

APPROXIMATION OF A MIXED FUNCTIONAL EQUATION IN QUASI-BANACH SPACES



Approximation of a Mixed
Functional Equation
Abbas Najati and
G. Zamani Eskandani
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Key words: Generalized Hyers-Ulam stability, additive function, cubic function, quasi-Banach space, p -Banach space.

Abstract: In this paper we establish the general solution of the functional equation

$$f(2x + y) + f(x + 2y) = 6f(x + y) + f(2x) + f(2y) - 5[f(x) + f(y)]$$

and investigate its generalized Hyers-Ulam stability in quasi-Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Th.M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

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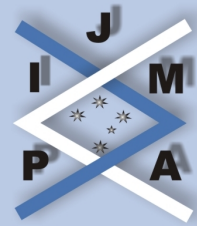
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1. Introduction and Preliminaries

In 1940, S.M. Ulam [30] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let (G_2, \diamond, d) be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In 1941, D.H. Hyers [9] considered the case of approximately additive functions $f : E \rightarrow E'$, where E and E' are Banach spaces and f satisfies Hyers inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

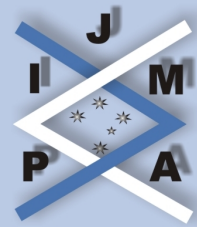
for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \rightarrow E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

T. Aoki [2] and Th.M. Rassias [27] provided a generalization of Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.



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Theorem 1.1 (Th.M. Rassias). Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$(1.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \rightarrow E'$ is the unique additive mapping which satisfies

$$(1.2) \quad \|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p$$

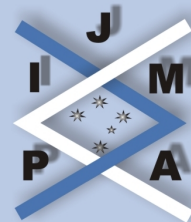
for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The inequality (1.1) has provided much influence in the development of what is now known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. P. Găvruta in [7] provided a further generalization of Th.M. Rassias' theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [4], [6], [8], [11], [13], [15] – [26]). We also refer the readers to the books [1], [5], [10], [14] and [28].

Jun and Kim [12] introduced the following cubic functional equation

$$(1.3) \quad f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

and they established the general solution and the generalized Hyers-Ulam stability problem for the functional equation (1.3). They proved that a function $f : E_1 \rightarrow E_2$



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satisfies the functional equation (1.3) if and only if there exists a function $B : E_1 \times E_1 \times E_1 \rightarrow E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables. The function B is given by

$$B(x, y, z) = \frac{1}{24}[f(x + y + z) + f(x - y - z) - f(x + y - z) - f(x - y + z)]$$

for all $x, y, z \in E_1$.

A. Najati and G.Z. Eskandani [25] established the general solution and investigated the generalized Hyers-Ulam stability of the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2f(2x) - 2f(x)$$

in quasi-Banach spaces.

In this paper, we deal with the following functional equation derived from cubic and additive functions:

$$(1.4) \quad f(2x + y) + f(x + 2y) = 6f(x + y) + f(2x) + f(2y) - 5[f(x) + f(y)].$$

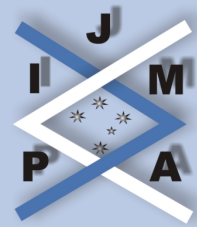
It is easy to see that the function $f(x) = ax^3 + cx$ is a solution of the functional equation (1.4).

The main purpose of this paper is to establish the general solution of (1.4) and investigate its generalized Hyers-Ulam stability.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.2 ([3, 29]). Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

- (i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;



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(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;

(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from condition (iii) that

$$\left\| \sum_{i=1}^{2n} x_i \right\| \leq K^n \sum_{i=1}^{2n} \|x_i\|, \quad \left\| \sum_{i=1}^{2n+1} x_i \right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \dots, x_{2n+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X . The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

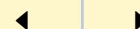
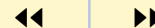
for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

By the Aoki-Rolewicz theorem [29] (see also [3]), each quasi-norm is equivalent to some *p-norm*. Since it is much easier to work with *p-norms* than quasi-norms, henceforth we restrict our attention mainly to *p-norms*.



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2. Solutions of Eq. (1.4)

Throughout this section, X and Y will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we need the following two lemmas.

Lemma 2.1. *If a function $f : X \rightarrow Y$ satisfies (1.4), then the function $g : X \rightarrow Y$ defined by $g(x) = f(2x) - 8f(x)$ is additive.*

Proof. Let $f : X \rightarrow Y$ satisfy the functional equation (1.4). Letting $x = y = 0$ in (1.4), we get that $f(0) = 0$. Replacing y by $2y$ in (1.4), we get

$$(2.1) \quad f(2x + 2y) + f(x + 4y) = 6f(x + 2y) + f(2x) + f(4y) - 5[f(x) + f(2y)]$$

for all $x, y \in X$. Replacing y by x and x by y in (2.1), we have

$$(2.2) \quad f(2x + 2y) + f(4x + y) = 6f(2x + y) + f(4x) + f(2y) - 5[f(2x) + f(y)]$$

for all $x, y \in X$. Adding (2.1) to (2.2) and using (1.4), we have

$$(2.3) \quad \begin{aligned} 2f(2x + 2y) + f(4x + y) + f(x + 4y) \\ = 36f(x + y) + f(4x) + f(4y) + 2[f(2x) + f(2y)] - 35[f(x) + f(y)] \end{aligned}$$

for all $x, y \in X$. Replacing y by $-x$ in (2.3), we get

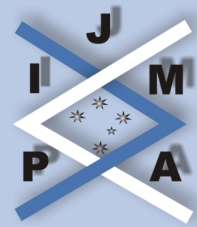
$$(2.4) \quad f(3x) + f(-3x) = f(4x) + f(-4x) + 2[f(2x) + f(-2x)] - 35[f(x) + f(-x)]$$

