

# LITTLEWOOD-PALEY DECOMPOSITION ASSOCIATED WITH THE DUNKL OPERATORS AND PARAPRODUCT OPERATORS

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*Abstract:* We define the Littlewood-Paley decomposition associated with the Dunkl operators; from this decomposition we give the characterization of the Sobolev, Hölder and Lebesgue spaces associated with the Dunkl operators. We construct the paraproduct operators associated with the Dunkl operators similar to those defined by J.M. Bony in [1]. Using the Littlewood-Paley decomposition we establish the Sobolev embedding, Gagliardo-Nirenberg inequality and we present the paraproduct algorithm.

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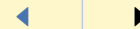
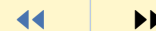
Littlewood-Paley Decomposition

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[Title Page](#)

[Contents](#)



Page 1 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>The Eigenfunction of the Dunkl Operators</b>	<b>5</b>
2.1	Reflection Groups, Root System and Multiplicity Functions . . . . .	5
2.2	Dunkl operators-Dunkl kernel and Dunkl intertwining operator . . .	6
2.3	The Dunkl Transform . . . . .	8
2.4	The Dunkl Convolution Operator . . . . .	10
<b>3</b>	<b>Littlewood-Paley Theory Associated with Dunkl Operators</b>	<b>13</b>
3.1	Dyadic Decomposition . . . . .	13
3.2	The Generalized Sobolev Spaces . . . . .	15
3.3	The Generalized Hölder Spaces . . . . .	22
<b>4</b>	<b>Applications</b>	<b>27</b>
4.1	Estimates of the Product of Two Functions . . . . .	27
4.2	Sobolev Embedding Theorem . . . . .	29
4.3	Gagliardo-Nirenberg Inequality . . . . .	35
<b>5</b>	<b>Paraproduct Associated with the Dunkl Operators</b>	<b>38</b>

Littlewood-Paley Decomposition

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vol. 9, iss. 4, art. 95, 2008

[Title Page](#)

[Contents](#)



Page 2 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

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issn: 1443-5756

[Title Page](#)[Contents](#)

Page 3 of 46

[Go Back](#)[Full Screen](#)[Close](#)

## 1. Introduction

The theory of function spaces appears at first to be a disconnected subject, because of the variety of spaces and the different considerations involved in their definitions. There are the Lebesgue spaces  $L^p(\mathbb{R}^d)$ , the Sobolev spaces  $H^s(\mathbb{R}^d)$ , the Besov spaces  $B_{p,q}^s(\mathbb{R}^d)$ , the BMO spaces (bounded mean oscillation) and others.

Nevertheless, several approaches lead to a unified viewpoint on these spaces, for example, approximation theory or interpolation theory. One of the most successful approaches is the Littlewood-Paley theory. This approach has been developed by the European school, which reached a similar unification of function space theory by a different path. Motivated by the methods of Hörmander in studying partial differential equations (see [6]), they used a Fourier transform approach. Pick Schwartz functions  $\phi$  and  $\chi$  on  $\mathbb{R}^d$  satisfying  $\text{supp } \widehat{\chi} \subset B(0, 2)$ ,  $\text{supp } \widehat{\phi} \subset \{\xi \in \mathbb{R}^d, \frac{1}{2} \leq \|\xi\| \leq 2\}$ , and the nondegeneracy condition  $|\widehat{\chi}(\xi)|, |\widehat{\phi}(\xi)| \geq C > 0$ . For  $j \in \mathbb{Z}$ , let  $\phi_j(x) = 2^{jd}\phi(2^jx)$ . In 1967 Peetre [10] proved that

$$(1.1) \quad \|f\|_{H^s(\mathbb{R}^d)} \simeq \|\chi * f\|_{L^2(\mathbb{R}^d)} + \left( \sum_{j \geq 1} 2^{2sj} \|\phi_j * f\|_{L^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$

Independently, Triebel [15] in 1973 and Lizorkin [8] in 1972 introduced  $F_{p,q}^s$  (the Triebel-Lizorkin spaces) defined originally for  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$  by the norm

$$(1.2) \quad \|f\|_{F_{p,q}^s} = \|\chi * f\|_{L^p(\mathbb{R}^d)} + \left\| \left( \sum_{j \geq 1} (2^{sj} |\phi_j * f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)}.$$

For the special case  $q = 1$  and  $s = 0$ , Triebel [16] proved that

$$(1.3) \quad L^p(\mathbb{R}^d) \simeq F_{p,2}^0.$$

Thus by the Littlewood-Paley decomposition we characterize the functional spaces  $L^p(\mathbb{R}^d)$ , Sobolev spaces  $H^s(\mathbb{R}^d)$ , Hölder spaces  $C^s(\mathbb{R}^d)$  and others. Using the



Title Page

Contents



Page 4 of 46

Go Back

Full Screen

Close

Littlewood-Paley decomposition J.M. Bony in [1], built the paraproduct operators which have been later successfully employed in various settings.

The purpose of this paper is to generalize the Littlewood-Paley theory, to unify and extend the paraproduct operators which allow the analysis of solutions to more general partial differential equations arising in applied mathematics and other fields. More precisely, we define the Littlewood-Paley decomposition associated with the Dunkl operators. We introduce the new spaces associated with the Dunkl operators, the Sobolev spaces  $H_k^s(\mathbb{R}^d)$ , the Hölder spaces  $C_k^s(\mathbb{R}^d)$  and the  $BMO_k(\mathbb{R}^d)$  that generalizes the corresponding classical spaces. The Dunkl operators are the differential-difference operators introduced by C.F. Dunkl in [3] and which played an important role in pure Mathematics and in Physics. For example they were a main tool in the study of special functions with root systems (see [4]).

As applications of the Littlewood-Paley decomposition we establish results analogous to (1.1) and (1.3), we prove the Sobolev embedding theorems, and the Gagliardo-Nirenberg inequality. Another tool of the Littlewood-Paley decomposition associated with the Dunkl operators is to generalize the paraproduct operators defined by J.M. Bony. We prove results similar to [2].

The paper is organized as follows. In Section 2 we recall the main results about the harmonic analysis associated with the Dunkl operators. We study in Section 3 the Littlewood-Paley decomposition associated with the Dunkl operators, we give the sufficient condition on  $u_p$  so that  $u := \sum u_p$  belongs to Sobolev or Hölder spaces associated with the Dunkl operators. We finish this section by the Littlewood-Paley decomposition of the Lebesgue spaces  $L_k^p(\mathbb{R}^d)$  associated with the Dunkl operators. In Section 4 we give some applications. More precisely we establish the Sobolev embedding theorems and the Gagliardo-Nirenberg inequality. Section 5 is devoted to defining the paraproduct operators associated with the Dunkl operators and to giving the paraproduct algorithm.



## 2. The Eigenfunction of the Dunkl Operators

In this section we collect some notations and results on Dunkl operators and the Dunkl kernel (see [3], [4] and [5]).

### 2.1. Reflection Groups, Root System and Multiplicity Functions

We consider  $\mathbb{R}^d$  with the euclidean scalar product  $\langle \cdot, \cdot \rangle$  and  $\|x\| = \sqrt{\langle x, x \rangle}$ . On  $\mathbb{C}^d$ ,  $\|\cdot\|$  denotes also the standard Hermitian norm, while  $\langle z, w \rangle = \sum_{j=1}^d z_j \bar{w}_j$ .

For  $\alpha \in \mathbb{R}^d \setminus \{0\}$ , let  $\sigma_\alpha$  be the reflection in the hyperplane  $H_\alpha \subset \mathbb{R}^d$  orthogonal to  $\alpha$ , i.e.

$$(2.1) \quad \sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{\|\alpha\|^2} \alpha.$$

A finite set  $R \subset \mathbb{R}^d \setminus \{0\}$  is called a root system if  $R \cap \mathbb{R} \cdot \alpha = \{\alpha, -\alpha\}$  and  $\sigma_\alpha R = R$  for all  $\alpha \in R$ . For a given root system  $R$  the reflections  $\sigma_\alpha$ ,  $\alpha \in R$ , generate a finite group  $W \subset O(d)$ , called the reflection group associated with  $R$ . All reflections in  $W$  correspond to suitable pairs of roots. For a given  $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$ , we fix the positive subsystem  $R_+ = \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$ , then for each  $\alpha \in R$  either  $\alpha \in R_+$  or  $-\alpha \in R_+$ . We will assume that  $\langle \alpha, \alpha \rangle = 2$  for all  $\alpha \in R_+$ .

A function  $k : R \rightarrow \mathbb{C}$  on a root system  $R$  is called a multiplicity function if it is invariant under the action of the associated reflection group  $W$ . If one regards  $k$  as a function on the corresponding reflections, this means that  $k$  is constant on the conjugacy classes of reflections in  $W$ . For brevity, we introduce the index

$$(2.2) \quad \gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha).$$

Title Page

Contents

◀▶

◀▶

Page 5 of 46

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)



Page 6 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

Moreover, let  $\omega_k$  denote the weight function

$$(2.3) \quad \omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree  $2\gamma$ . We introduce the Mehta-type constant

$$(2.4) \quad c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx.$$

## 2.2. Dunkl operators-Dunkl kernel and Dunkl intertwining operator

**Notations.** We denote by

- $C(\mathbb{R}^d)$  (resp.  $C_c(\mathbb{R}^d)$ ) the space of continuous functions on  $\mathbb{R}^d$  (resp. with compact support).
- $\mathcal{E}(\mathbb{R}^d)$  the space of  $C^\infty$ -functions on  $\mathbb{R}^d$ .
- $\mathcal{S}(\mathbb{R}^d)$  the space of  $C^\infty$ -functions on  $\mathbb{R}^d$  which are rapidly decreasing as their derivatives.
- $D(\mathbb{R}^d)$  the space of  $C^\infty$ -functions on  $\mathbb{R}^d$  which are of compact support.

We provide these spaces with the classical topology.

Consider also the following spaces

- $\mathcal{E}'(\mathbb{R}^d)$  the space of distributions on  $\mathbb{R}^d$  with compact support. It is the topological dual of  $\mathcal{E}(\mathbb{R}^d)$ .



[Title Page](#)

[Contents](#)



Page 7 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

–  $\mathcal{S}'(\mathbb{R}^d)$  the space of temperate distributions on  $\mathbb{R}^d$ . It is the topological dual of  $\mathcal{S}(\mathbb{R}^d)$ .

The Dunkl operators  $T_j$ ,  $j = 1, \dots, d$ , on  $\mathbb{R}^d$  associated with the finite reflection group  $W$  and multiplicity function  $k$  are given by

$$(2.5) \quad T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^d).$$

In the case  $k = 0$ , the  $T_j$ ,  $j = 1, \dots, d$ , reduce to the corresponding partial derivatives. In this paper, we will assume throughout that  $k \geq 0$ .

For  $y \in \mathbb{R}^d$ , the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & j = 1, \dots, d, \\ u(0, y) = 1, & \text{for all } y \in \mathbb{R}^d \end{cases}$$

admits a unique analytic solution on  $\mathbb{R}^d$ , which will be denoted by  $K(x, y)$  and called the Dunkl kernel. This kernel has a unique holomorphic extension to  $\mathbb{C}^d \times \mathbb{C}^d$ . The Dunkl kernel possesses the following properties.

