journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 2, Article 54, 7 pp.

INVARIANCE IN THE CLASS OF WEIGHTED LEHMER MEANS

IULIA COSTIN AND GHEORGHE TOADER

DEPARTMENT OF COMPUTER SCIENCE
TECHNICAL UNIVERSITY OF CLUJ, ROMANIA
iulia.costin@cs.utcluj.ro

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF CLUJ, ROMANIA
gheorghe.toader@math.utcluj.ro

Received 15 July, 2007; accepted 26 March, 2008
Communicated by L. Losonczi

ABSTRACT. We study invariance in the class of weighted Lehmer means. Thus we look at triples of weighted Lehmer means with the property that one is invariant with respect to the other two.

Key words and phrases: Lehmer means, Generalized means, Invariant means, Complementary means, Double sequences.

2000 *Mathematics Subject Classification.* 26E60.

1. MEANS

The abstract definitions of means are usually given as:

Definition 1.1. A **mean** is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad \forall a,b > 0.$$

A mean M is called **symmetric** if

$$M(a,b) = M(b,a), \quad \forall a,b > 0.$$

In [12] the following definition was given:

Definition 1.2. The function M is called a **generalized mean** if it has the property

$$M(a,a) = a, \qquad \forall a > 0.$$

A generalized mean is called in [10] a **pre-mean**, which seems more adequate.
Of course, each mean is reflexive, thus it is a generalized mean.
In what follows, we use the weighted Lehmer means $\mathcal{C}_{p;\lambda}$ defined by

$$\mathcal{C}_{p;\lambda}(a,b) = \frac{\lambda \cdot a^p + (1-\lambda)\cdot b^p}{\lambda \cdot a^{p-1} + (1-\lambda)\cdot b^{p-1}},$$

236-07

with $\lambda \in [0,1]$ fixed. Important special cases are the weighted arithmetic mean and the weighted harmonic mean, given respectively by

$$\mathcal{A}_\lambda = \mathcal{C}_{1;\lambda} \quad \text{and} \quad \mathcal{H}_\lambda = \mathcal{C}_{0;\lambda}.$$

For $\lambda = 1/2$ we get the symmetric means denoted by $\mathcal{C}_p, \mathcal{A}$ and $\mathcal{H}$. Note that the geometric mean can also be obtained, but the weighted geometric mean cannot:

$$\mathcal{C}_{1/2} = \mathcal{G} \quad \text{but} \quad \mathcal{C}_{1/2;\lambda} \neq \mathcal{G}_\lambda \quad \text{for} \quad \lambda \neq 1/2.$$

For $\lambda = 0$ and $\lambda = 1$ we have

$$\mathcal{C}_{p;0} = \Pi_2 \quad \text{respectively} \quad \mathcal{C}_{p;1} = \Pi_1, \quad \forall p \in \mathbb{R},$$

where Π_1 and Π_2 are the first and the second projections, defined respectively by

$$\Pi_1(a,b) = a, \quad \Pi_2(a,b) = b, \quad \forall a, b \geq 0.$$

If $\lambda \notin [0,1]$ the functions $\mathcal{C}_{p;\lambda}$ are generalized means only.

2. INVARIANT MEANS

Given three means P, Q and R , their **compound**

$$P(Q,R)(a,b) = P(Q(a,b), R(a,b)), \quad \forall a, b > 0,$$

defines also a mean $P(Q,R)$.

Definition 2.1. A mean P is called $(Q,R)-$**invariant** if it verifies

$$P(Q,R) = P.$$

Remark 1. Using the property of $(\mathcal{A}, \mathcal{G})$ $-$invariance of the mean

$$M(a,b) = \frac{\pi}{2} \cdot \left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} \right]^{-1},$$

Gauss showed that this mean gives the limit of the arithmetic-geometric double sequence. As was proved in [1], this property is generally valid: the mean P which is $(Q,R)-$invariant gives the limit of the double sequence of Gauss type defined with the means Q and R :

$$a_{n+1} = Q(a_n, b_n), \quad b_{n+1} = R(a_n, b_n), \quad n \geq 0.$$

Moreover, the validity of this property for generalized means is proved in [14] (if the limit L exists and $P(L,L)$ is defined).