for all $x \in X$. Letting $y = x$ in (1.4), we get

$$(2.5) \quad f(3x) = 4f(2x) - 5f(x)$$

for all $x \in X$. Letting $y = -x$ in (1.4), we have

$$(2.6) \quad f(2x) + f(-2x) = 6[f(x) + f(-x)]$$



for all $x \in X$. It follows from (2.4), (2.5) and (2.6) that $f(-x) = -f(x)$ for all $x \in X$, i.e., f is odd. Replacing x by $x + y$ and y by $-y$ in (1.4) and using the oddness of f , we have

$$(2.7) \quad f(2x + y) + f(x - y) = 6f(x) + f(2x + 2y) - f(2y) - 5[f(x + y) - f(y)]$$

for all $x, y \in X$. Replacing y by x and y by x in (2.7), we get

$$(2.8) \quad f(x + 2y) - f(x - y) = 6f(y) + f(2x + 2y) - f(2x) - 5[f(x + y) - f(x)]$$

for all $x, y \in X$. Adding (2.7) to (2.8), we have

$$(2.9) \quad \begin{aligned} f(2x + y) + f(x + 2y) \\ = 2f(2x + 2y) - f(2x) - f(2y) - 10f(x + y) + 11[f(x) + f(y)] \end{aligned}$$

for all $x, y \in X$. It follows from (1.4) and (2.9) that

$$(2.10) \quad f(2x + 2y) - 8f(x + y) = f(2x) + f(2y) - 8[f(x) + f(y)]$$

for all $x, y \in X$, which means that the function $g : X \rightarrow Y$ is additive. \square

Lemma 2.2. *If a function $f : X \rightarrow Y$ satisfies the functional equation (1.4), then the function $h : X \rightarrow Y$ defined by $h(x) = f(2x) - 2f(x)$ is cubic.*

Proof. Let $g : X \rightarrow Y$ be a function defined by $g(x) = f(2x) - 8f(x)$ for all $x \in X$. By Lemma 2.1 and its proof, the function f is odd and the function g is additive. It is clear that

$$(2.11) \quad h(x) = g(x) + 6f(x), \quad f(2x) = g(x) + 8f(x)$$

for all $x \in X$. Replacing x by $x - y$ in (1.4), we have

$$(2.12) \quad f(2x - y) + f(x + y) = 6f(x) + f(2x - 2y) + f(2y) - 5[f(x - y) + f(y)]$$

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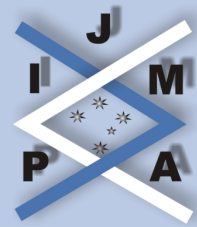
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for all $x, y \in X$. Replacing y by $-y$ in (2.12), we have

$$(2.13) \quad f(2x + y) + f(x - y) = 6f(x) + f(2x + 2y) - f(2y) - 5[f(x + y) - f(y)]$$

for all $x, y \in X$. Adding (2.12) to (2.13), we get

$$(2.14) \quad f(2x - y) + f(2x + y) \\ = 12f(x) + f(2x + 2y) + f(2x - 2y) - 6[f(x + y) + f(x - y)]$$

for all $x, y \in X$. Since g is additive, it follows from (2.11) and (2.14) that

$$h(2x + y) + h(2x - y) = 2[h(x + y) + h(x - y)] + 12h(x)$$

for all $x, y \in X$. So the function h is cubic. □

Theorem 2.3. *A function $f : X \rightarrow Y$ satisfies (1.4) if and only if there exist functions $C : X \times X \times X \rightarrow Y$ and $A : X \rightarrow Y$ such that*

$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables and the function A is additive.

Proof. We first assume that the function $f : X \rightarrow Y$ satisfies (1.4). Let $g, h : X \rightarrow Y$ be functions defined by

$$g(x) := f(2x) - 8f(x), \quad h(x) := f(2x) - 2f(x)$$

for all $x \in X$. By Lemmas 2.1 and 2.2, we achieve that the functions g and h are additive and cubic, respectively, and

$$(2.15) \quad f(x) = \frac{1}{6}[h(x) - g(x)]$$

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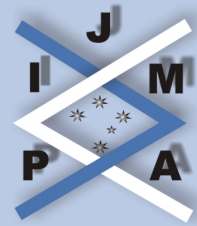
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for all $x \in X$. Therefore by Theorem 2.1 of [12] there exists a function $C : X \times X \times X \rightarrow Y$ such that $h(x) = 6C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. So

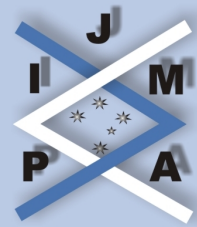
$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where $A(x) = -\frac{1}{6}g(x)$ for all $x \in X$.

Conversely, let

$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and additive for fixed two variables and the function A is additive. By a simple computation one can show that the functions $x \mapsto C(x, x, x)$ and A satisfy the functional equation (1.4). So the function f satisfies (1.4). \square



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3. Generalized Hyers-Ulam stability of Eq. (1.4)

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that Y is a p -Banach space with p -norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$.

In this section, using an idea of Găvruta [7] we prove the stability of the functional equation (1.4) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given function $f : X \rightarrow Y$:

$$Df(x, y) := f(2x + y) + f(x + 2y) - 6f(x + y) - f(2x) - f(2y) + 5[f(x) + f(y)]$$

for all $x, y \in X$.

