamma)}f(z) \quad (n-1\le\gamma<n; n\in\mathbb{N}).$$

It may be observed that for

$$\alpha = \lambda + \mu, \quad \alpha' = \beta' = 0, \quad \beta = -\eta, \quad \gamma = \lambda, \tag{1.9}$$

we obtain the relationship

$$I_{0,z}^{(\lambda+\mu,0,-\eta,0,\lambda)} = I_{0,z}^{\lambda,\mu,\eta} \tag{1.10}$$

in terms of the Saigo type fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ ([12]). On the other hand, if

$$\alpha = \mu - \lambda, \quad \alpha' = \beta' = 0, \quad \beta = -\eta, \quad \gamma = \lambda, \tag{1.11}$$

then we get

$$D_{0,z}^{(\mu-\lambda,0,-\eta,0,\lambda)} = J_{0,z}^{\lambda,\mu,\eta}, \tag{1.12}$$

where $J_{0,z}^{(\lambda,\mu,\eta}$ is the Saigo type fractional derivative operator ([6]; see also [7]). Further, when

$$\alpha = \beta' = 0, \quad \alpha' = 1 - \mu, \quad \gamma = \lambda \text{ (or } -\lambda), \tag{1.13}$$

then the operators $I_{0,z}^{(0,1-\mu,0,0,\lambda)}$ and $D_{0,z}^{(0,1-\mu,0,0,-\lambda)}$ correspond to the differential-integral operators Q_μ^λ due to Dziok [2].

Let $\mathcal{H}^p$ $(0 \leq p < \infty)$ be the class of analytic functions in $\mathbb{U}$ such that

$$\|f\|_p = \lim_{r\to 1-} \{M_p(r, f)\} < \infty, \tag{1.14}$$

where

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p\right)^{\frac{1}{p}} & (0 < p < \infty), \\ \sup\limits_{|z|\leq r} |f(z)|. \end{cases} \tag{1.15}$$

In this paper we first define a new function class in terms of the fractional derivative operators (with the Appell hypergeometric function in the kernel) and then establish a coefficient bound inequality for this function class. Also, we prove an inclusion theorem for a class of fractional integral operators involving the Hardy space of analytic functions. The relevance of the main results and possibilities of further work by using the new classes of fractional calculus operators are briefly pointed out in the concluding section of this paper.

2. A SET OF COEFFICIENT BOUNDS

We begin by proving the following coefficient bounds inequality for a function $f(z)$ to be in the class $\Delta_n^{(\alpha,\alpha',\beta,\beta',\gamma)}(\sigma)$.

Theorem 2.1. *Let $f(z)$ defined by (1.1) be in the class $\Delta_n^{(\alpha,\alpha',\beta,\beta',\gamma)}(\sigma)$, then*

$$\sum_{k=n+1}^{\infty} \frac{|a_k|}{\chi_k(\alpha,\alpha',\beta,\beta',\gamma)} \leq \frac{1-\sigma}{\chi_1(\alpha,\alpha',\beta,\beta',\gamma)}, \tag{2.1}$$

where $\chi_m(\alpha,\alpha',\beta,\beta',\gamma)$ is defined by (1.3). The result is sharp.

Proof. Assume that

$$\operatorname{Re}\left\{\chi_1(\alpha,\alpha',\beta,\beta',\gamma) z^{\alpha+\alpha'-\gamma-1} D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\} > \sigma \quad (z \in \mathbb{U}).$$

Using (1.1) and the formula (see, e.g. [9, p. 394]):

$$D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)} z^q = \frac{\Gamma(1+q)\Gamma(1+q-\alpha'+\beta')\Gamma(1+q-\alpha-\beta-\gamma)}{\Gamma(1+q+\beta')\Gamma(1+q-\alpha'-\beta-\gamma)\Gamma(1+q-\alpha-\alpha'-\gamma)} z^{q-\alpha-\alpha'-\gamma}, \tag{2.2}$$

$$(0 \leq \gamma < 1; \alpha, \alpha', \beta, \beta' \in \mathbb{R}; q > \max(0, \alpha'-\beta', \alpha+\beta+\gamma) - 1)$$

we obtain

$$\operatorname{Re}\left\{1 + \sum_{k=n+1}^{\infty} \frac{\chi_1(\alpha,\alpha',\beta,\beta',\gamma)}{\chi_k(\alpha,\alpha',\beta,\beta',\gamma)} a_k z^{k-1}\right\} > \sigma \quad (z \in \mathbb{U}), \tag{2.3}$$

and for $f(z) \in \mathcal{V}_n(\theta_k;\rho)$ ($z = re^{i\theta}$), the inequality thus obtainable from (2.3) on letting $r \to 1-$ therein, readily yields

$$\operatorname{Re}\left\{1 + \sum_{k=n+1}^{\infty} \frac{\chi_1(\alpha,\alpha',\beta,\beta',\gamma)}{\chi_k(\alpha,\alpha',\beta,\beta',\gamma)} |a_k| \exp\left(i\left(\theta_k + (k-1)\rho\right)\right)\right\} > \sigma. \tag{2.4}$$