Remark 2. In this paper, we are interested in the **problem of invariance** in a family $\mathcal{M}$ of means. It consists of determining all the triples of means (P,Q,R) from $\mathcal{M}$ such that P is $(Q,R)-$invariant. This problem was considered for the first time for the class of quasi-arithmetic means by Sutô in [11] and many years later by J. Matkowski in [8]. It was called the problem of Matkowski-Sutô and was completely solved in [4]. The invariance problem was also solved for the class of weighted quasi-arithmetic means in [6], for the class of Greek means in [13] and for the class of Gini-Beckenbach means in [9]. In this paper we are interested in the problem of invariance in the class of weighted Lehmer means. We use the method of series expansion of means, as in [13]. The other papers mentioned before have used functional equations methods.

3. SERIES EXPANSION OF MEANS

For the study of some problems related to a mean M, in [7] the power series expansions of the normalized function $M(1,1-x)$ is used. For some means it is very difficult, or even impossible to determine all the coefficients. In these cases, a recurrence relation for the coefficients is very useful. Such a formula is presented in [5] as **Euler's formula**.

Theorem 3.1. *If the function f has the Taylor series*

$$f(x)=\sum_{n-0}^{\infty}a_n\cdot x^n,$$

p is a real number and

$$[f(x)]^p=\sum_{n-0}^{\infty}b_n\cdot x^n,$$

then we have the recurrence relation

$$\sum_{k=0}^{n}[k(p+1)-n]\cdot a_k\cdot b_{n-k}=0,\quad n\geq 0.$$

Using it in [3], the series expansion of the weighted Lehmer mean is given by:

$$\begin{aligned}&\mathcal{C}_{p;\lambda}(1,1-x)\\&\quad=1-(1-\lambda)\,x+\lambda\,(1-\lambda)\,(p-1)\,x^2-\lambda\,(1-\lambda)\,(p-1)\,[2\lambda\,(p-1)-p]\frac{x^3}{2}\\&\qquad+\lambda\,(1-\lambda)\,(p-1)\left[6\lambda^2\,(p-1)^2-6\lambda p\,(p-1)+p\,(p+1)\right]\cdot\frac{x^4}{6}+\cdots.\end{aligned}$$

4. $\mathcal{C}_{p,\lambda}-$COMPLEMENTARY OF MEANS

If the mean P is $(Q,R)-$invariant, the mean R is called **complementary to** Q **with respect to** P (or $P-$**complementary** to Q). If a given mean Q has a unique $P-$complementary mean R, we denote it by $R=Q^P$.

Some obvious general examples are given in the following

Proposition 4.1. *For every mean M we have*

$$M^M=M,\quad \Pi_1^M=\Pi_2,\quad M^{\Pi_2}=\Pi_2.$$

If M is a symmetric mean we have also

$$\Pi_2^M=\Pi_1.$$

*We shall call these results **trivial cases** of complementariness.*

Denote the $\mathcal{C}_{p;\lambda}-$complementary of the mean M by $M^{\mathcal{C}(p;\lambda)}$, or by $M^{\mathcal{C}(p)}$ if $\lambda=1/2$. Using Euler's formula, we can establish the following.