We will use the following lemma in this section.

Lemma 3.1 ([23]). *Let $0 \leq p \leq 1$ and let x_1, x_2, \dots, x_n be non-negative real numbers. Then*

$$(3.1) \quad \left(\sum_{i=1}^n x_i \right)^p \leq \sum_{i=1}^n x_i^p.$$

Theorem 3.2. *Let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that*

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0$$

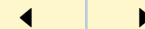
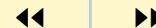
for all $x, y \in X$, and

$$(3.3) \quad M(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi^p(2^i x, 2^i y) < \infty$$



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for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$(3.4) \quad \|Df(x, y)\|_Y \leq \varphi(x, y),$$

$$(3.5) \quad \|f(x) + f(-x)\|_Y \leq \varphi(x, 0)$$

for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n \rightarrow \infty} \frac{f(2^{n+1}x) - 8f(2^n x)}{2^n}$$

exists for all $x \in X$, and the function $A : X \rightarrow Y$ is a unique additive function satisfying

$$(3.6) \quad \|f(2x) - 8f(x) - A(x) + f(0)\|_Y \leq \frac{K}{2} [\tilde{\varphi}(x)]^{\frac{1}{p}}$$

for all $x \in X$, where

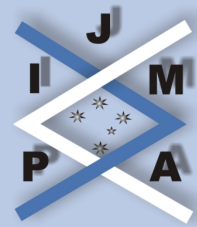
$$\tilde{\varphi}(x) = K^{2p} M(2x, -x) + \frac{K^{2p}}{2^p} M(x, x) + K^p M(2x, 0) + 5^p M(x, 0).$$

Proof. Letting $y = x$ in (3.4), we have

$$(3.7) \quad \|f(3x) - 4f(2x) + 5f(x)\|_Y \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in X$. Replacing x by $2x$ and y by $-x$ in (3.4), we have

$$(3.8) \quad \|f(3x) - f(4x) + 5f(2x) - f(-2x) - 6f(x) + 5f(-x) + f(0)\|_Y \leq \varphi(2x, -x).$$



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Using (3.5), (3.7) and (3.8), we have

$$(3.9) \quad \|g(2x) - 2g(x) - f(0)\|_Y \leq \phi(x)$$

for all $x \in X$, where

$$\phi(x) = K \left[K^2 \varphi(2x, -x) + \frac{K^2}{2} \varphi(x, x) + K \varphi(2x, 0) + 5 \varphi(x, 0) \right]$$

and $g(x) = f(2x) - 8f(x)$. By Lemma 3.1 and (3.3), we infer that

$$(3.10) \quad \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \phi^p(2^i x) < \infty$$

for all $x \in X$. Replacing x by $2^n x$ in (3.9) and dividing both sides of (3.9) by 2^{n+1} , we get

$$(3.11) \quad \left\| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x) - \frac{1}{2^{n+1}} f(0) \right\|_Y \leq \frac{1}{2^{n+1}} \phi(2^n x)$$

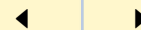
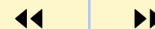
for all $x \in X$ and all non-negative integers n . Since Y is a p -Banach space, we have

$$(3.12) \quad \begin{aligned} & \left\| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^m} g(2^m x) - \sum_{i=m}^n \frac{1}{2^{i+1}} f(0) \right\|_Y^p \\ & \leq \sum_{i=m}^n \left\| \frac{1}{2^{i+1}} g(2^{i+1}x) - \frac{1}{2^i} g(2^i x) - \frac{1}{2^{i+1}} f(0) \right\|_Y^p \\ & \leq \frac{1}{2^p} \sum_{i=m}^n \frac{1}{2^{ip}} \phi^p(2^i x) \end{aligned}$$



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for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.10) and (3.12) that the sequence $\{\frac{1}{2^n}g(2^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}g(2^n x)\}$ converges in Y for all $x \in X$. So we can define the function $A : X \rightarrow Y$ by

$$(3.13) \quad A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} g(2^n x)$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \rightarrow \infty$ in (3.12), we get (3.6). Now, we show that A is an additive function. It follows from (3.10), (3.11) and (3.13) that

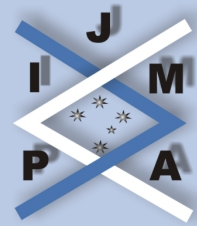
$$\begin{aligned} & \|A(2x) - 2A(x)\|_Y \\ &= \lim_{n \rightarrow \infty} \left\| \frac{1}{2^n} g(2^{n+1}x) - \frac{1}{2^{n-1}} g(2^n x) \right\|_Y \\ &\leq 2K \lim_{n \rightarrow \infty} \left(\left\| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x) - \frac{1}{2^{n+1}} f(0) \right\|_Y + \frac{1}{2^{n+1}} \|f(0)\|_Y \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{2^n} \phi(2^n x) = 0 \end{aligned}$$

for all $x \in X$. So

$$(3.14) \quad A(2x) = 2A(x)$$

for all $x \in X$. On the other hand, it follows from (3.2), (3.4) and (3.13) that

$$\begin{aligned} \|DA(x, y)\|_Y &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|Dg(2^n x, 2^n y)\|_Y \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{2^n} \{ \|Df(2^{n+1}x, 2^{n+1}y)\|_Y + 8 \|Df(2^n x, 2^n y)\|_Y \} \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{2^n} [\varphi(2^{n+1}x, 2^{n+1}y) + 8\varphi(2^n x, 2^n y)] = 0 \end{aligned}$$



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for all $x, y \in X$. Hence the function A satisfies (1.4). So by Lemma 2.1, the function $x \mapsto A(2x) - 8A(x)$ is additive. Therefore (3.14) implies that the function A is additive.