If we apply (1.5), then (2.4) gives

$$1 - \sum_{k=n+1}^{\infty} \frac{\chi_1(\alpha,\alpha',\beta,\beta',\gamma)}{\chi_k(\alpha,\alpha',\beta,\beta',\gamma)} |a_k| > \sigma, \tag{2.5}$$

which leads to the desired inequality (2.1). We also observe that the equality sign in (2.1) is attained for the function $f(z)$ defined by

$$f(z) = z + \frac{(1-\sigma)\chi_k(\alpha,\alpha',\beta,\beta',\gamma)}{\chi_1(\alpha,\alpha',\beta,\beta',\gamma)} z^k \exp(i\theta_k) \quad (k \geq n+1; n \in \mathbb{N}), \tag{2.6}$$

and this completes the proof of Theorem 2.1. □

3. INCLUSION RELATIONS

Under the hypotheses of Definition 1.1, let

$$\gamma > 0; \min(\gamma-\alpha-\alpha', \gamma-\alpha'-\beta, \beta', \gamma-\alpha-\alpha'-\beta, \beta'-\alpha') > -2; \quad \alpha, \alpha', \beta, \beta' \in \mathbb{R}, \tag{3.1}$$

then the fractional integral operator

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} : \mathcal{A} \to \mathcal{A} \quad (\mathcal{A}(1) = \mathcal{A})$$

is defined by

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) = \chi_1(\alpha,\alpha',\beta,\beta',-\gamma) z^{\alpha+\alpha'+\gamma} I_{(0,z)}^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z). \tag{3.2}$$

where $\chi_1(\alpha,\alpha',\beta,\beta',-\gamma)$ is given by (1.3).

By using the formula ([9, p. 394]; see also [4, p. 170, Lemma 9])

$$I_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)} z^q = \frac{\Gamma(1+q)\Gamma(1+q-\alpha'+\beta')\Gamma(1+q-\alpha-\alpha'-\beta+\gamma)}{\Gamma(1+q+\beta')\Gamma(1+q-\alpha'-\beta+\gamma)\Gamma(1+q-\alpha-\alpha'+\gamma)} z^{q-\alpha-\alpha'+\gamma}, \tag{3.3}$$

$$(\gamma > 0; \alpha, \alpha', \beta, \beta' \in \mathbb{R}; q > \max(0, \alpha'-\beta', \alpha+\beta-\gamma) - 1)$$

it follows from (1.1), (3.2) and (3.3) that

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) = z + \chi_1(\alpha,\alpha',\beta,\beta',-\gamma) \sum_{k=2}^{\infty} \frac{a_k}{\chi_k(\alpha,\alpha',\beta,\beta',-\gamma)} z^k, \tag{3.4}$$

where (as before) $\chi_k(\alpha,\alpha',\beta,\beta',-\gamma)$ is given by (1.3).

Before stating and proving our main inclusion theorem, we recall here the following known results concerning the class $\mathcal{R}(\rho)$ in $\mathcal{A}$ which satisfies the inequality that $\Re\{f'(z)\} > \rho (0 \le \rho < 1)$, where $\mathcal{R}(1)$ is denoted by $\mathcal{R}$.

Lemma 3.1 ([3, p. 141])**.** *Let $f(z) \in \mathcal{R}$, then*

$$f(z) \in \mathcal{H}^p \quad : (0 < p < \infty). \tag{3.5}$$

Lemma 3.2 ([5, p. 533])**.** *Let $f(z)$ defined by (1.1) be in the class $\mathcal{R}(\rho)$ $(0 \le \rho < 1)$, then*

$$|a_k| \le \frac{2}{k} \quad (k = 2, 3, 4, ...). \tag{3.6}$$

Theorem 3.3. *Let $f(z) \in \mathcal{R}$, then (under the constraints stated in (3.1))*

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \in \mathcal{H}^p \quad (0 < p < \infty) \tag{3.7}$$

and

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \in \mathcal{H}^\infty \quad (\gamma > 1). \tag{3.8}$$

Proof. In view of (1.6) and (3.2), we obtain

$$\begin{aligned} \Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) &= \chi_1(\alpha,\alpha',\beta,\beta',-\gamma) \\ &\quad \times \int_0^1 (1-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t,1-\frac{1}{t}\right) f(zt)dt. \end{aligned} \tag{3.9}$$

This implies that

$$\begin{aligned} &\operatorname{Re}\left\{\frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\} \\ &= \chi_1(\alpha,\alpha',\beta,\beta',-\gamma) \int_0^1 (1-t)^{\gamma-1} t^{1-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t,1-\frac{1}{t}\right) \Re\left\{f'(zt)\right\} dt. \end{aligned} \tag{3.10}$$

Since $f(z) \in \mathcal{R}$, therefore, we infer from (3.10) that

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \in \mathcal{R}, \tag{3.11}$$

and applying Lemma 3.1, (3.11) gives the inclusion relation (3.7) under the conditions stated in (3.1).