**Proposition 2.1.** *Let  $z, w \in \mathbb{C}^d$ , and  $x, y \in \mathbb{R}^d$ .*

i)

$$(2.6) \quad \begin{aligned} K(z, w) &= K(w, z), & K(z, 0) &= 1 \quad \text{and} \\ K(\lambda z, w) &= K(z, \lambda w), & & \text{for all } \lambda \in \mathbb{C}. \end{aligned}$$

ii) *For all  $\nu \in \mathbb{N}^d$ ,  $x \in \mathbb{R}^d$  and  $z \in \mathbb{C}^d$ , we have*

$$(2.7) \quad |D_z^\nu K(x, z)| \leq \|x\|^{|\nu|} \exp(\|x\| \| \operatorname{Re} z \|),$$



Title Page

Contents



Page 8 of 46

Go Back

Full Screen

Close

and for all  $x, y \in \mathbb{R}^d$ :

$$(2.8) \quad |K(ix, y)| \leq 1,$$

with  $D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} \dots \partial z_d^{\nu_d}}$  and  $|\nu| = \nu_1 + \dots + \nu_d$ .

iii) For all  $x, y \in \mathbb{R}^d$  and  $w \in W$  we have

$$(2.9) \quad K(-ix, y) = \overline{K(ix, y)} \quad \text{and} \quad K(wx, wy) = K(x, y).$$

The Dunkl intertwining operator  $V_k$  is defined on  $C(\mathbb{R}^d)$  by

$$(2.10) \quad V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad \text{for all } x \in \mathbb{R}^d,$$

where  $d\mu_x$  is a probability measure given on  $\mathbb{R}^d$ , with support in the closed ball  $B(0, \|x\|)$  of center 0 and radius  $\|x\|$ .

### 2.3. The Dunkl Transform

The results of this subsection are given in [7] and [18].

**Notations.** We denote by

–  $L_k^p(\mathbb{R}^d)$  the space of measurable functions on  $\mathbb{R}^d$  such that

$$\|f\|_{L_k^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) dx \right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \leq p < \infty,$$

$$\|f\|_{L_k^\infty(\mathbb{R}^d)} = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < \infty.$$





[Title Page](#)

[Contents](#)



Page 9 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

- $H(\mathbb{C}^d)$  the space of entire functions on  $\mathbb{C}^d$ , rapidly decreasing of exponential type.
- $\mathcal{H}(\mathbb{C}^d)$  the space of entire functions on  $\mathbb{C}^d$ , slowly increasing of exponential type.

We provide these spaces with the classical topology.

The Dunkl transform of a function  $f$  in  $D(\mathbb{R}^d)$  is given by

$$(2.11) \quad \mathcal{F}_D(f)(y) = \frac{1}{c_k} \int_{\mathbb{R}^d} f(x) K(-iy, x) \omega_k(x) dx, \quad \text{for all } y \in \mathbb{R}^d.$$

It satisfies the following properties:

i) For  $f$  in  $L_k^1(\mathbb{R}^d)$  we have

$$(2.12) \quad \|\mathcal{F}_D(f)\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{1}{c_k} \|f\|_{L_k^1(\mathbb{R}^d)}.$$

ii) For  $f$  in  $\mathcal{S}(\mathbb{R}^d)$  we have

$$(2.13) \quad \forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y), \quad j = 1, \dots, d.$$

iii) For all  $f$  in  $L_k^1(\mathbb{R}^d)$  such that  $\mathcal{F}_D(f)$  is in  $L_k^1(\mathbb{R}^d)$ , we have the inversion formula

$$(2.14) \quad f(y) = \int_{\mathbb{R}^d} \mathcal{F}_D(f)(x) K(ix, y) \omega_k(x) dx, \quad a.e.$$

**Theorem 2.2.** *The Dunkl transform  $\mathcal{F}_D$  is a topological isomorphism.*

i) *From  $\mathcal{S}(\mathbb{R}^d)$  onto itself.*



Title Page

Contents



Page 10 of 46

Go Back

Full Screen

Close

ii) From  $D(\mathbb{R}^d)$  onto  $H(\mathbb{C}^d)$ .

The inverse transform  $\mathcal{F}_D^{-1}$  is given by

$$(2.15) \quad \forall y \in \mathbb{R}^d, \quad \mathcal{F}_D^{-1}(f)(y) = \mathcal{F}_D(f)(-y), \quad f \in S(\mathbb{R}^d).$$

**Theorem 2.3.** The Dunkl transform  $\mathcal{F}_D$  is a topological isomorphism.

i) From  $\mathcal{S}'(\mathbb{R}^d)$  onto itself.

ii) From  $\mathcal{E}'(\mathbb{R}^d)$  onto  $\mathcal{H}(\mathbb{C}^d)$ .

**Theorem 2.4.**

i) Plancherel formula for  $\mathcal{F}_D$ . For all  $f$  in  $\mathcal{S}(\mathbb{R}^d)$  we have

$$(2.16) \quad \int_{\mathbb{R}^d} |f(x)|^2 \omega_k(x) dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi.$$

ii) Plancherel theorem for  $\mathcal{F}_D$ . The Dunkl transform  $f \rightarrow \mathcal{F}_D(f)$  can be uniquely extended to an isometric isomorphism on  $L_k^2(\mathbb{R}^d)$ .

## 2.4. The Dunkl Convolution Operator

**Definition 2.5.** Let  $y$  be in  $\mathbb{R}^d$ . The Dunkl translation operator  $f \mapsto \tau_y f$  is defined on  $\mathcal{S}(\mathbb{R}^d)$  by

$$(2.17) \quad \mathcal{F}_D(\tau_y f)(x) = K(ix, y) \mathcal{F}_D(f)(x), \quad \text{for all } x \in \mathbb{R}^d.$$

*Example 2.1.* Let  $t > 0$ , we have

$$\tau_x(e^{-t\|\xi\|^2})(y) = e^{-t(\|x\|^2 + \|y\|^2)} K(2tx, y), \quad \text{for all } x \in \mathbb{R}^d.$$



Title Page

Contents



Page 11 of 46

Go Back

Full Screen

Close

*Remark 1.* The operator  $\tau_y$ ,  $y \in \mathbb{R}^d$ , can also be defined on  $\mathcal{E}(\mathbb{R}^d)$  by

$$(2.18) \quad \tau_y f(x) = (V_k)_x (V_k)_y [(V_k)^{-1}(f)(x+y)], \quad \text{for all } x \in \mathbb{R}^d$$

(see [18]).

At the moment an explicit formula for the Dunkl translation operators is known only in the following two cases. (See [11] and [13]).

- 1<sup>st</sup> case:  $d = 1$  and  $W = \mathbb{Z}_2$ .
- 2<sup>nd</sup> case: For all  $f$  in  $\mathcal{E}(\mathbb{R}^d)$  radial we have

$$(2.19) \quad \tau_y f(x) = V_k \left[ f_0 \left( \sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, \cdot \rangle} \right) \right] (x), \quad \text{for all } x \in \mathbb{R}^d,$$

with  $f_0$  the function on  $[0, \infty[$  given by

$$f(x) = f_0(\|x\|).$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [11] and [18]).

**Definition 2.6.** *The Dunkl convolution product of  $f$  and  $g$  in  $D(\mathbb{R}^d)$  is the function  $f *_D g$  defined by*

$$(2.20) \quad f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy, \quad \text{for all } x \in \mathbb{R}^d.$$

This convolution is commutative, associative and satisfies the following properties. (See [13]).



### Proposition 2.7.

i) For  $f$  and  $g$  in  $D(\mathbb{R}^d)$  (resp.  $\mathcal{S}(\mathbb{R}^d)$ ) the function  $f *_D g$  belongs to  $D(\mathbb{R}^d)$  (resp.  $\mathcal{S}(\mathbb{R}^d)$ ) and we have

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y), \quad \text{for all } y \in \mathbb{R}^d.$$

ii) Let  $1 \leq p, q, r \leq \infty$ , such that  $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ . If  $f$  is in  $L_k^p(\mathbb{R}^d)$  and  $g$  is a radial element of  $L_k^q(\mathbb{R}^d)$ , then  $f *_D g \in L_k^r(\mathbb{R}^d)$  and we have

$$(2.21) \quad \|f *_D g\|_{L_k^r(\mathbb{R}^d)} \leq \|f\|_{L_k^p(\mathbb{R}^d)} \|g\|_{L_k^q(\mathbb{R}^d)}.$$

iii) Let  $W = \mathbb{Z}_2^d$ . We have the same result for all  $f \in L_k^p(\mathbb{R}^d)$  and  $g \in L_k^q(\mathbb{R}^d)$ .

Title Page

Contents



Page 12 of 46

Go Back

Full Screen

Close

### 3. Littlewood-Paley Theory Associated with Dunkl Operators

We consider now a dyadic decomposition of  $\mathbb{R}^d$ .

#### 3.1. Dyadic Decomposition

For  $p \geq 0$  be a natural integer, we set

$$(3.1) \quad C_p = \{\xi \in \mathbb{R}^d; 2^{p-1} \leq \|\xi\| \leq 2^{p+1}\} = 2^p C_0$$

and

$$(3.2) \quad C_{-1} = B(0, 1) = \{\xi \in \mathbb{R}^d; \|\xi\| \leq 1\}.$$

Clearly  $\mathbb{R}^d = \bigcup_{p=-1}^{\infty} C_p$ .

*Remark 2.* We remark that

$$(3.3) \quad \text{card} \left\{ q; C_p \cap C_q \neq \emptyset \right\} \leq 2.$$

Now, let us define a dyadic partition of unity that we shall use throughout this paper.

**Lemma 3.1.** *There exist positive functions  $\varphi$  and  $\psi$  in  $D(\mathbb{R}^d)$ , radial with  $\text{supp } \psi \subset C_{-1}$ , and  $\text{supp } \varphi \subset C_0$ , such that for any  $\xi \in \mathbb{R}^d$  and  $n \in \mathbb{N}$ , we have*

$$\psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi) = 1$$

and

$$\psi(\xi) + \sum_{p=0}^n \varphi(2^{-p}\xi) = \psi(2^{-n}\xi).$$



Title Page

Contents



Page 13 of 46

Go Back

Full Screen

Close

[Title Page](#)[Contents](#)

Page 14 of 46

[Go Back](#)[Full Screen](#)[Close](#)

*Remark 3.* It is not hard to see that for any  $\xi \in \mathbb{R}^d$

$$(3.4) \quad \frac{1}{2} \leq \psi^2(\xi) + \sum_{p=0}^{\infty} \varphi^2(2^{-p}\xi) \leq 2.$$

**Definition 3.2.** Let  $\lambda \in \mathbb{R}$ . For  $\chi$  in  $\mathcal{S}(\mathbb{R}^d)$ , we define the pseudo-differential-difference operator  $\chi(\lambda T)$  by

$$\mathcal{F}_D(\chi(\lambda T)u) = \chi(\lambda\xi)\mathcal{F}_D(u), \quad u \in \mathcal{S}'(\mathbb{R}^d).$$

**Definition 3.3.** For  $u$  in  $\mathcal{S}'(\mathbb{R}^d)$ , we define its Littlewood-Paley decomposition associated with the Dunkl operators (or dyadic decomposition)  $\{\Delta_p u\}_{p=-1}^{\infty}$  as  $\Delta_{-1}u = \psi(T)u$  and for  $q \geq 0$ ,  $\Delta_q u = \varphi(2^{-q}T)u$ .

Now we go to see in which case we can have the identity

$$Id = \sum_{p \geq -1} \Delta_p.$$

This is described by the following proposition.

**Proposition 3.4.** For  $u$  in  $\mathcal{S}'(\mathbb{R}^d)$ , we have  $u = \sum_{p=-1}^{\infty} \Delta_p u$ , in the sense of  $\mathcal{S}'(\mathbb{R}^d)$ .