To prove the uniqueness of A , let $T : X \rightarrow Y$ be another additive function satisfying (3.6). It follows from (3.3) that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{np}} M(2^n x, 2^n y) = \lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} \frac{1}{2^{ip}} \varphi^p(2^i x, 2^i y) = 0$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Hence $\lim_{n \rightarrow \infty} \frac{1}{2^{np}} \tilde{\varphi}(2^n x) = 0$ for all $x \in X$. So it follows from (3.6) and (3.13) that

$$\begin{aligned} \|A(x) - T(x)\|_Y^p &= \lim_{n \rightarrow \infty} \frac{1}{2^{np}} \|g(2^n x) - T(2^n x) + f(0)\|_Y^p \\ &\leq \frac{K^p}{2^p} \lim_{n \rightarrow \infty} \frac{1}{2^{np}} \tilde{\varphi}(2^n x) = 0 \end{aligned}$$

for all $x \in X$. So $A = T$. □

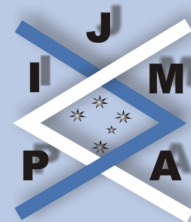
Corollary 3.3. *Let θ be non-negative real number . Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities*

$$(3.15) \quad \|Df(x, y)\|_Y \leq \theta, \quad \|f(x) + f(-x)\|_Y \leq \theta$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \rightarrow Y$ satisfying

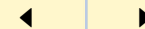
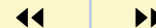
$$\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{K^2 \theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{2^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{4}$$

for all $x \in X$.



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Proof. It follows from (3.15) that $\|f(0)\|_Y \leq \theta/4$. So the result follows from Theorem 3.2. \square

Theorem 3.4. Let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} 2^n \varphi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) = 0$$

for all $x, y \in X$, and

$$(3.16) \quad M(x, y) := \sum_{i=1}^{\infty} 2^{ip} \varphi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \leq \varphi(x, y), \quad \|f(x) + f(-x)\|_Y \leq \varphi(x, 0)$$

for all $x, y \in X$. Then the limit

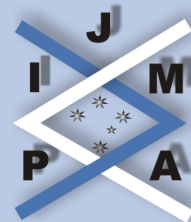
$$A(x) = \lim_{n \rightarrow \infty} 2^n \left[f \left(\frac{x}{2^{n-1}} \right) - 8f \left(\frac{x}{2^n} \right) \right]$$

exists for all $x \in X$ and the function $A : X \rightarrow Y$ is a unique additive function satisfying

$$(3.17) \quad \|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{K}{2} [\tilde{\varphi}(x)]^{\frac{1}{p}}$$

for all $x \in X$, where

$$\tilde{\varphi}(x) = K^{2p} M(2x, -x) + \frac{K^{2p}}{2^p} M(x, x) + K^p M(2x, 0) + 5^p M(x, 0).$$



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Proof. It follows from (3.16) that $\varphi(0, 0) = 0$ and so $f(0) = 0$. We introduce the same definitions for $g : X \rightarrow Y$ and $\phi(x)$ as in the proof of Theorem 3.2. Similar to the proof of Theorem 3.2, we have

$$(3.18) \quad \|g(2x) - 2g(x)\|_Y \leq \phi(x)$$

for all $x \in X$. By Lemma 3.1 and (3.16), we infer that

$$(3.19) \quad \sum_{i=1}^{\infty} 2^{ip} \phi^p\left(\frac{x}{2^i}\right) < \infty$$

for all $x \in X$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.18) and multiplying both sides of (3.18) by 2^n , we get

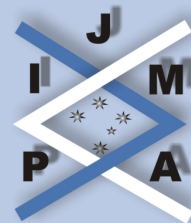
$$\left\| 2^{n+1}g\left(\frac{x}{2^{n+1}}\right) - 2^n g\left(\frac{x}{2^n}\right) \right\|_Y \leq 2^n \phi\left(\frac{x}{2^{n+1}}\right)$$

for all $x \in X$ and all non-negative integers n . Since Y is a p -Banach space, we have

$$(3.20) \quad \left\| 2^{n+1}g\left(\frac{x}{2^{n+1}}\right) - 2^m g\left(\frac{x}{2^m}\right) \right\|_Y^p \leq \sum_{i=m}^n \left\| 2^{i+1}g\left(\frac{x}{2^{i+1}}\right) - 2^i g\left(\frac{x}{2^i}\right) \right\|_Y^p \\ \leq \sum_{i=m}^n 2^{ip} \phi^p\left(\frac{x}{2^{i+1}}\right)$$

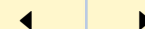
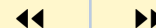
for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.19) and (3.20) that the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{2^n g(x/2^n)\}$ converges in Y for all $x \in X$. So we can define the function $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n g\left(\frac{x}{2^n}\right)$$



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for all $x \in X$. Letting $m = 0$ and passing the limit when $n \rightarrow \infty$ in (3.20) and applying Lemma 3.1, we get (3.17).

The rest of the proof is similar to the proof of Theorem 3.2 and we omit the details. \square

Corollary 3.5. *Let θ, r, s be non-negative real numbers such that $r, s > 1$ or $0 < r, s < 1$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities*

$$(3.21) \quad \|Df(x, y)\|_Y \leq \theta(\|x\|_X^r + \|y\|_X^s), \quad \|f(x) + f(-x)\|_Y \leq \theta\|x\|_X^r$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \rightarrow Y$ satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{K\theta}{2} \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K)^p + 10^p}{|2^p - 2^{rp}|} \|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|2^p - 2^{sp}|} \|x\|_X^{sp} \right\}^{\frac{1}{p}}$$

for all $x \in X$.