To prove the result (3.8), we observe the following three-term recurrence relation:

$$\begin{aligned} &\frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \\ &= z^{-1}\left\{(\gamma-\alpha'-\beta+1)\,\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma-1)} f(z) - (\gamma-\alpha'-\beta)\,\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\}, \end{aligned} \tag{3.12}$$

which yields the inequality

$$\begin{aligned} \left|\frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right|^p &\le r^{-p}\left\{(\gamma-\alpha'-\beta+1)^p \left|\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma-1)} f(z)\right|^p \right. \\ &\quad \left. - (\gamma-\alpha'-\beta)^p \left|\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right|^p\right\} \quad (|\mathrm{z}| = r), \end{aligned} \tag{3.13}$$

provided that

$$\gamma > 1; \min\left(1+\gamma-\alpha-\alpha', 1+\gamma-\alpha'-\beta, 1+\beta', 1+\gamma-\alpha-\alpha'-\beta, 1+\beta'-\alpha'\right) > -1; \quad \alpha, \alpha', \beta, \beta' \in \mathbb{R} \tag{3.14}$$

and $0 < p < \infty$.

Making use of (1.14) and (1.15), the above inequality (3.13) (with $p = 1$) yields

$$M_1\left(r, \frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right) \leq r^{-1}\left\{(\gamma-\alpha'-\beta+1)\, M_1\left(r, \Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma-1)} f(z)\right) - (\gamma-\alpha'-\beta)\, M_1\left(r, \Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right)\right\} \tag{3.15}$$

and

$$\left\|\frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\|_1 \leq (\gamma-\alpha'-\beta+1)\left\|\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma-1)} f(z)\right\|_1 - (\gamma-\alpha'-\beta)\left\|\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right\|_1. \tag{3.16}$$

Applying (3.7), we infer (under the constraints stated in (3.14)) that

$$\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma-1)} f(z) \in \mathcal{H}^1 \quad \text{and} \quad \Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \in \mathcal{H}^1 \quad (\gamma > 1), \tag{3.17}$$

and consequently (3.16) implies that

$$\frac{d}{dz}\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z) \in \mathcal{H}^1,$$

provided that the conditions stated in (3.14) are satisfied. By appealing to a known result [1, p. 42, Theorem 3.11], we infer from (3.17) that $\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)$ is continuous in $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } |z| \geq 1\}$. But $\mathbb{U}^*$ being compact, we finally conclude that $\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)$ is a bounded analytic function in $\mathbb{U}$, and the proof of the second assertion (3.8) of Theorem 3.3 is complete.

The assertion (3.8) of Theorem 3.3 can also be proved by applying Lemma 3.2 (see also [3, p. 145]). Indeed, it follows from (3.4) and (3.6) that

$$\begin{aligned}\left|\Omega_z^{(\alpha,\alpha',\beta,\beta',\gamma)} f(z)\right| &\leq |z| + \chi_1(\alpha,\alpha',\beta,\beta',-\gamma)\sum_{k=2}^{\infty}\frac{|a_k|}{\chi_k(\alpha,\alpha',\beta,\beta',-\gamma)}\left|z^k\right| \\ &\leq 1 + 2 : \chi_1(\alpha,\alpha',\beta,\beta',-\gamma)\sum_{k=2}^{\infty}\frac{\Gamma(k)}{\Gamma(k+1)\chi_k(\alpha,\alpha',\beta,\beta',-\gamma)} \\ &= 1 + \frac{2(2-\alpha'+\beta')(2-\alpha-\alpha'-\beta+\gamma)}{(2+\beta')(2-\alpha-\alpha'+\gamma)(2-\alpha'-\beta+\gamma)} \\ &\quad \times {}_4F_3\left[\begin{array}{c} 1, 2, 3-\alpha'+\beta', 3-\alpha-\alpha'-\beta+\gamma; \\ 3+\beta', 3-\alpha-\alpha'+\gamma, 3-\alpha'-\beta+\gamma; \end{array}\middle| 1\right]\end{aligned}$$

in terms of the generalized hypergeometric function.

Now, for fixed values of the parameters $\alpha, \alpha', \beta, \beta', \gamma$ satisfying the conditions stated in (3.1), we observe that by using the asymptotic formula [10, p. 109],

$$\frac{\Gamma(k)}{\Gamma(k+1)\chi_k(\alpha,\alpha',\beta,\beta',-\gamma)} = o\left(k^{-\gamma-1}\right) \qquad (k \to \infty),$$

and since $\gamma > 1$, this proves our assertion (3.8). □

4. CONCLUDING REMARKS

In view of the relationships (1.10) and (1.12), the main results (Theorems 2.1 and 3.3) of this paper would correspond to the results due to Raina and Srivastava [8, p. 75, Theorem 1; p. 79, Theorem 7]. Furthermore, in view of the relationship (1.13), we can easily apply Theorems 2.1 and 3.3 to obtain the corresponding results associated with Dziok's differential-integral operators [2]. The family of fractional calculus operators (fractional integrals and fractional derivatives) defined by (1.6) and (1.8) can fruitfully be used in Geometric Function Theory. Several new analytic, multivalent (or meromorphic) function classes can be defined and the various properties of coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity for such contemplated classes investigated.

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