*Proof.* For any  $f$  in  $\mathcal{S}(\mathbb{R}^d)$ , it is easy to see that  $\mathcal{F}_D(f) = \sum_{p=-1}^{\infty} \mathcal{F}_D(\Delta_p f)$  in the sense of  $\mathcal{S}(\mathbb{R}^d)$ . Then for any  $u$  in  $\mathcal{S}'(\mathbb{R}^d)$ , we have

$$\begin{aligned} \langle u, f \rangle &= \langle \mathcal{F}_D(u), \mathcal{F}_D(f) \rangle = \sum_{p=-1}^{\infty} \langle \mathcal{F}_D(u), \mathcal{F}_D(\Delta_p f) \rangle \\ &= \sum_{p=-1}^{\infty} \langle \mathcal{F}_D(\Delta_p u), \mathcal{F}_D(f) \rangle \end{aligned}$$

$$= \left\langle \sum_{p=-1}^{\infty} \mathcal{F}_D(\Delta_p u), \mathcal{F}_D(f) \right\rangle = \left\langle \sum_{p=-1}^{\infty} \Delta_p u, f \right\rangle.$$

The proof is finished. □

### 3.2. The Generalized Sobolev Spaces

In this subsection we will give a characterization of Sobolev spaces associated with the Dunkl operators by a Littlewood-Paley decomposition. First, we recall the definition of these spaces (see [9]).

**Definition 3.5.** Let  $s$  be in  $\mathbb{R}$ , we define the space  $H_k^s(\mathbb{R}^d)$  by

$$\{u \in \mathcal{S}'(\mathbb{R}^d) : (1 + \|\xi\|^2)^{\frac{s}{2}} \mathcal{F}_D(u) \in L_k^2(\mathbb{R}^d)\}.$$

We provide this space by the scalar product

$$(3.5) \quad \langle u, v \rangle_{H_k^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s \mathcal{F}_D(u)(\xi) \overline{\mathcal{F}_D(v)(\xi)} \omega_k(\xi) d\xi,$$

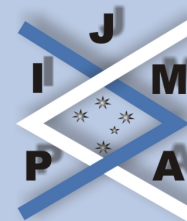
and the norm

$$(3.6) \quad \|u\|_{H_k^s(\mathbb{R}^d)}^2 = \langle u, u \rangle_{H_k^s(\mathbb{R}^d)}.$$

Another proposition will be useful. Let  $S_q u = \sum_{p \leq q-1} \Delta_p u$ .

**Proposition 3.6.** For all  $s$  in  $\mathbb{R}$  and for all distributions  $u$  in  $H_k^s(\mathbb{R}^d)$ , we have

$$\lim_{n \rightarrow \infty} S_n u = u.$$



Littlewood-Paley Decomposition

Hatem Mejjali

vol. 9, iss. 4, art. 95, 2008

Title Page

Contents



Page 15 of 46

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b



Title Page

Contents



Page 16 of 46

Go Back

Full Screen

Close

*Proof.* For all  $\xi$  in  $\mathbb{R}^d$ , we have

$$\mathcal{F}_D(S_n u - u)(\xi) = (\psi(2^{-n}\xi) - 1)\mathcal{F}_D(u)(\xi).$$

Hence

$$\lim_{n \rightarrow \infty} \mathcal{F}_D(S_n u - u)(\xi) = 0.$$

On the other hand

$$(1 + \|\xi\|^2)^s |\mathcal{F}_D(S_n u - u)(\xi)|^2 \leq 2(1 + \|\xi\|^2)^s |\mathcal{F}_D(u)(\xi)|^2.$$

Thus the result follows from the dominated convergence theorem. □

The first application of the Littlewood-Paley decomposition associated with the Dunkl operators is the characterization of the Sobolev spaces associated with these operators through the behavior on  $q$  of  $\|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}$ . More precisely, we now define a norm equivalent to the norm  $\|\cdot\|_{H_k^s(\mathbb{R}^d)}$  in terms of the dyadic decomposition.

**Proposition 3.7.** *There exists a positive constant  $C$  such that for all  $s$  in  $\mathbb{R}$ , we have*

$$\frac{1}{C^{|s|+1}} \|u\|_{H_k^s(\mathbb{R}^d)}^2 \leq \sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}^2 \leq C^{|s|+1} \|u\|_{H_k^s(\mathbb{R}^d)}^2.$$

*Proof.* Since  $\text{supp } \mathcal{F}_D(\Delta_q u) \subset C_q$ , from the definition of the norm  $\|\cdot\|_{H_k^s(\mathbb{R}^d)}$ , there exists a positive constant  $C$  such that we have

$$(3.7) \quad \frac{1}{C^{|s|+1}} 2^{qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)} \leq \|\Delta_q u\|_{H_k^s(\mathbb{R}^d)} \leq C^{|s|+1} 2^{qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}.$$

From (3.4) we deduce that

$$\begin{aligned} \frac{1}{2} \|u\|_{H_k^s(\mathbb{R}^d)}^2 &\leq \int_{\mathbb{R}^d} \left[ \psi^2(\xi) + \sum_{q=0}^{\infty} \varphi^2(2^{-q}\xi) \right] (1 + \|\xi\|^2)^s |\mathcal{F}_D(u)(\xi)|^2 \omega_k(\xi) d\xi \\ &\leq 2 \|u\|_{H_k^s(\mathbb{R}^d)}^2. \end{aligned}$$





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 17 of 46

Go Back

Full Screen

Close

Hence

$$\frac{1}{2} \|u\|_{H_k^s(\mathbb{R}^d)}^2 \leq \sum_{q \geq -1} \|\Delta_q u\|_{H_k^s(\mathbb{R}^d)}^2 \leq 2 \|u\|_{H_k^s(\mathbb{R}^d)}^2.$$

Thus from this and (3.7) we deduce the result.  $\square$

The following theorem is a consequence of Proposition 3.7.

**Theorem 3.8.** *Let  $u$  be in  $\mathcal{S}'(\mathbb{R}^d)$  and  $u = \sum_{q \geq -1} \Delta_q u$  its Littlewood-Paley decomposition. The following are equivalent:*

- i)  $u \in H_k^s(\mathbb{R}^d)$ .
- ii)  $\sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}^2 < \infty$ .
- iii)  $\|\Delta_q u\|_{L_k^2(\mathbb{R}^d)} \leq c_q 2^{-qs}$ , with  $\{c_q\} \in l^2$ .

*Remark 4.* Since for  $u$  in  $\mathcal{S}'(\mathbb{R}^d)$  we have  $\Delta_p u$  in  $\mathcal{S}'(\mathbb{R}^d)$  and  $\text{supp } \mathcal{F}_D(\Delta_p u) \subset C_p$ , from Theorem 2.3 ii) we deduce that  $\Delta_p u$  is in  $\mathcal{E}(\mathbb{R}^d)$ .

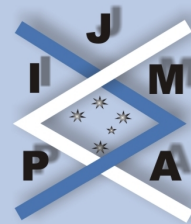
The following propositions will be very useful.

**Proposition 3.9.** *Let  $\tilde{C}$  be an annulus in  $\mathbb{R}^d$  and  $s$  in  $\mathbb{R}$ . Let  $(u_p)_{p \in \mathbb{N}}$  be a sequence of smooth functions. If the sequence  $(u_p)_{p \in \mathbb{N}}$  satisfies*

$$\text{supp } \mathcal{F}_D(u_p) \subset 2^p \tilde{C} \quad \text{and} \quad \|u_p\|_{L_k^2(\mathbb{R}^d)} \leq C c_p 2^{-ps}, \{c_p\} \in l^2,$$

then we have

$$u = \sum_{p \geq 0} u_p \in H_k^s(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H_k^s(\mathbb{R}^d)} \leq C(s) \left( \sum_{p \geq 0} 2^{2ps} \|u_p\|_{L_k^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$



Title Page

Contents

◀ ▶

◀ ▶

Page 18 of 46

Go Back

Full Screen

Close

*Proof.* Since  $\tilde{C}$  and  $C_0$  are two annuli, there exists an integer  $N_0$  so that

$$|p - q| > N_0 \implies 2^p C_0 \cap 2^q \tilde{C} = \emptyset.$$

It is clear that

$$|p - q| > N_0 \implies \mathcal{F}_D(\Delta_q u_p) = 0.$$

Then

$$\Delta_q u = \sum_{|p-q| \leq N_0} \Delta_q u_p.$$

By the triangle inequality and definition of  $\Delta_q u_p$  we deduce that

$$\|\Delta_q u\|_{L_k^2(\mathbb{R}^d)} \leq \sum_{|p-q| \leq N_0} \|u_p\|_{L_k^2(\mathbb{R}^d)}.$$

Thus the Cauchy-Schwartz inequality implies that

$$\sum_{q \geq 0} 2^{2qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}^2 \leq C \left( \sum_{q/|p-q| \leq N_0} 2^{2(q-p)s} \right) \left( \sum_{p \geq 0} 2^{2ps} \|u_p\|_{L_k^2(\mathbb{R}^d)}^2 \right).$$

From Theorem 3.8 we deduce that if  $\|u_p\|_{L_k^2(\mathbb{R}^d)} \leq C c_p 2^{-ps}$  then  $u \in H_k^s(\mathbb{R}^d)$ .  $\square$

**Proposition 3.10.** *Let  $K > 0$  and  $s > 0$ . Let  $(u_p)_{p \in \mathbb{N}}$  be a sequence of smooth functions. If the sequence  $(u_p)_{p \in \mathbb{N}}$  satisfies*

$$\text{supp } \mathcal{F}_D(u_p) \subset B(0, K2^p) \quad \text{and} \quad \|u_p\|_{L_k^2(\mathbb{R}^d)} \leq C c_p 2^{-ps}, \{c_p\} \in l^2,$$

then we have

$$u = \sum_{p \geq 0} u_p \in H_k^s(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H_k^s(\mathbb{R}^d)} \leq C(s) \left( \sum_{q \geq 0} 2^{2qs} \|u_q\|_{L_k^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$



Title Page

Contents



Page 19 of 46

Go Back

Full Screen

Close

*Proof.* Since  $\text{supp } \mathcal{F}_D(u_p) \subset B(0, K2^p)$ , there exists  $N_1$  such that

$$\Delta_q u = \sum_{p \geq q - N_1} \Delta_q u_p.$$

So, we get that

$$\begin{aligned} 2^{qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)} &\leq \sum_{p \geq q - N_1} 2^{qs} \|u_p\|_{L_k^2(\mathbb{R}^d)} \\ &= \sum_{p \geq q - N_1} 2^{(q-p)s} 2^{ps} \|u_p\|_{L_k^2(\mathbb{R}^d)}. \end{aligned}$$

Since  $s > 0$ , the Cauchy-Schwartz inequality implies

$$\sum_q 2^{2qs} \|\Delta_q u\|_{L_k^2(\mathbb{R}^d)}^2 \leq \frac{2^{2N_1 s}}{1 - 2^{-s}} \sum_p 2^{2ps} \|u_p\|_{L_k^2(\mathbb{R}^d)}^2.$$

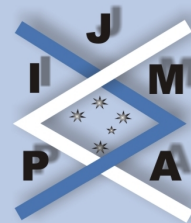
From Theorem 3.8 we deduce the result. □