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Theorems 3.2 and 3.4. \square

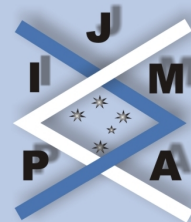
Corollary 3.6. *Let $\theta \geq 0$ and $r, s > 0$ be real numbers such that $\lambda := r + s \neq 1$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality*

$$(3.22) \quad \|Df(x, y)\|_Y \leq \theta\|x\|_X^r\|y\|_Y^s$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \rightarrow Y$ satisfying

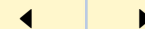
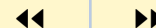
$$\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{K^3\theta}{2} \left\{ \frac{1 + 2^{p(r+1)}}{|2^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda$$

for all $x \in X$.



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Proof. $f(0) = 0$, since f is odd. Hence the result follows from Theorems 3.2 and 3.4. \square

Theorem 3.7. Let $\psi : X \times X \rightarrow [0, \infty)$ be a function such that

$$(3.23) \quad \lim_{n \rightarrow \infty} \frac{1}{8^n} \psi(2^n x, 2^n y) = 0$$

for all $x, y \in X$, and

$$(3.24) \quad M(x, y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}} \psi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$(3.25) \quad \|Df(x, y)\|_Y \leq \psi(x, y), \quad \|f(x) + f(-x)\|_Y \leq \psi(x, 0)$$

for all $x, y \in X$. Then the limit

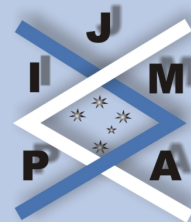
$$C(x) = \lim_{n \rightarrow \infty} \frac{1}{8^n} [f(2^{n+1}x) - 2f(2^n x)]$$

exists for all $x \in X$, and $C : X \rightarrow Y$ is a unique cubic function satisfying

$$(3.26) \quad \left\| f(2x) - 2f(x) - C(x) + \frac{1}{7}f(0) \right\|_Y \leq \frac{K}{8} [\tilde{\psi}(x)]^{\frac{1}{p}}$$

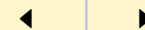
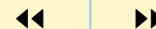
for all $x \in X$, where

$$\tilde{\psi}(x) := K^{2p} M(2x, -x) + \frac{K^{2p}}{2^p} M(x, x) + K^p M(2x, 0) + 5^p M(x, 0).$$



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Proof. Similar to the proof of Theorem 3.2, we have

$$(3.27) \quad \|f(4x) - 10f(2x) + 16f(x) - f(0)\|_Y \leq \phi(x)$$

for all $x \in X$, where

$$\phi(x) = K \left[K^2 \psi(2x, -x) + \frac{K^2}{2} \psi(x, x) + K \psi(2x, 0) + 5 \psi(x, 0) \right].$$

Let $h : X \rightarrow Y$ be a function defined by $h(x) = f(2x) - 2f(x)$. Hence (3.27) means

$$(3.28) \quad \|h(2x) - 8h(x) - f(0)\|_Y \leq \phi(x)$$

for all $x \in X$. By Lemma 3.1 and (3.24), we infer that

$$(3.29) \quad \sum_{i=0}^{\infty} \frac{1}{8^{ip}} \phi^p(2^i x) < \infty$$

for all $x \in X$. Replacing x by $2^n x$ in (3.28) and dividing both sides of (3.28) by 8^{n+1} , we get

$$(3.30) \quad \left\| \frac{1}{8^{n+1}} h(2^{n+1}x) - \frac{1}{8^n} h(2^n x) - \frac{1}{8^{n+1}} f(0) \right\|_Y \leq \frac{1}{8^{n+1}} \phi(2^n x)$$

for all $x \in X$ and all non-negative integers n . Since Y is a p -Banach space, we have

$$(3.31) \quad \left\| \frac{1}{8^{n+1}} h(2^{n+1}x) - \frac{1}{8^m} h(2^m x) - \sum_{i=m}^n \frac{1}{8^{i+1}} f(0) \right\|_Y^p \\ \leq \sum_{i=m}^n \left\| \frac{1}{8^{i+1}} h(2^{i+1}x) - \frac{1}{8^i} h(2^i x) - \frac{1}{8^{i+1}} f(0) \right\|_Y^p$$

$$\leq \frac{1}{8^p} \sum_{i=m}^n \frac{1}{8^{ip}} \phi^p(2^i x)$$

for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.29) and (3.31) that the sequence $\left\{ \frac{1}{8^n} h(2^n x) \right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\left\{ \frac{1}{8^n} h(2^n x) \right\}$ converges for all $x \in X$. So we can define the function $C : X \rightarrow Y$ by:

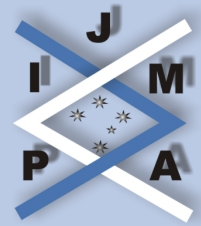
$$(3.32) \quad C(x) = \lim_{n \rightarrow \infty} \frac{1}{8^n} h(2^n x)$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \rightarrow \infty$ in (3.31), we get (3.26). Now, we show that the function C is cubic. It follows from (3.29), (3.30) and (3.32) that

$$\begin{aligned} & \|C(2x) - 8C(x)\|_Y \\ &= \lim_{n \rightarrow \infty} \left\| \frac{1}{8^n} h(2^{n+1}x) - \frac{1}{8^{n-1}} h(2^n x) \right\|_Y \\ &\leq 8K \lim_{n \rightarrow \infty} \left(\left\| \frac{1}{8^{n+1}} h(2^{n+1}x) - \frac{1}{8^n} h(2^n x) - \frac{1}{8^{n+1}} f(0) \right\|_Y + \frac{1}{8^{n+1}} \|f(0)\|_Y \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{8^n} \phi(2^n x) = 0 \end{aligned}$$

for all $x \in X$. Therefore we have

$$(3.33) \quad C(2x) = 8C(x)$$



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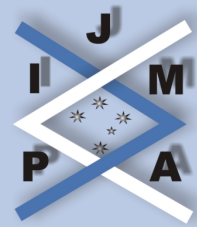
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for all $x \in X$. On the other hand, it follows from (3.23), (3.25) and (3.32) that

$$\begin{aligned} \|DC(x, y)\|_Y &= \lim_{n \rightarrow \infty} \frac{1}{8^n} \|Dh(2^n x, 2^n y)\|_Y \\ &= \lim_{n \rightarrow \infty} \frac{1}{8^n} \{ \|Df(2^{n+1}x, 2^{n+1}y) - 2Df(2^n x, 2^n y)\|_Y \} \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{8^n} \{ \|Df(2^{n+1}x, 2^{n+1}y)\|_Y + 2\|Df(2^n x, 2^n y)\|_Y \} \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{8^n} [\psi(2^{n+1}x, 2^{n+1}y) + 2\psi(2^n x, 2^n y)] = 0 \end{aligned}$$

for all $x, y \in X$. Hence the function C satisfies (1.4). So by Lemma 2.2, the function $x \mapsto C(2x) - 2C(x)$ is cubic. Hence (3.33) implies that the function C is cubic. To prove the uniqueness of C , let $T : X \rightarrow Y$ be another cubic function satisfying (3.26). It follows from (3.24) that

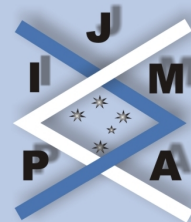
$$\lim_{n \rightarrow \infty} \frac{1}{8^{np}} M(2^n x, 2^n y) = \lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} \frac{1}{8^{ip}} \psi^p(2^i x, 2^i y) = 0$$

for all $x \in X$ and $y \in \{0, x, -x/2\}$. Hence $\lim_{n \rightarrow \infty} \frac{1}{8^{np}} \tilde{\psi}(2^n x) = 0$ for all $x \in X$. So it follows from (3.26) and (3.32) that

$$\begin{aligned} \|C(x) - T(x)\|_Y^p &= \lim_{n \rightarrow \infty} \frac{1}{8^{np}} \|h(2^n x) - T(2^n x) + \frac{1}{7}f(0)\|_Y^p \\ &\leq \frac{K^p}{8^p} \lim_{n \rightarrow \infty} \frac{1}{8^{np}} \tilde{\psi}(2^n x) = 0 \end{aligned}$$

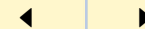
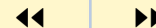
for all $x \in X$. So $C = T$. □

Corollary 3.8. *Let θ be non-negative real number. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities (3.15). Then there exists a unique cubic function*



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$C : X \rightarrow Y$ satisfying

$$\|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{8^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{28}$$

for all $x \in X$.

Proof. We get from (3.15) that $\|f(0)\| \leq \theta/4$. So the result follows from Theorem 3.7. \square

Theorem 3.9. Let $\psi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} 8^n \psi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) = 0$$

for all $x, y \in X$, and

$$(3.34) \quad M(x, y) := \sum_{i=1}^{\infty} 8^{ip} \psi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

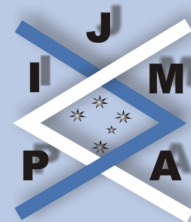
$$\|Df(x, y)\|_Y \leq \psi(x, y), \quad \|f(x) + f(-x)\|_Y \leq \psi(x, 0)$$

for all $x, y \in X$. Then the limit

$$C(x) = \lim_{n \rightarrow \infty} 8^n \left[f \left(\frac{x}{2^{n-1}} \right) - 2f \left(\frac{x}{2^n} \right) \right]$$

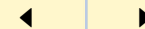
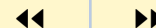
exists for all $x \in X$ and the function $C : X \rightarrow Y$ is a unique cubic function satisfying

$$(3.35) \quad \|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{K}{8} [\tilde{\psi}(x)]^{\frac{1}{p}}$$



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for all $x \in X$, where

$$\tilde{\psi}(x) = K^{2p}M(2x, -x) + \frac{K^{2p}}{2^p}M(x, x) + K^pM(2x, 0) + 5^pM(x, 0).$$

Proof. It follows from (3.34) that $\psi(0, 0) = 0$ and so $f(0) = 0$. We introduce the same definitions for $h : X \rightarrow Y$ and $\phi(x)$ as in the proof of Theorem 3.7. Similar to the proof of Theorem 3.7, we have

$$(3.36) \quad \|h(2x) - 8h(x)\|_Y \leq \phi(x)$$

for all $x \in X$. By Lemma 3.1 and (3.34), we infer that

$$(3.37) \quad \sum_{i=1}^{\infty} 8^{ip} \phi^p \left(\frac{x}{2^i} \right) < \infty$$

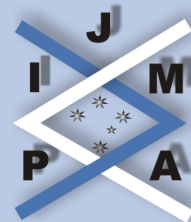
for all $x \in X$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.36) and multiplying both sides of (3.36) to 8^n , we get

$$\left\| 8^{n+1}h \left(\frac{x}{2^{n+1}} \right) - 8^n h \left(\frac{x}{2^n} \right) \right\|_Y \leq 8^n \phi \left(\frac{x}{2^{n+1}} \right)$$

for all $x \in X$ and all non-negative integers n . Since Y is a p -Banach space, we have

$$(3.38) \quad \left\| 8^{n+1}h \left(\frac{x}{2^{n+1}} \right) - 8^m h \left(\frac{x}{2^m} \right) \right\|_Y^p \leq \sum_{i=m}^n \left\| 8^{i+1}h \left(\frac{x}{2^{i+1}} \right) - 8^i h \left(\frac{x}{2^i} \right) \right\|_Y^p \\ \leq \sum_{i=m}^n 8^{ip} \phi^p \left(\frac{x}{2^{i+1}} \right)$$

for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefor we conclude from (3.37) and (3.38) that the sequence $\{8^n h(x/2^n)\}$ is a Cauchy sequence



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in Y for all $x \in X$. Since Y is complete, the sequence $\{8^n h(x/2^n)\}$ converges in Y for all $x \in X$. So we can define the function $C : X \rightarrow Y$ by

$$C(x) := \lim_{n \rightarrow \infty} 8^n h\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \rightarrow \infty$ in (3.38) and applying Lemma 3.1, we get (3.35).

The rest of the proof is similar to the proof of Theorem 3.7 and we omit the details. \square

Corollary 3.10. *Let θ, r, s be non-negative real numbers such that $r, s > 3$ or $0 < r, s < 3$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities (3.21). Then there exists a unique cubic function $C : X \rightarrow Y$ satisfying*

$$\begin{aligned} & \|f(2x) - 2f(x) - C(x)\|_Y \\ & \leq \frac{K\theta}{2} \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|8^p - 2^{rp}|} \|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|8^p - 2^{sp}|} \|x\|_X^{sp} \right\}^{\frac{1}{p}} \end{aligned}$$

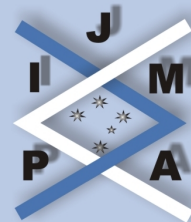
for all $x \in X$.

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Theorems 3.7 and 3.9. \square

Corollary 3.11. *Let θ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \neq 3$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality (3.22). Then there exists a unique cubic function $C : X \rightarrow Y$ satisfying*

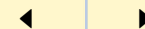
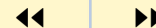
$$\|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{K^3\theta}{2} \left\{ \frac{1 + 2^{(r+1)p}}{|8^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} \|x\|_X^\lambda$$

for all $x \in X$.



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Proof. $f(0) = 0$, since f is odd. Hence the result follows from Theorems 3.7 and 3.9. \square

Theorem 3.12. Let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0$$

for all $x, y \in X$, and

$$M_a(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \leq \varphi(x, y) \quad \|f(x) + f(-x)\|_Y \leq \varphi(x, 0)$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

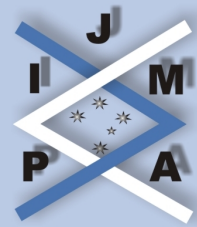
$$(3.39) \quad \left\| f(x) - A(x) - C(x) - \frac{1}{7}f(0) \right\|_Y \leq \frac{K^2}{48} \left\{ 4[\widetilde{\varphi}_a(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$, where

$$M_c(x, y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}} \varphi^p(2^i x, 2^i y),$$

$$\widetilde{\varphi}_c(x) := K^{2p} M_c(2x, -x) + \frac{K^{2p}}{2^p} M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0),$$

$$\widetilde{\varphi}_a(x) := K^{2p} M_a(2x, -x) + \frac{K^{2p}}{2^p} M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0).$$



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Proof. By Theorems 3.2 and 3.7, there exists an additive function $A_0 : X \rightarrow Y$ and a cubic function $C_0 : X \rightarrow Y$ such that

$$\|A_0(x) - f(2x) + 8f(x) - f(0)\|_Y \leq \frac{K}{2} [\widetilde{\varphi}_a(x)]^{\frac{1}{p}},$$

$$\|C_0(x) - f(2x) + 2f(x) - \frac{1}{7}f(0)\|_Y \leq \frac{K}{8} [\widetilde{\varphi}_c(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore it follows from the last inequalities that

$$\left\| f(x) + \frac{1}{6}A_0(x) - \frac{1}{6}C_0(x) - \frac{1}{7}f(0) \right\|_Y \leq \frac{K^2}{48} \left\{ 4[\widetilde{\varphi}_a(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$. So we obtain (3.39) by letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ for all $x \in X$.

To prove the uniqueness of A and C , let $A_1, C_1 : X \rightarrow Y$ be further additive and cubic functions satisfying (3.39). Let $A' = A - A_1$ and $C' = C - C_1$. Then

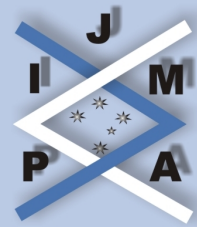
$$(3.40) \quad \|A'(x) + C'(x)\|_Y \leq K \left[\left\| f(x) - A(x) - C(x) - \frac{1}{7}f(0) \right\|_Y \right. \\ \left. + \left\| f(x) - A_1(x) - C_1(x) - \frac{1}{7}f(0) \right\|_Y \right] \\ \leq \frac{K^3}{24} \left\{ 4[\widetilde{\varphi}_a(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$. Since

$$\lim_{n \rightarrow \infty} \frac{1}{8^{np}} \widetilde{\varphi}_c(2^n x) = \lim_{n \rightarrow \infty} \frac{1}{2^{np}} \widetilde{\varphi}_a(2^n x) = 0$$

for all $x \in X$, then (3.40) implies that

$$\lim_{n \rightarrow \infty} \frac{1}{8^n} \|A'(2^n x) + C'(2^n x)\|_Y = 0$$



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for all $x \in X$. Since A' is additive and C' is cubic, we get $C' = 0$. So it follows from (3.40) that

$$\|A'(x)\|_Y \leq \frac{5K^3}{24} [\widetilde{\varphi}_a(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore $A' = 0$. □