**Proposition 3.11.** *Let  $s > 0$  and  $(u_p)_{p \in \mathbb{N}}$  be a sequence of smooth functions. If the sequence  $(u_p)_{p \in \mathbb{N}}$  satisfies*

$$u_p \in \mathcal{E}(\mathbb{R}^d) \quad \text{and} \quad \text{for all } \mu \in \mathbb{N}^d, \quad \|T^\mu u_p\|_{L_k^2(\mathbb{R}^d)} \leq C c_{p,\mu} 2^{-p(s-|\mu|)}, \quad \{c_{p,\mu}\} \in l^2,$$

then we have

$$u = \sum_{p \geq 0} u_p \in H_k^s(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H_k^s(\mathbb{R}^d)} \leq C(s) \left( \sum_{p \geq 0} 2^{2ps} \|u_p\|_{L_k^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$



Title Page

Contents



Page 20 of 46

Go Back

Full Screen

Close

*Proof.* By the assumption we first have  $u = \sum u_p \in L_k^2(\mathbb{R}^d)$ . Take  $\mu \in \mathbb{N}^d$  with  $|\mu| = s_0 > s > 0$ , and  $\chi_p(\xi) = \chi(2^{-p}\xi) \in D(\mathbb{R}^d)$  with  $\text{supp } \chi \subset B(0, 2)$ ,  $\chi(\xi) = 1$ ,  $\|\xi\| \leq 1$  and  $0 \leq \chi \leq 1$ , then

$$\text{supp } \chi_p(1 - \chi_p) \subset \{\xi \in \mathbb{R}^d; 2^p \leq \|\xi\| \leq 2^{p+2}\}.$$

Set

$$\begin{aligned} \mathcal{F}_D(u_p)(\xi) &= \chi_p(\xi)\mathcal{F}_D(u_p)(\xi) + (1 - \chi_p(\xi))\mathcal{F}_D(u_p)(\xi) \\ &= \mathcal{F}_D(u_p^{(1)})(\xi) + \mathcal{F}_D(u_p^{(2)})(\xi), \end{aligned}$$

and we have

$$\begin{aligned} \|u_p\|_{L_k^2(\mathbb{R}^d)}^2 &= \|\mathcal{F}_D(u_p)\|_{L_k^2(\mathbb{R}^d)}^2 \\ &= \left[ \int_{\mathbb{R}^d} |\mathcal{F}_D(u_p^{(1)})(\xi)|^2 \omega_k(\xi) d\xi + \int_{\mathbb{R}^d} |\mathcal{F}_D(u_p^{(2)})(\xi)|^2 \omega_k(\xi) d\xi \right. \\ &\quad \left. + 2 \int_{\mathbb{R}^d} |\mathcal{F}_D(u_p)(\xi)|^2 \chi_p(\xi)(1 - \chi_p(\xi)) \omega_k(\xi) d\xi \right]. \end{aligned}$$

Since  $0 \leq \chi_p(\xi)(1 - \chi_p(\xi)) \leq 1$ , we deduce that

$$\|u_p^{(1)}\|_{L_k^2(\mathbb{R}^d)}^2 + \|u_p^{(2)}\|_{L_k^2(\mathbb{R}^d)}^2 \leq \|u_p\|_{L_k^2(\mathbb{R}^d)}^2 \leq c_p^2 2^{-2ps}.$$

Similarly, using Theorem 3.1 of [9], we obtain

$$\|u_p^{(1)}\|_{H_k^{s_0}(\mathbb{R}^d)}^2 + \|u_p^{(2)}\|_{H_k^{s_0}(\mathbb{R}^d)}^2 \leq \|u_p\|_{H_k^{s_0}(\mathbb{R}^d)}^2 \leq c_p^2 2^{-2p(s-s_0)}.$$

Set  $u^{(1)} = \sum_p u_p^{(1)}$ ,  $u^{(2)} = \sum_p u_p^{(2)}$ , then  $u = u^{(1)} + u^{(2)}$ , and from Proposition 3.10



[Title Page](#)

[Contents](#)



Page 21 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

we deduce that  $u^{(1)}$  belongs to  $H_k^s(\mathbb{R}^d)$ . For  $u^{(2)}$  the definition of  $u_p^{(2)}$  gives that

$$\|\Delta_q(u^{(2)})\|_{L_k^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left| \sum_{p \leq q+1} \varphi(2^{-q}\xi) \mathcal{F}_D(u_p^{(2)})(\xi) \right|^2 \omega_k(\xi) d\xi.$$

Thus by the Cauchy-Schwartz inequality we have

$$\begin{aligned} & \|\Delta_q(u^{(2)})\|_{L_k^2(\mathbb{R}^d)}^2 \\ & \leq \left( \sum_{p \leq q+1} 2^{-2p(s-s_0)} \right) \left( \int_{\mathbb{R}^d} \sum_{p \leq q+1} 2^{2p(s-s_0)} |\varphi(2^{-q}\xi) \mathcal{F}_D(u_p^{(2)})(\xi)|^2 \omega_k(\xi) d\xi \right) \\ & \leq \frac{1 - 2^{-2(q+2)(s-s_0)}}{1 - 2^{-(s-s_0)}} 2^{-2qs_0} \sum_{p \leq q+1} 2^{2p(s-s_0)} \|\Delta_q(u_p^{(2)})\|_{H_k^{s_0}(\mathbb{R}^d)}^2. \end{aligned}$$

Moreover, since  $s_0 > s > 0$ ,

$$\frac{1 - 2^{-2(q+2)(s-s_0)}}{1 - 2^{-(s-s_0)}} 2^{-2qs_0} \leq C 2^{-2qs},$$

and  $C$  is independent of  $q$ . Now set

$$c_q^2 = \sum_{p \leq q+1} 2^{2p(s-s_0)} \|\Delta_q(u_p^{(2)})\|_{H_k^{s_0}(\mathbb{R}^d)}^2,$$

then

$$\sum_{q \geq -1} 2^{2qs} \|\Delta_q(u^{(2)})\|_{L_k^2(\mathbb{R}^d)}^2 \leq \sum_{q \geq -1} c_q^2 \leq \sum_p 2^{2p(s-s_0)} \|u_p^{(2)}\|_{H_k^{s_0}(\mathbb{R}^d)}^2 < \infty.$$

Thus by Theorem 3.8 we deduce that  $u^{(2)} = \sum_q \Delta_q(u^{(2)})$  belongs to  $H_k^s(\mathbb{R}^d)$ .  $\square$

**Corollary 3.12.** *The spaces  $H_k^s(\mathbb{R}^d)$  do not depend on the choice of the function  $\varphi$  and  $\psi$  used in the Definition 3.3.*

[Title Page](#)[Contents](#)

Page 22 of 46

[Go Back](#)[Full Screen](#)[Close](#)

### 3.3. The Generalized Hölder Spaces

**Definition 3.13.** For  $\alpha$  in  $\mathbb{R}$ , we define the Hölder space  $C_k^\alpha(\mathbb{R}^d)$  associated with the Dunkl operators as the set of  $u \in \mathcal{S}'(\mathbb{R}^d)$  satisfying

$$\|u\|_{C_k^\alpha(\mathbb{R}^d)} = \sup_{p \geq -1} 2^{p\alpha} \|\Delta_p u\|_{L_k^\infty(\mathbb{R}^d)} < \infty,$$

where  $u = \sum_{p \geq -1} \Delta_p u$  is its Littlewood-Paley decomposition.

In the following proposition we give sufficient conditions so that the series  $\sum_q u_q$  belongs to the Hölder spaces associated with the Dunkl operators.

#### Proposition 3.14.

i) Let  $\tilde{C}$  be an annulus in  $\mathbb{R}^d$  and  $\alpha \in \mathbb{R}$ . Let  $(u_p)_{p \in \mathbb{N}}$  be a sequence of smooth functions. If the sequence  $(u_p)_{p \in \mathbb{N}}$  satisfies

$$\text{supp } \mathcal{F}_D(u_p) \subset 2^p \tilde{C} \quad \text{and} \quad \|u_p\|_{L_k^\infty(\mathbb{R}^d)} \leq C 2^{-p\alpha},$$

then we have

$$u = \sum_{p \geq 0} u_p \in C_k^\alpha(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{C_k^\alpha(\mathbb{R}^d)} \leq C(\alpha) \sup_{p \geq 0} 2^{p\alpha} \|u_p\|_{L_k^\infty(\mathbb{R}^d)}.$$

ii) Let  $K > 0$  and  $\alpha > 0$ . Let  $(u_p)_{p \in \mathbb{N}}$  be a sequence of smooth functions. If the sequence  $(u_p)_{p \in \mathbb{N}}$  satisfies

$$\text{supp } \mathcal{F}_D(u_p) \subset B(0, K 2^p) \quad \text{and} \quad \|u_p\|_{L_k^\infty(\mathbb{R}^d)} \leq C 2^{-p\alpha},$$

then we have

$$u = \sum_{p \geq 0} u_p \in C_k^\alpha(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{C_k^\alpha(\mathbb{R}^d)} \leq C(\alpha) \sup_{p \geq 0} 2^{p\alpha} \|u_p\|_{L_k^\infty(\mathbb{R}^d)}.$$



*Proof.* The proof uses the same idea as for Propositions 3.9 and 3.10. □

**Proposition 3.15.** *The distribution defined by*

$$g(x) = \sum_{p \geq 0} K(ix, 2^p e), \quad \text{with } e = (1, \dots, 1),$$

*belongs to  $C_k^0(\mathbb{R}^d)$  and does not belong to  $L_k^\infty(\mathbb{R}^d)$ .*

**Proposition 3.16.** *Let  $\varepsilon \in ]0, 1[$  and  $f$  in  $C_k^\varepsilon(\mathbb{R}^d)$ , then there exists a positive constant  $C$  such that*

$$\|f\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{C}{\varepsilon} \|f\|_{C_k^0(\mathbb{R}^d)} \log \left( e + \frac{\|f\|_{C_k^\varepsilon(\mathbb{R}^d)}}{\|f\|_{C_k^0(\mathbb{R}^d)}} \right).$$

*Proof.* Since  $f = \sum_{p \geq -1} \Delta_p f$ ,

$$\|f\|_{L_k^\infty(\mathbb{R}^d)} \leq \sum_{p \leq N-1} \|\Delta_p f\|_{L_k^\infty(\mathbb{R}^d)} + \sum_{p \geq N} \|\Delta_p f\|_{L_k^\infty(\mathbb{R}^d)},$$

with  $N$  is a positive integer that will be chosen later. Since  $f \in C_k^\varepsilon(\mathbb{R}^d)$ , using the definition of generalized Hölder norms, we deduce that

$$\|f\|_{L_k^\infty(\mathbb{R}^d)} \leq (N+1) \|f\|_{C_k^0(\mathbb{R}^d)} + \frac{2^{-(N-1)\varepsilon}}{2^\varepsilon - 1} \|f\|_{C_k^\varepsilon(\mathbb{R}^d)}.$$

We take

$$N = 1 + \left\lceil \frac{1}{\varepsilon} \log_2 \frac{\|f\|_{C_k^\varepsilon(\mathbb{R}^d)}}{\|f\|_{C_k^0(\mathbb{R}^d)}} \right\rceil,$$

we obtain

$$\|f\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{C}{\varepsilon} \|f\|_{C_k^0(\mathbb{R}^d)} \left[ 1 + \log \left( \frac{\|f\|_{C_k^\varepsilon(\mathbb{R}^d)}}{\|f\|_{C_k^0(\mathbb{R}^d)}} \right) \right].$$

This implies the result. □

Littlewood-Paley Decomposition

Hatem Meijaoui

vol. 9, iss. 4, art. 95, 2008

Title Page

Contents



Page 23 of 46

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

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Title Page

Contents



Page 24 of 46

Go Back

Full Screen

Close

Now we give the characterization of  $L_k^p(\mathbb{R}^d)$  spaces by using the dyadic decomposition.

If  $(f_j)_{j \in \mathbb{N}}$  is a sequence of  $L_k^p(\mathbb{R}^d)$ -functions, we set

$$\|(f_j)\|_{L_k^p(l^2)} = \left\| \left( \sum_{j \in \mathbb{N}} |f_j(x)|^2 \right)^{\frac{1}{2}} \right\|_{L_k^p(\mathbb{R}^d)},$$

the norm in  $L_k^p(\mathbb{R}^d, l^2(\mathbb{N}))$ .

**Theorem 3.17 (Littlewood-Paley decomposition of  $L_k^p(\mathbb{R}^d)$ ).** *Let  $f$  be in  $S'(\mathbb{R}^d)$  and  $1 < p < \infty$ . Then the following assertions are equivalent*

i)  $f \in L_k^p(\mathbb{R}^d)$ ,

ii)  $S_0 f \in L_k^p(\mathbb{R}^d)$  and  $\left( \sum_{j \in \mathbb{N}} |\Delta_j f(x)|^2 \right)^{\frac{1}{2}} \in L_k^p(\mathbb{R}^d)$ .

Moreover, the following norms are equivalent :

$$\|f\|_{L_k^p(\mathbb{R}^d)} \quad \text{and} \quad \|S_0 f\|_{L_k^p(\mathbb{R}^d)} + \left\| \left( \sum_{j \in \mathbb{N}} |\Delta_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L_k^p(\mathbb{R}^d)}.$$

*Proof.* If  $f$  is in  $L_k^2(\mathbb{R}^d)$ , then from Proposition 3.7 we have

$$\left\| \left( \sum_{j \in \mathbb{N}} |\Delta_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L_k^2(\mathbb{R}^d)} \leq \|f\|_{L_k^2(\mathbb{R}^d)}.$$

Thus the mapping

$$\Lambda_1 : f \mapsto (\Delta_j f)_{j \in \mathbb{N}},$$





Title Page

Contents

◀ ▶

◀ ▶

Page 25 of 46

Go Back

Full Screen

Close

is bounded from  $L_k^2(\mathbb{R}^d)$  into  $L_k^2(\mathbb{R}^d, l^2(\mathbb{N}))$ .

On the other hand, from properties of  $\varphi$  we see that

$$\begin{aligned}\|(\tilde{\varphi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \\ \|(\partial_{y_i}\tilde{\varphi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \quad i = 1, \dots, d,\end{aligned}$$

where

$$\tilde{\varphi}_j(x) = 2^{j(d+2\gamma)} \mathcal{F}_D^{-1}(\varphi)(2^j x).$$

We may then apply the theory of singular integrals to this mapping  $\Lambda_1$  (see [14]).

Thus we deduce that

$$\|\Delta_j f\|_{L_k^p(l^2)} \leq C_{p,k} \|f\|_{L_k^p(\mathbb{R}^d)}, \quad \text{for } 1 < p < \infty.$$

The converse uses the same idea. Indeed we put

$$\tilde{\phi}_j = \sum_{i=-1}^1 \tilde{\varphi}_{j+i}.$$

From Proposition 3.7 the mapping

$$\Lambda_2 : (f_j)_{j \in \mathbb{N}} \mapsto \sum_{j \in \mathbb{N}} f_j *_D \tilde{\phi}_j,$$

is bounded from  $L_k^2(\mathbb{R}^d, l^2(\mathbb{N}))$  into  $L_k^2(\mathbb{R}^d)$ .

On the other hand, from properties of  $\varphi$  we see that

$$\begin{aligned}\|(\tilde{\phi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \\ \|(\partial_{y_i}\tilde{\phi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \quad i = 1, \dots, d.\end{aligned}$$

We may then apply the theory of singular integrals to this mapping  $\Lambda_2$  (see [14]).  
Thus we obtain

$$\left\| \sum_{j \in \mathbb{N}} \Delta_j f \right\|_{L_k^p(\mathbb{R}^d)} \leq C_{p,k} \|\Delta_j f\|_{L_k^p(l^2)}.$$

□



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## Littlewood-Paley Decomposition

Hatem Mezzaoui

vol. 9, iss. 4, art. 95, 2008

---

[Title Page](#)

[Contents](#)



Page 26 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



Title Page

Contents



Page 27 of 46

Go Back

Full Screen

Close

## 4. Applications

### 4.1. Estimates of the Product of Two Functions

#### Proposition 4.1.

i) Let  $u, v \in C_k^\alpha(\mathbb{R}^d)$  and  $\alpha > 0$  then  $uv \in C_k^\alpha(\mathbb{R}^d)$ , and

$$\|uv\|_{C_k^\alpha(\mathbb{R}^d)} \leq C \left[ \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{C_k^\alpha(\mathbb{R}^d)} + \|v\|_{L_k^\infty(\mathbb{R}^d)} \|u\|_{C_k^\alpha(\mathbb{R}^d)} \right].$$

ii) Let  $u, v \in H_k^s(\mathbb{R}^d) \cap L_k^\infty(\mathbb{R}^d)$  and  $s > 0$  then  $uv \in H_k^s(\mathbb{R}^d)$ , and

$$\|uv\|_{H_k^s(\mathbb{R}^d)} \leq C \left[ \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)} + \|v\|_{L_k^\infty(\mathbb{R}^d)} \|u\|_{H_k^s(\mathbb{R}^d)} \right].$$

*Proof.* Let  $u = \sum_p \Delta_p u$  and  $v = \sum_q \Delta_q v$  be their Littlewood-Paley decompositions. Then we have

$$\begin{aligned} uv &= \sum_{p,q} \Delta_p u \Delta_q v \\ &= \sum_q \sum_{p \leq q-1} \Delta_p u \Delta_q v + \sum_q \sum_{p \geq q} \Delta_p u \Delta_q v \\ &= \sum_q \sum_{p \leq q-1} \Delta_p u \Delta_q v + \sum_p \sum_{q \leq p} \Delta_p u \Delta_q v \\ &= \sum_q S_q u \Delta_q v + \sum_p S_{p+1} v \Delta_p u \\ &= \sum_1 + \sum_2. \end{aligned}$$



Title Page

Contents



Page 28 of 46

Go Back

Full Screen

Close

We have

$$\text{supp}(\mathcal{F}_D(S_q u \Delta_q v)) = \text{supp}(\mathcal{F}_D(\Delta_q v) *_D \mathcal{F}_D(S_q u)).$$

Hence from Theorem 2.2 we deduce that  $\text{supp}(\mathcal{F}_D(S_q u \Delta_q v)) \subset B(0, C2^q)$ .

i) If  $u$  and  $v$  are in  $C_k^\alpha(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \|S_q u \Delta_q v\|_{L_k^\infty(\mathbb{R}^d)} &\leq \|S_q u\|_{L_k^\infty(\mathbb{R}^d)} \|\Delta_q v\|_{L_k^\infty(\mathbb{R}^d)}, \\ &\leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{C_k^\alpha(\mathbb{R}^d)} 2^{-q\alpha}. \end{aligned}$$

From Proposition 3.14 ii) we deduce

$$\left\| \sum_1 \right\|_{C_k^\alpha(\mathbb{R}^d)} \leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{C_k^\alpha(\mathbb{R}^d)}.$$

Similarly we prove that

$$\left\| \sum_2 \right\|_{C_k^\alpha(\mathbb{R}^d)} \leq C \|v\|_{L_k^\infty(\mathbb{R}^d)} \|u\|_{C_k^\alpha(\mathbb{R}^d)},$$

and this implies the result.

ii) If  $u$  and  $v$  are in  $H_k^s(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \|S_q u \Delta_q v\|_{L_k^2(\mathbb{R}^d)} &\leq \|S_q u\|_{L_k^\infty(\mathbb{R}^d)} \|\Delta_q v\|_{L_k^2(\mathbb{R}^d)}, \\ &\leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)} C_q 2^{-qs}. \end{aligned}$$

Thus Proposition 3.10 gives

$$\left\| \sum_1 \right\|_{H_k^s(\mathbb{R}^d)} \leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)}.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 29 of 46

Go Back

Full Screen

Close

Similarly, we prove that

$$\left\| \sum_2 \right\|_{H_k^s(\mathbb{R}^d)} \leq C \|v\|_{L_k^\infty(\mathbb{R}^d)} \|u\|_{H_k^s(\mathbb{R}^d)},$$

and this implies the result.  $\square$

**Corollary 4.2.** For  $s > \frac{d}{2} + \gamma$ ,  $H_k^s(\mathbb{R}^d)$  is an algebra.

## 4.2. Sobolev Embedding Theorem

Using the Littlewood-Paley decomposition, we have a very simple proof of Sobolev embedding theorems:

**Theorem 4.3.** For any  $s > \gamma + \frac{d}{2}$ , we have the continuous embedding

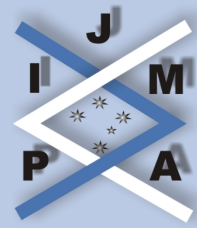
$$H_k^s(\mathbb{R}^d) \hookrightarrow C_k^{s-\gamma-\frac{d}{2}}(\mathbb{R}^d).$$

*Proof.* Let  $u$  be in  $H_k^s(\mathbb{R}^d)$ ,  $u = \sum_{p \geq -1} \Delta_p u$  the Littlewood-Paley decomposition. Take  $\phi$  in  $D(\mathbb{R}^d)$  such that  $\phi(\xi) = 1$  on  $C_0$ , and

$$\text{supp } \phi \subset C'_0 = \left\{ \xi \in \mathbb{R}^d, \frac{1}{3} \leq \|\xi\| \leq 3 \right\}.$$

Setting  $\phi_p(\xi) = \phi(2^{-p}\xi)$ , we obtain

$$\mathcal{F}_D(\Delta_p u)(\xi) = \mathcal{F}_D(\Delta_p u)(\xi) \phi(2^{-p}\xi).$$



Title Page

Contents



Page 30 of 46

Go Back

Full Screen

Close

Hence

$$\Delta_p u(x) = \int_{\mathbb{R}^d} \mathcal{F}_D(\Delta_p u)(\xi) \phi(2^{-p}\xi) K(ix, \xi) \omega_k(\xi) d\xi,$$
$$|\Delta_p u(x)| \leq \int_{\mathbb{R}^d} |\mathcal{F}_D(\Delta_p u)(\xi)| \phi(2^{-p}\xi) |\omega_k(\xi)| d\xi.$$

The Cauchy-Schwartz inequality and Theorem 3.8 give that

$$\begin{aligned} \|\Delta_p u\|_{L_k^\infty(\mathbb{R}^d)} &\leq \left( \int_{\mathbb{R}^d} |\mathcal{F}_D(\Delta_p u)(\xi)|^2 \omega_k(\xi) d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |\phi(2^{-p}\xi)|^2 \omega_k(\xi) d\xi \right)^{\frac{1}{2}} \\ &\leq C 2^{p(\gamma + \frac{d}{2})} \|\Delta_p u\|_{L_k^2(\mathbb{R}^d)} \\ &\leq C 2^{-p(s - \gamma - \frac{d}{2})} c_p. \end{aligned}$$

Then from Definition 3.13 we deduce that  $u \in C_k^{s - \gamma - \frac{d}{2}}(\mathbb{R}^d)$ . □

**Theorem 4.4.** For any  $0 < s < \gamma + \frac{d}{2}$ , we have the continuous embedding

$$H_k^s(\mathbb{R}^d) \hookrightarrow L_k^p(\mathbb{R}^d),$$

where  $p = \frac{2(2\gamma + d)}{2\gamma + d - 2s}$ .