Corollary 3.13. *Let θ be a non-negative real number. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities (3.15). Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying*

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{K}{6} (\delta_a + \delta_c)$$

for all $x \in X$, where

$$\delta_a = \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + k^{2p} + 10^p}{2^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{4},$$

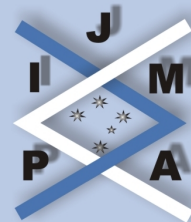
$$\delta_c = \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{8^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{28}.$$

Theorem 3.14. *Let $\psi : X \times X \rightarrow [0, \infty)$ be a function such that*

$$\lim_{n \rightarrow \infty} 8^n \psi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) = 0$$

for all $x, y \in X$, and

$$M_c(x, y) := \sum_{i=1}^{\infty} 8^{ip} \psi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right) < \infty$$



for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \leq \psi(x, y) \quad \|f(x) + f(-x)\|_Y \leq \psi(x, 0)$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

$$(3.41) \quad \|f(x) - A(x) - C(x)\|_Y \leq \frac{K^2}{48} \left\{ 4[\widetilde{\psi}_a(x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$, where

$$M_a(x, y) := \sum_{i=1}^{\infty} 2^{ip} \psi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right),$$

$$\widetilde{\psi}_c(x) := K^{2p} M_c(2x, -x) + \frac{K^{2p}}{2^p} M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0),$$

$$\widetilde{\psi}_a(x) := K^{2p} M_a(2x, -x) + \frac{K^{2p}}{2^p} M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0).$$

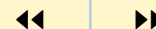
Proof. Applying Theorems 3.4 and 3.9, we get (3.41). \square

Corollary 3.15. Let θ, r, s be non-negative real numbers such that $r, s > 3$ or $0 < r, s < 1$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities (3.21). Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

$$(3.42) \quad \|f(x) - A(x) - C(x)\|_Y \leq \frac{K^2 \theta}{12} [\delta_a(x) + \delta_c(x)]$$

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for all $x \in X$, where

$$\delta_a(x) = \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|2^p - 2^{rp}|} \|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|2^p - 2^{sp}|} \|x\|_X^{sp} \right\}^{\frac{1}{p}},$$

$$\delta_c(x) = \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|8^p - 2^{rp}|} \|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|8^p - 2^{sp}|} \|x\|_X^{sp} \right\}^{\frac{1}{p}}.$$

Corollary 3.16. Let $\theta \geq 0$ and $r, s > 0$ be real numbers such that $\lambda := r + s \in (0, 1) \cup (3, +\infty)$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality (3.22). Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

$$(3.43) \quad \|f(x) - A(x) - C(x)\|_Y \leq \frac{K^4\theta}{12} \left[\left\{ \frac{1 + 2^{p(r+1)}}{|2^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} + \left\{ \frac{1 + 2^{p(r+1)}}{|8^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} \right] \|x\|_X^\lambda$$

for all $x \in X$.

Theorem 3.17. Let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

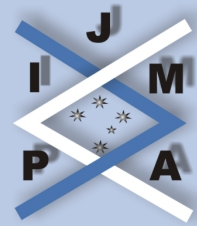
$$\lim_{n \rightarrow \infty} \frac{1}{8^n} \varphi(2^n x, 2^n y) = 0, \quad \lim_{n \rightarrow \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$, and

$$M_a(x, y) := \sum_{i=1}^{\infty} 2^{ip} \varphi^p\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \quad M_c(x, y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}} \varphi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \leq \varphi(x, y) \quad \|f(x) + f(-x)\|_Y \leq \varphi(x, 0)$$



for all $x, y \in X$. Then there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{K^2}{48} \left\{ 4[\widetilde{\varphi}_a(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$, where

$$\begin{aligned} \widetilde{\varphi}_c(x) &:= K^{2p} M_c(2x, -x) + \frac{K^{2p}}{2^p} M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0), \\ \widetilde{\varphi}_a(x) &:= K^{2p} M_a(2x, -x) + \frac{K^{2p}}{2^p} M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0). \end{aligned}$$

Proof. By the assumption, we get $f(0) = 0$. So the result follows from Theorem 3.4 and Theorem 3.7. \square

Corollary 3.18. Let θ, r, s be non-negative real numbers such that $1 < r, s < 3$. Suppose that a function $f : X \rightarrow Y$ satisfies the inequalities (3.21) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ satisfying (3.42).

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Corollaries 3.5 and 3.10. \square

Corollary 3.19. Let θ, r, s be non-negative real numbers such that $1 < \lambda := r + s < 3$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality (3.22) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ satisfying (3.43).

Proof. $f(0) = 0$, since f is odd. Hence the result follows from Corollaries 3.6 and 3.11. \square

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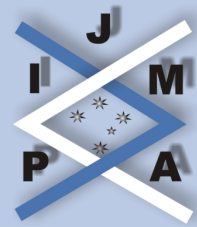
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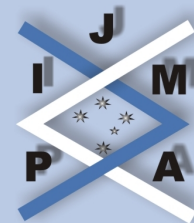
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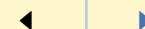
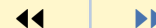
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