*Proof.* Let  $f$  be in  $\mathcal{S}(\mathbb{R}^d)$ , we have, due to Fubini's theorem,

$$(4.1) \quad \|f\|_{L_k^p(\mathbb{R}^d)}^p = p \int_0^\infty \lambda^{p-1} m_k \{ |f| \geq \lambda \} d\lambda,$$

where

$$m_k \{ |f| \geq \lambda \} = \int_{\{x; |f(x)| \geq \lambda\}} \omega_k(x) dx.$$



[Title Page](#)

[Contents](#)



Page 31 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

For  $A > 0$ , we set  $f = f_{1,A} + f_{2,A}$  with  $f_{1,A} = \sum_{2^j < A} \Delta_j f$  and  $f_{2,A} = \sum_{2^j \geq A} \Delta_j f$ .  
We have

$$\|f_{1,A}\|_{L_k^\infty(\mathbb{R}^d)} \leq \sum_{2^j < A} \|\Delta_j f\|_{L_k^\infty(\mathbb{R}^d)} \leq \sum_{2^j < A} \|\mathcal{F}_D(\Delta_j f)\|_{L_k^1(\mathbb{R}^d)}.$$

Using the Cauchy-Schwartz inequality, the Parseval's identity associated with the Dunkl operators and Theorem 3.8, we obtain

$$\|f_{1,A}\|_{L_k^\infty(\mathbb{R}^d)} \leq \sum_{2^j < A} 2^{j(\gamma + \frac{d}{2} - s)} c_j \|f\|_{H_k^s(\mathbb{R}^d)} \leq CA^{\gamma + \frac{d}{2} - s} \|f\|_{H_k^s(\mathbb{R}^d)}.$$

On the other hand for all  $\lambda > 0$ , we have

$$(4.2) \quad m_k \left\{ |f| \geq \lambda \right\} \leq m_k \left\{ |f_{1,A}| \geq \frac{\lambda}{2} \right\} + m_k \left\{ |f_{2,A}| \geq \frac{\lambda}{2} \right\}.$$

From (4.2) we infer that if we take

$$A = A_\lambda = \left( \frac{\lambda}{4C \|f\|_{H_k^s(\mathbb{R}^d)}} \right)^{\frac{1}{\gamma + \frac{d}{2} - s}},$$

then

$$\|f_{1,A_\lambda}\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{\lambda}{4}.$$

Hence

$$m_k \left\{ |f_{1,A_\lambda}| \geq \frac{\lambda}{2} \right\} = 0.$$



[Title Page](#)

[Contents](#)



Page 32 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

From (4.1) and (4.2) we deduce that

$$\|f\|_{L_k^p(\mathbb{R}^d)}^p \leq p \int_0^\infty \lambda^{p-1} m_k \left\{ 2|f_{2,A_\lambda}| \geq \lambda \right\} d\lambda.$$

Moreover the Bienaymé-Tchebychev inequality yields

$$m_k \left\{ 2|f_{2,A_\lambda}| \geq \lambda \right\} \leq \frac{4}{\lambda^2} \|f_{2,A_\lambda}\|_{L_k^2(\mathbb{R}^d)}^2.$$

Thus we obtain

$$\|f\|_{L_k^p(\mathbb{R}^d)}^p \leq p \int_0^\infty \lambda^{p-3} \|f_{2,A_\lambda}\|_{L_k^2(\mathbb{R}^d)}^2 d\lambda.$$

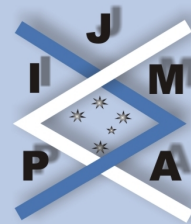
On the other hand, by using the Cauchy-Schwartz inequality for all  $\varepsilon > 0$ , we have

$$\begin{aligned} \|f_{2,A_\lambda}\|_{L_k^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} \left| \sum_{2^j \geq A_\lambda} \Delta_j f(x) \right|^2 \omega_k(x) dx \\ &\leq \left( \int_{\mathbb{R}^d} \sum_{2^j \geq A_\lambda} 2^{2j\varepsilon} |\Delta_j f(x)|^2 \omega_k(x) dx \right) \left( \sum_{2^j \geq A_\lambda} 2^{-2j\varepsilon} \right) \\ &\leq A_\lambda^{-2\varepsilon} \sum_{2^j \geq A_\lambda} 2^{2j\varepsilon} \|\Delta_j f\|_{L_k^2(\mathbb{R}^d)}^2. \end{aligned}$$

So by using the definition of  $A_\lambda$  and the Fubini theorem, we can write

$$\|f\|_{L_k^p(\mathbb{R}^d)}^p \leq p \int_0^\infty \lambda^{p-3} A_\lambda^{-2\varepsilon} \sum_{2^j \geq A_\lambda} 2^{2j\varepsilon} \|\Delta_j f\|_{L_k^2(\mathbb{R}^d)}^2 d\lambda$$





Title Page

Contents



Page 33 of 46

Go Back

Full Screen

Close

$$\begin{aligned} &\leq C \sum_{j \geq -1} \int_0^{4C2^{j(\gamma+\frac{d}{2}-s)}} \|f\|_{H_k^s(\mathbb{R}^d)} \lambda^{p-3-\frac{2\varepsilon}{\gamma+\frac{d}{2}-s}} d\lambda \left(4C\|f\|_{H_k^s(\mathbb{R}^d)}\right)^{\frac{2\varepsilon}{\gamma+\frac{d}{2}-s}} 2^{2j\varepsilon} \|\Delta_j f\|_{L_k^2(\mathbb{R}^d)}^2 \\ &\leq C \|f\|_{H_k^s(\mathbb{R}^d)}^{p-2} \sum_{j \geq -1} 2^{j(p-2)(\gamma+\frac{d}{2}-s)} \|\Delta_j f\|_{L_k^2(\mathbb{R}^d)}^2 \\ &\leq C \|f\|_{H_k^s(\mathbb{R}^d)}^{p-2} \sum_{j \geq -1} 2^{2js} \|\Delta_j f\|_{L_k^2(\mathbb{R}^d)}^2 \leq C \|f\|_{H_k^s(\mathbb{R}^d)}^p. \end{aligned}$$

This implies the result. □

**Definition 4.5.** We define the space  $BMO_k$  as the set of functions  $u \in L_{loc,k}^1(\mathbb{R}^d)$  satisfying

$$\sup_B \frac{1}{\text{mes}_k(B)} \int_B |u(x) - u_B| \omega_k(x) dx < \infty,$$

where

$$B = B(x_0, R), \quad u_B = \frac{1}{\text{mes}_k(B)} \int_B u(x) \omega_k(x) dx$$

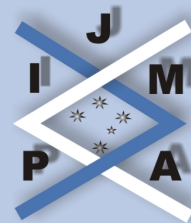
denote the average of  $u$  on  $B$  and  $\text{mes}_k(B) = \int_B \omega_k(x) dx$ .

**Theorem 4.6.** We have the continuous embedding

$$H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d) \hookrightarrow BMO_k.$$

*Proof.* For  $R > 0$  small enough, let  $N$  be such that  $2^N = [\frac{1}{R}]$ . Let  $u$  be in  $H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)$ . Set  $u = u^{(1)} + u^{(2)}$  with

$$u^{(1)} = \sum_{p=-1}^{N-1} \Delta_p u \quad \text{and} \quad u^{(2)} = \sum_{p \geq N} \Delta_p u.$$



Title Page

Contents



Page 34 of 46

Go Back

Full Screen

Close

From the Cauchy-Schwartz inequality we have

$$\left( \frac{1}{\text{mes}_k(B)} \int_B |u(x) - u_B| \omega_k(x) dx \right)^2 \leq \frac{1}{\text{mes}_k(B)} \int_B |u(x) - u_B|^2 \omega_k(x) dx.$$

It is easy to see that this implies

$$\begin{aligned} & \left( \frac{1}{\text{mes}_k(B)} \int_B |u(x) - u_B| \omega_k(x) dx \right)^2 \\ & \leq \frac{2}{\text{mes}_k(B)} \left[ \int_B |u^{(1)}(x) - u_B^{(1)}|^2 \omega_k(x) dx + \int_B |u^{(2)}(x) - u_B^{(2)}|^2 \omega_k(x) dx \right]. \end{aligned}$$

Moreover, from the mean value theorem, we have

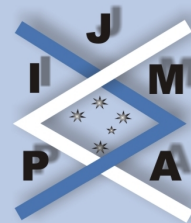
$$\begin{aligned} \frac{1}{\text{mes}_k(B)} \int_B |u^{(1)}(x) - u_B^{(1)}|^2 \omega_k(x) dx & \leq \frac{R^2}{\text{mes}_k(B)} \int_B |Du^{(1)}(x)|^2 \omega_k(x) dx, \\ & \leq R^2 \|Du^{(1)}\|_{L_k^\infty(\mathbb{R}^d)}^2. \end{aligned}$$

By (2.7) we deduce that

$$\|Du^{(1)}\|_{L_k^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \|\xi\| |\mathcal{F}_D(u^{(1)})(\xi)| \omega_k(\xi) d\xi.$$

By recalling that  $\text{supp } \mathcal{F}_D(\Delta_p u) \subset C_p$  and  $|\mathcal{F}_D(\Delta_p u)(\xi)| \leq |\mathcal{F}_D(u)(\xi)|$ , we apply the Parseval identity associated with the Dunkl operators and the Cauchy-Schwartz inequality. We deduce that

$$\frac{1}{\text{mes}_k(B)} \int_B |u^{(1)}(x) - u_B^{(1)}|^2 \omega_k(x) dx \leq R^2 \left( \sum_{p=-1}^{N-1} \int_{C_p} \|\xi\| |\mathcal{F}_D(\Delta_p u)(\xi)| \omega_k(\xi) d\xi \right)^2$$



Title Page

Contents



Page 35 of 46

Go Back

Full Screen

Close

$$\leq R^2 \left( \int_{B(0,2^N)} \|\xi\|^{2-2\gamma-d} \omega_k(\xi) d\xi \right) \|u\|_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}^2 \leq C2^{2N} R^2 \|u\|_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}^2.$$

For the second term, we have

$$\begin{aligned} \frac{1}{\text{mes}_k(B)} \int_B |u^{(2)}(x) - u_B^{(2)}|^2 \omega_k(x) dx &\leq \frac{2}{\text{mes}_k(B)} \int_B |u^{(2)}(x)|^2 \omega_k(x) dx \\ &\leq CR^{-d-2\gamma} \int_{\|\xi\| \geq 2^N} |\mathcal{F}_D(u)(\xi)|^2 \omega_k(\xi) d\xi \\ &\leq C(2^N R)^{-d-2\gamma} \|u\|_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}^2. \end{aligned}$$

Hence,

$$(4.3) \quad \left( \frac{1}{\text{mes}_k(B)} \int_B |u(x) - u_B| \omega_k(x) dx \right)^2 \leq C \|u\|_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}^2.$$

We have proved (4.3) for small  $R$ , since  $u \in H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d) \subset L_k^2(\mathbb{R}^d)$ , (4.3) is evident for  $R > R_0$  with constant  $C = C(R_0)$ . This implies the continuous embedding

$$H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d) \hookrightarrow BMO_k.$$

□

### 4.3. Gagliardo-Nirenberg Inequality

We will use the generalized Sobolev space  $W_k^{s,r}(\mathbb{R}^d)$  associated with the Dunkl operators defined as

$$W_k^{s,r}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : (-\Delta_k)^{\frac{s}{2}} u \in L_k^r(\mathbb{R}^d) \right\},$$

with

$$\Delta_k u = \sum_{j=1}^d T_j^2 u.$$

The main result of this subsection is the following theorem.

**Theorem 4.7.** *Let  $f$  be in  $W_k^{s,r}(\mathbb{R}^d) \cap L_k^q(\mathbb{R}^d)$  with  $q, r \in [1, \infty]$  and  $s \geq 0$ . Then  $f$  belongs to  $W_k^{t,p}(\mathbb{R}^d)$ , and we have*

$$\left\| (-\Delta_k)^{\frac{t}{2}} f \right\|_{L_k^p(\mathbb{R}^d)} \leq C \|f\|_{L_k^q(\mathbb{R}^d)}^\theta \left\| (-\Delta_k)^{\frac{s}{2}} f \right\|_{L_k^r(\mathbb{R}^d)}^{1-\theta},$$

where  $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$ ,  $t = (1-\theta)s$  and  $\theta \in ]0, 1[$ .

*Proof.* First, we prove this theorem for  $q$  and  $r$  in  $]1, \infty]$ . Let  $f$  be in  $\mathcal{S}(\mathbb{R}^d)$ . It is easy to see that

$$(-\Delta_k)^{\frac{t}{2}} f = \sum_{j \leq A} (-\Delta_k)^{\frac{t}{2}} \Delta_j f + \sum_{j > A} (-\Delta_k)^{\frac{t-s}{2}} \Delta_j \left( (-\Delta_k)^{\frac{s}{2}} f \right),$$

where  $A$  will be chosen later.

On the other hand, by a simple calculation, if  $a$  is a homogenous function in  $C^\infty(\mathbb{R}^*)$  of degree  $m$ , we can write

$$(4.4) \quad a \left( (-\Delta_k)^{\frac{1}{2}} \right) \Delta_j f = 2^{jm+j(d+2\gamma)} b(\delta_{2^j}) *_D \sum_{|j-j'| \leq 1} \Delta_{j'} f,$$

where  $\delta_{2^j}$  is defined by  $\delta_{2^j} x = 2^j x$ ,  $x \in \mathbb{R}^d$  and  $b$  is in  $\mathcal{S}(\mathbb{R}^d)$  such that

$$\mathcal{F}_D(b)(\xi) = \varphi(\xi) a(\|\xi\|).$$



Title Page

Contents

◀ ▶

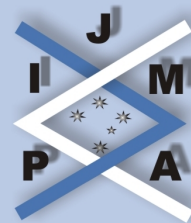
◀ ▶

Page 36 of 46

Go Back

Full Screen

Close



Title Page

Contents

◀ ▶

◀ ▶

Page 37 of 46

Go Back

Full Screen

Close

We proceed as in [12, p. 21] to obtain

$$(4.5) \quad \left| a \left( (-\Delta_k)^{\frac{1}{2}} \right) \Delta_j f(x) \right| \leq C 2^{jm} M_k f(x),$$

where  $M_k(f)$  is a maximal function of  $f$  associated with the Dunkl operators (see [13]).

Hence by applying (4.5) for  $a(r) = r^t$  and  $a(r) = r^{\frac{t-s}{2}}$ , we get

$$\begin{aligned} \left| (-\Delta_k)^{\frac{t}{2}} f(x) \right| &\leq C \left( \sum_{j \leq A} 2^{jt} M_k f(x) + \sum_{j > A} 2^{j(t-s)} M_k \left( (-\Delta_k f)^{\frac{s}{2}} \right) (x) \right) \\ &\leq C 2^{tA} M_k f(x) + C 2^{(t-s)A} M_k \left( (-\Delta_k)^{\frac{s}{2}} f \right) (x). \end{aligned}$$

We minimize over  $A$  to obtain

$$\left| (-\Delta_k)^{\frac{t}{2}} f(x) \right| \leq C \left( M_k f(x) \right)^{1-\frac{t}{s}} \left( M_k \left( (-\Delta_k)^{\frac{s}{2}} f \right) (x) \right)^{\frac{t}{s}}.$$

By this inequality and the Hölder inequality, we have

$$\left\| (-\Delta_k)^{\frac{t}{2}} f \right\|_{L_k^p(\mathbb{R}^d)} \leq C \|M_k f\|_{L_k^q(\mathbb{R}^d)}^\theta \|M_k \left( (-\Delta_k)^{\frac{s}{2}} f \right)\|_{L_k^r(\mathbb{R}^d)}^{1-\theta},$$

with  $\theta = 1 - \frac{t}{s}$ .

Now, we apply Theorem 6.1 of [13] to deduce the result if  $q$  and  $r \in [1, \infty]$ .

Now, we assume  $q = r = 1$ . Let  $f$  be in  $\mathcal{S}(\mathbb{R}^d)$ . We have

$$\begin{aligned} \left\| (-\Delta_k)^{\frac{t}{2}} f \right\|_{L_k^1(\mathbb{R}^d)} &\leq \left\| \sum_{j \leq A} (-\Delta_k)^{\frac{t}{2}} \Delta_j f \right\|_{L_k^1(\mathbb{R}^d)} + \left\| \sum_{j > A} (-\Delta_k)^{\frac{t-s}{2}} \Delta_j \left( (-\Delta_k)^{\frac{s}{2}} f \right) \right\|_{L_k^1(\mathbb{R}^d)} \\ &\leq C 2^{(1-\theta)sA} \|f\|_{L_k^1(\mathbb{R}^d)} + C 2^{-\theta sA} \left\| (-\Delta_k)^{\frac{s}{2}} f \right\|_{L_k^1(\mathbb{R}^d)}. \end{aligned}$$

By minimizing over  $A$ , we obtain the result. □



## 5. Paraproduct Associated with the Dunkl Operators

In this section, we are going to study how the product acts on Sobolev and Hölder spaces associated with the Dunkl operators. This could be very useful in nonlinear partial differential-difference equations. Of course, we shall use the Littlewood-Paley decomposition associated with the Dunkl operators. Let us consider two temperate distributions  $u$  and  $v$ . We write

$$u = \sum_p \Delta_p u \quad \text{and} \quad v = \sum_q \Delta_q v.$$

Formally, the product can be written as

$$uv = \sum_{p,q} \Delta_p u \Delta_q v.$$

Now we introduce the paraproduct operator associated with the Dunkl operators.

**Definition 5.1.** We define the paraproduct operator  $\Pi_a : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$  by

$$\Pi_a u = \sum_{q \geq 1} (S_{q-2} a) \Delta_q u,$$

where  $u \in \mathcal{S}'(\mathbb{R}^d)$ ;  $\{\Delta_q a\}$  and  $\{\Delta_q u\}$  are the Littlewood-Paley decompositions and  $S_q a = \sum_{p \leq q-1} \Delta_p a$ .

Let  $R$  indicate the following bilinear symmetric operator on  $\mathcal{S}'(\mathbb{R}^d)$  defined by

$$R(u, v) = \sum_{|p-q| \leq 1} \Delta_p u \Delta_q v, \quad \text{for all } u, v \in \mathcal{S}'(\mathbb{R}^d).$$

Title Page

Contents



Page 38 of 46

Go Back

Full Screen

Close

Obviously from Definition 5.1 it is clear that

$$uv = \Pi_u v + \Pi_v u + R(u, v).$$

The following theorems describe the action of the paraproduct and remainder on the Sobolov and the Hölder spaces associated with the Dunkl operators.

**Theorem 5.2.** *There exists a positive constant  $C$  such that the operator  $\Pi$  has the following properties:*

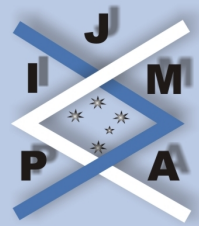
1.  $\|\Pi\|_{\mathcal{L}(L_k^\infty(\mathbb{R}^d) \times C_k^s(\mathbb{R}^d), C_k^s(\mathbb{R}^d))} \leq C^{s+1}, \quad \text{for all } s > 0.$
2.  $\|\Pi\|_{\mathcal{L}(L_k^\infty(\mathbb{R}^d) \times H_k^s(\mathbb{R}^d), H_k^s(\mathbb{R}^d))} \leq C^{s+1}, \quad \text{for all } s > 0.$
3.  $\|\Pi\|_{\mathcal{L}(C_k^t(\mathbb{R}^d) \times H_k^s(\mathbb{R}^d), H_k^{s+t}(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{-t}, \quad \text{for all } s, t \text{ with } s+t > 0 \text{ and } t < 0.$
4.  $\|\Pi\|_{\mathcal{L}(C_k^t(\mathbb{R}^d) \times C_k^s(\mathbb{R}^d), C_k^{s+t}(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{-t}, \quad \text{for all } s, t \text{ with } s+t > 0 \text{ and } t < 0.$
5.  $\|\Pi\|_{\mathcal{L}(H_k^s(\mathbb{R}^d) \times H_k^t(\mathbb{R}^d), H_k^{s+t-\gamma-\frac{d}{2}}(\mathbb{R}^d))} \leq C^{s+t-\gamma-\frac{d}{2}+1}, \quad \text{with } s+t > \gamma + \frac{d}{2} \text{ and } s < \frac{d}{2} + \gamma.$

*Proof.* Let  $u$  and  $v$  be in  $\mathcal{S}'(\mathbb{R}^d)$ . We have

$$\text{supp}(\mathcal{F}_D(S_{q-2}u\Delta_q v)) = \text{supp}(\mathcal{F}_D(\Delta_q v) *_D \mathcal{F}_D(S_{q-2}u)) \subset B(0, C2^q).$$

1) If  $u$  in  $L_k^\infty(\mathbb{R}^d)$  and  $v$  in  $C_k^s(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \|S_{q-2}u\Delta_q v\|_{L_k^\infty(\mathbb{R}^d)} &\leq \|S_{q-2}u\|_{L_k^\infty(\mathbb{R}^d)} \|\Delta_q v\|_{L_k^\infty(\mathbb{R}^d)} \\ &\leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{C_k^s(\mathbb{R}^d)} 2^{-qs}. \end{aligned}$$



Title Page

Contents



Page 39 of 46

Go Back

Full Screen

Close



Title Page

Contents

◀ ▶

◀ ▶

Page 40 of 46

Go Back

Full Screen

Close

Applying Proposition 3.14 ii), we obtain

$$\|\Pi_u v\|_{C_k^s(\mathbb{R}^d)} \leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{C_k^s(\mathbb{R}^d)}.$$

2) If  $u$  is in  $L_k^\infty(\mathbb{R}^d)$  and  $v$  is in  $H_k^s(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \|S_{q-2} u \Delta_q v\|_{L_k^2(\mathbb{R}^d)} &\leq \|S_{q-2} u\|_{L_k^\infty(\mathbb{R}^d)} \|\Delta_q v\|_{L_k^2(\mathbb{R}^d)} \\ &\leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)} c_q 2^{-qs}. \end{aligned}$$

Thus Proposition 3.10 gives

$$\|\Pi_u v\|_{H_k^s(\mathbb{R}^d)} \leq C \|u\|_{L_k^\infty(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)},$$

this implies the result.

3) Let  $u$  be in  $C_k^t(\mathbb{R}^d)$  and  $v$  in  $H_k^s(\mathbb{R}^d)$ . We have

$$\|S_{q-2} u \Delta_q v\|_{L_k^2(\mathbb{R}^d)} \leq \|S_{q-2} u\|_{L_k^\infty(\mathbb{R}^d)} \|\Delta_q v\|_{L_k^2(\mathbb{R}^d)}.$$

Since  $t < 0$ , we estimate  $\|S_{q-2} u\|_{L_k^\infty(\mathbb{R}^d)}$  in a different way. In fact,  $S_{q-2} u = \sum_{p \leq q-3} \Delta_p u$ . Since  $u \in C_k^t(\mathbb{R}^d)$  and  $t < 0$ , we obtain

$$\|S_{q-2} u\|_{L_k^\infty(\mathbb{R}^d)} \leq \|u\|_{C_k^t(\mathbb{R}^d)} \sum_{p \leq q-3} 2^{-pt} \leq \frac{C}{-t} 2^{-qt} \|u\|_{C_k^t(\mathbb{R}^d)}.$$

Hence

$$\|S_{q-2} u \Delta_q v\|_{L_k^2(\mathbb{R}^d)} \leq \frac{C}{-t} 2^{-q(t+s)} c_q \|u\|_{C_k^t(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)}, \quad c_q \in l^2.$$

Thus Proposition 3.10 gives the result.

The proof of 4) uses the same idea.





[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 41 of 46

[Go Back](#)

[Full Screen](#)

[Close](#)

5) We have

$$\begin{aligned} \|S_{q-2}u\|_{L_k^\infty(\mathbb{R}^d)} &\leq \|\mathcal{F}_D(S_{q-2}u)\|_{L_k^1(\mathbb{R}^d)} \\ &\leq \int_{\mathbb{R}^d} |\psi(2^{-(q-2)}\xi)|(1 + \|\xi\|^2)^{-\frac{s}{2}} |\mathcal{F}_D(u)(\xi)|(1 + \|\xi\|^2)^{\frac{s}{2}} \omega_k(\xi) d\xi. \end{aligned}$$

The Cauchy-Schwartz inequality implies that

$$\begin{aligned} \|S_{q-2}u\|_{L_k^\infty(\mathbb{R}^d)} &\leq \left( \int_{\mathbb{R}^d} |\psi(2^{-(q-2)}\xi)|^2 (1 + \|\xi\|^2)^{-s} \omega_k(\xi) d\xi \right)^{\frac{1}{2}} \|u\|_{H_k^s(\mathbb{R}^d)}, \\ &\leq C2^{q(\frac{d}{2}+\gamma)} \left( \int_{B(0,1)} |\psi(t)|^2 (1 + 2^{2(q-2)}\|t\|^2)^{-s} \omega_k(t) dt \right)^{\frac{1}{2}} \|u\|_{H_k^s(\mathbb{R}^d)}. \end{aligned}$$

If  $s \geq 0$ ,

$$\begin{aligned} \|S_{q-2}u\|_{L_k^\infty(\mathbb{R}^d)} &\leq C2^{q(\frac{d}{2}+\gamma-s)} \left( \int_{B(0,1)} |\psi(t)|^2 \|t\|^{-2s} \omega_k(t) dt \right)^{\frac{1}{2}} \\ &\leq C2^{q(\frac{d}{2}+\gamma-s)}. \end{aligned}$$

If  $s \leq 0$ ,

$$\begin{aligned} \|S_{q-2}u\|_{L_k^\infty(\mathbb{R}^d)} &\leq C2^{q(\frac{d}{2}+\gamma-s)} \left( \int_{B(0,1)} |\psi(t)|^2 (1 + \|t\|^2)^{-s} \omega_k(t) dt \right)^{\frac{1}{2}} \\ &\leq C2^{q(\frac{d}{2}+\gamma-s)}. \end{aligned}$$

By proceeding as in the previous cases we deduce the result.  $\square$

**Theorem 5.3.** *There exists a positive constant  $C$  such that the operator  $\Pi$  has the following properties:*



Title Page

Contents



Page 42 of 46

Go Back

Full Screen

Close

1. If  $a \in L_k^\infty(\mathbb{R}^d)$  is radial, then for any  $s$  in  $\mathbb{R}$ , we have

$$\|\Pi_a\|_{\mathcal{L}(C_k^s(\mathbb{R}^d), C_k^s(\mathbb{R}^d))} \leq C\|a\|_{L_k^\infty(\mathbb{R}^d)}, \quad \|\Pi_a\|_{\mathcal{L}(H_k^s(\mathbb{R}^d), H_k^s(\mathbb{R}^d))} \leq C\|a\|_{L_k^\infty(\mathbb{R}^d)}.$$

2. If  $a \in C_k^t(\mathbb{R}^d)$  is radial with  $t < 0$ , then for all  $s$ , we have

$$\|\Pi_a\|_{\mathcal{L}(H_k^s(\mathbb{R}^d), H_k^{s+t}(\mathbb{R}^d))} \leq C\|a\|_{C_k^t(\mathbb{R}^d)}, \quad \|\Pi_a\|_{\mathcal{L}(C_k^s(\mathbb{R}^d), C_k^{s+t}(\mathbb{R}^d))} \leq C\|a\|_{C_k^t(\mathbb{R}^d)}.$$

3. If  $a \in H_k^t(\mathbb{R}^d)$  is radial, then for all  $s, t$  with  $s < \frac{d}{2} + \gamma$ , we have

$$\|\Pi_a\|_{\mathcal{L}(H_k^s(\mathbb{R}^d), H_k^{s+t-\gamma-\frac{d}{2}}(\mathbb{R}^d))} \leq C\|a\|_{H_k^t(\mathbb{R}^d)}.$$

*Proof.* From the relation (2.19) and Definition 2.6 we deduce that there exists an annulus  $\tilde{C}_0$  such that  $\text{supp } \mathcal{F}_D(S_{q-2}a\Delta_q u) \subset 2^q\tilde{C}_0$ . Thus we proceed as in the proof of Theorem 4.3 and using Propositions 3.9 and 3.14 i), we obtain the result.  $\square$

*Remark 5.* In the case  $W = \mathbb{Z}_2^d$  the assumption that  $a$  is radial is not necessary.

**Theorem 5.4.** *There exists a positive constant  $C$  such that the operator  $R$  has the following properties:*

$$1. \quad \|R\|_{\mathcal{L}(C_k^t(\mathbb{R}^d) \times H_k^s(\mathbb{R}^d), H_k^{s+t}(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{s+t}, \quad \text{for all } s, t \text{ with } s+t > 0.$$

$$2. \quad \|R\|_{\mathcal{L}(C_k^t(\mathbb{R}^d) \times C_k^s(\mathbb{R}^d), C_k^{s+t}(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{s+t}, \quad \text{for all } s, t \text{ with } s+t > 0.$$

$$3. \quad \|R\|_{\mathcal{L}(H_k^t(\mathbb{R}^d) \times H_k^s(\mathbb{R}^d), H_k^{s+t-\gamma-\frac{d}{2}}(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{s+t-\gamma-\frac{d}{2}}, \quad \text{for all } s, t \text{ with } s+t > \gamma + \frac{d}{2}.$$

*Proof.* By the definition of the remainder operator

$$R(u, v) = \sum_q R_q \quad \text{with} \quad R_q = \sum_{i=-1}^1 \Delta_{q-i} u \Delta_q v.$$



By the definition of  $\Delta_q$ , the support of the Dunkl transform of  $R_q$  is included in  $B(0, C2^q)$ . Then, to prove 1) it is sufficient to estimate  $\|R_q\|_{L_k^2(\mathbb{R}^d)}$ . In fact, we have

$$\|R_q\|_{L_k^2(\mathbb{R}^d)} \leq \|\Delta_q v\|_{L_k^2(\mathbb{R}^d)} \sum_{i=-1}^1 \|\Delta_{q-i} u\|_{L_k^\infty(\mathbb{R}^d)}.$$

Using the facts that  $u \in C_k^t(\mathbb{R}^d)$  and  $v \in H_k^s(\mathbb{R}^d)$ , we obtain

$$\begin{aligned} \|R_q\|_{L_k^2(\mathbb{R}^d)} &\leq 2^{-qs} c_q \|v\|_{H_k^s(\mathbb{R}^d)} \sum_{i=-1}^1 2^{-(q-i)t} \|u\|_{C_k^t(\mathbb{R}^d)}, \quad c_q \in l^2 \\ &\leq C c_q 2^{-q(s+t)} \|u\|_{C_k^t(\mathbb{R}^d)} \|v\|_{H_k^s(\mathbb{R}^d)}. \end{aligned}$$

Now we apply Proposition 3.10 to conclude the proof. The proof of the second case uses the same idea. We want to prove 3). We have  $R(u, v) = \sum_q R_q$ . We proceed as in 1), so

$$\begin{aligned} \|R_q\|_{L_k^2(\mathbb{R}^d)} &\leq C \|\mathcal{F}_D(\Delta_q v)\|_{L_k^2(\mathbb{R}^d)} \sum_{i=-1}^1 \|\Delta_{q-i} u\|_{L_k^\infty(\mathbb{R}^d)}, \\ &\leq C \|\mathcal{F}_D(\Delta_q v)\|_{L_k^2(\mathbb{R}^d)} \sum_{i=-1}^1 \|\mathcal{F}_D(\Delta_{q-i} u)\|_{L_k^1(\mathbb{R}^d)}. \end{aligned}$$

Using the fact that  $v \in H_k^s(\mathbb{R}^d)$ , we obtain

$$\|R_q\|_{L_k^2(\mathbb{R}^d)} \leq C c_q 2^{-qt} \|v\|_{H_k^s(\mathbb{R}^d)} \sum_{i=-1}^1 \|\mathcal{F}_D(\Delta_{q-i} u)\|_{L_k^1(\mathbb{R}^d)}, \quad c_q \in l^2.$$

Title Page

Contents



Page 43 of 46

Go Back

Full Screen

Close



Title Page

Contents



Page 44 of 46

Go Back

Full Screen

Close

On the other hand, by the Cauchy-Schwartz inequality we have

$$\begin{aligned}\|\mathcal{F}_D(\Delta_{q-i}u)\|_{L_k^1(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} |\varphi(2^{-(q-i)}\xi)|(1 + \|\xi\|^2)^{\frac{-s}{2}}(1 + \|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_D(u)(\xi)\omega_k(\xi)d\xi \\ &\leq \|u\|_{H_k^s(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |\varphi(2^{-(q-i)}\xi)|^2(1 + \|\xi\|^2)^{-s}\omega_k(\xi)d\xi \right)^{\frac{1}{2}} \\ &\leq C2^{q(\gamma+\frac{d}{2})}\|u\|_{H_k^s(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |\varphi(t)|^2(1 + 2^{2(q-i)}\|t\|^2)^{-s}\omega_k(t)dt \right)^{\frac{1}{2}} \\ &\leq C2^{q(\gamma+\frac{d}{2}-s)}.\end{aligned}$$

Hence

$$\|R_q\|_{L_k^2(\mathbb{R}^d)} \leq C2^{q(\gamma+\frac{d}{2}-s-t)}c_q, \quad c_q \in l^2.$$

Then we conclude the result by using Proposition 3.10. □

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Littlewood-Paley Decomposition

Hatem Mejsaoli

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Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 45 of 46

Go Back

Full Screen

Close

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Title Page

Contents



Page 46 of 46

Go Back

Full Screen

Close