Theorem 4.2. *If the mean M has the series expansion*

$$M(1,1-x)=1+\sum_{n=0}^{\infty}a_nx^n,$$

then the first terms of the series expansion of $M^{\mathcal{C}(p;\lambda)}$, for $\lambda \neq 0,1$, are

$$\begin{aligned}
&M^{\mathcal{C}(p;\lambda)}(1,1-x)\\
&\quad = 1-\frac{1-\lambda+\lambda a_1}{1-\lambda}x-\frac{\lambda}{(1-\lambda)^2}\left[(p-1)\,a_1\,(a_1+2\,(1-\lambda))+a_2\,(1-\lambda)\right]\cdot x^2\\
&\qquad -\frac{\lambda}{2\,(1-\lambda)^3}\Big[a_1\,(p-1)\left(2\lambda^3 p-\lambda^2\,(p+2)-4\lambda\,(p-1)+3p-2\right)\\
&\qquad\quad +a_1^2\,(p-1)\left(2\lambda^2\,(1-3p)+\lambda\,(3p+2)+3p-4\right)+a_1^3\,(p-1)\,(2\lambda p+p-2)\\
&\qquad\qquad +4a_2\,(p-1)\,(1-\lambda)^2+4a_1a_2\,(p-1)\,(1-\lambda)+2a_3\,(1-\lambda)^2\Big]\cdot x^3+\cdots .
\end{aligned}$$

Corollary 4.3. *The first terms of the series expansion of $\mathcal{C}_{r;\mu}^{\mathcal{C}(p;\lambda)}$ are*

$$\begin{aligned}
&\mathcal{C}_{r;\mu}^{\mathcal{C}(p;\lambda)}(1,1-x)\\
&= 1-\frac{1-2\lambda+\lambda\mu}{1-\mu}x+\frac{\lambda\,(1-\mu)}{(1-\lambda)^2}[p\,(1-2\lambda+\mu)+\mu r\,(\lambda-1)-1\\
&\quad +2\lambda-\lambda\mu]x^2+\frac{\lambda\,(1-\mu)}{(1-\lambda)^3}\left[p^2\left(2\lambda^3+2\lambda\mu^2-6\lambda^2\mu-\lambda\mu+5\lambda^2+\mu^2+\mu-5\lambda+1\right)\right.\\
&\quad +4pr\left(\lambda\mu^2+\lambda\mu-\lambda^2\mu-\mu^2\right)+r^2\left(2\lambda\mu-4\lambda\mu^2-\lambda^2\mu-\mu+2\mu^2\right)+p\left(2\lambda^2\mu^2\right.\\
&\quad \left.+12\lambda^2\mu-6\lambda\mu^2-2\lambda^3-9\lambda^2+\mu^2-\lambda\mu+7\lambda-\mu-1\right)+r\left(5\lambda^2\mu-4\lambda^2\mu^2\right.\\
&\quad \left.\left.+4\lambda\mu^2-6\lambda\mu+\mu\right)+2\lambda^2\mu^2+4\lambda^2-6\lambda^2\mu+2\lambda\mu-2\lambda\right]x^3+\cdots .
\end{aligned}$$

Using them we can prove the following main result.

Corollary 4.4. *We have*

$$\mathcal{C}_{p;\lambda}(\mathcal{C}_{r;\mu},\mathcal{C}_{u;\nu})=\mathcal{C}_{p;\lambda}$$

if we are in one of the following non-trivial cases:

$i)$	$\mathcal{C}_{1;\lambda}(\mathcal{C}_{1;(2\lambda-1)/\lambda},\mathcal{C}_{u;1})=\mathcal{C}_{1;\lambda};$
$ii)$	$\mathcal{C}_{0;\lambda}(\mathcal{C}_{0;(2\lambda-1)/\lambda},\mathcal{C}_{u;1})=\mathcal{C}_{0;\lambda};$
$iii)$	$\mathcal{C}_0(\mathcal{C}_{r;\mu},\mathcal{C}_{-r;1-\mu})=\mathcal{C}_0;$
$iv)$	$\mathcal{C}_{1/2}(\mathcal{C}_{r;\mu},\mathcal{C}_{1-r;1-\mu})=\mathcal{C}_{1/2};$
$v)$	$\mathcal{C}_1(\mathcal{C}_{r;\mu},\mathcal{C}_{2-r;1-\mu})=\mathcal{C}_1;$
$vi)$	$\mathcal{C}_{0;\lambda}(\mathcal{C}_{0;(3\lambda-1)/2\lambda},\mathcal{C}_{0;1/2})=\mathcal{C}_{0;\lambda};$
$vii)$	$\mathcal{C}_{1;\lambda}(\mathcal{C}_{1;(3\lambda-1)/2\lambda},\mathcal{C}_1)=\mathcal{C}_{1;\lambda};$
$viii)$	$\mathcal{C}_{0,1/3}(\mathcal{C}_{r;0},\mathcal{C}_0)=\mathcal{C}_{0;1/3};$
$ix)$	$\mathcal{C}_{1,1/3}(\mathcal{C}_{r;0},\mathcal{C}_1)=\mathcal{C}_{1;1/3};$
$x)$	$\mathcal{C}_{2,1/4}(\mathcal{C}_{1;-1/2},\mathcal{C}_1)=\mathcal{C}_{2,1/4};$
$xi)$	$\mathcal{C}_{-1,1/4}(\mathcal{C}_{0;-1/2},\mathcal{C}_0)=\mathcal{C}_{-1,1/4};$
$xii)$	$\mathcal{C}_{0;\lambda}(\mathcal{C}_0,\mathcal{C}_{0;\lambda/(2-2\lambda)})=\mathcal{C}_{0;\lambda};$
$xiii)$	$\mathcal{C}_{1;\lambda}(\mathcal{C}_1,\mathcal{C}_{1;\lambda/(2-2\lambda)})=\mathcal{C}_{1;\lambda};$
$xiv)$	$\mathcal{C}_{-1;3/4}(\mathcal{C}_0,\mathcal{C}_{0;3/2})=\mathcal{C}_{-1;3/4};$
$xv)$	$\mathcal{C}_{2;3/4}(\mathcal{C}_1,\mathcal{C}_{1;3/2})=\mathcal{C}_{2;3/4}.$

Proof. We consider the equivalent condition $\mathcal{C}_{r;\mu}^{\mathcal{C}(p;\lambda)} = \mathcal{C}_{u;\nu}$ which gives

$$\mathcal{C}_{r;\mu}^{\mathcal{C}(p;\lambda)}(1,1-x) = \mathcal{C}_{u;\nu}(1,1-x).$$

Equating the coefficients of $x^k, k = 1,2,\dots,5$, we get the following table of solutions with corresponding conclusions:

Case	λ	μ	ν	p	r	u	$\mathcal{C}_{r;\mu}^{\mathcal{C}(p;\lambda)} = \mathcal{C}_{u;\nu}$	Case
1	0	μ	0	p	r	u	$\mathcal{C}_{r;\mu}^{\Pi(2)} = \Pi_2$	Trivial
2	λ	1	0	p	r	u	$\Pi_1^{\mathcal{C}(p;\lambda)} = \Pi_2$	Trivial
3	$\frac{1}{2}$	0	1	p	r	u	$\Pi_2^{\mathcal{C}(p)} = \Pi_1$	Trivial
4	λ	$\frac{2\lambda-1}{\lambda}$	1	1	1	u	$\mathcal{A}_{\frac{2\lambda-1}{\lambda}}^{\mathcal{A}(\lambda)} = \Pi_1$	i)
5	λ	$\frac{2\lambda-1}{\lambda}$	1	0	0	u	$\mathcal{H}_{\frac{2\lambda-1}{\lambda}}^{\mathcal{H}(\lambda)} = \Pi_1$	ii)
6	$\frac{1}{2}$	μ	$1-\mu$	0	r	$-r$	$\mathcal{C}_{r;\mu}^{\mathcal{H}} = \mathcal{C}_{-r;1-\mu}$	iii)
7	$\frac{1}{2}$	μ	$1-\mu$	$\frac{1}{2}$	r	$1-r$	$\mathcal{C}_{r;\mu}^{\mathcal{G}} = \mathcal{C}_{1-r;1-\mu}$	iv)
8	$\frac{1}{2}$	μ	$1-\mu$	1	r	$2-r$	$\mathcal{C}_{r;\mu}^{\mathcal{A}} = \mathcal{C}_{2-r;1-\mu}$	v)
9	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	p	p	p	$\mathcal{C}_p^{\mathcal{C}(p)} = \mathcal{C}_p$	Trivial
10	λ	$\frac{3\lambda-1}{2\lambda}$	$\frac{1}{2}$	0	0	0	$\mathcal{H}_{\frac{3\lambda-1}{2\lambda}}^{\mathcal{H}(\lambda)} = \mathcal{H}$	vi)
11	λ	$\frac{3\lambda-1}{2\lambda}$	$\frac{1}{2}$	1	1	1	$\mathcal{A}_{\frac{3\lambda-1}{2\lambda}}^{\mathcal{A}(\lambda)} = \mathcal{A}$	vii)
12	$\frac{1}{3}$	0	$\frac{1}{2}$	0	r	0	$\Pi_2^{\mathcal{H}(1/3)} = \mathcal{H}$	viii)
13	$\frac{1}{3}$	0	$\frac{1}{2}$	1	r	1	$\Pi_2^{\mathcal{A}(1/3)} = \mathcal{A}$	ix)
14	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	2	1	1	$\mathcal{A}_{-1/2}^{\mathcal{C}(2;1/4)} = \mathcal{A}$	x)
15	$\frac{1}{4}$	-$\frac{1}{2}$	$\frac{1}{2}$	-1	0	0	$\mathcal{H}_{-1/2}^{\mathcal{C}(2;1/4)} = \mathcal{H}$	xi)
16	λ	$\frac{1}{2}$	$\frac{\lambda}{2(1-\lambda)}$	0	0	0	$\mathcal{H}^{\mathcal{H}(\lambda)} = \mathcal{H}_{\frac{\lambda}{2(1-\lambda)}}$	xii)
17	λ	$\frac{1}{2}$	$\frac{\lambda}{2(1-\lambda)}$	1	1	1	$\mathcal{A}^{\mathcal{A}(\lambda)} = \mathcal{A}_{\frac{\lambda}{2(1-\lambda)}}$	xiii)
18	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$	-1	0	0	$\mathcal{H}^{\mathcal{C}(-1;3/4)} = \mathcal{H}_{3/2}$	xiv)
19	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$	2	1	1	$\mathcal{A}^{\mathcal{C}(2;3/4)} = \mathcal{A}_{3/2}$	xv)

$\square$

Remark 3. Equating the coefficients of $x^1, x^2, ..., x^n$, we have a system of n equations with six unknowns (the parameters of the means). For $n = 2,3,4,$ solving the system, we get relations among the parameters such as:

$$\nu = \frac{\lambda(1-\mu)}{1-\lambda}, \qquad u = \frac{\lambda\mu r - \mu r + p\mu - 2\lambda p + p}{1 - 2\lambda + \lambda\mu}, \qquad r = \frac{Z}{\lambda - 1},$$

where

$$Z^2\mu(\mu-1)+2p\mu Z(\lambda-\lambda\mu+\mu-1)+\lambda^2p-2\lambda^2\mu^2p-\lambda^2p^2+2\lambda^3p^2-2\lambda^3p$$
$$+3\lambda^2\mu^2p^2-\lambda\mu^3p^2-\lambda^3\mu p^2+\lambda^3\mu p+\lambda\mu^3p+4\lambda^2\mu p+4\lambda\mu p^2$$
$$-5\lambda^2\mu p^2-2\lambda\mu p-2\lambda\mu^2p+\mu^2p-\mu p^2=0.$$

For $n=5$ we obtained the table of solutions given in the previous corollary. For $n=6$, however, the system could not even be solved using Maple. As a result, we are not certain that we have obtained all the solutions for the problem of invariance.

Remark 4. The cases i)-ii), vi)-vii), xii)-xiii) and xiv)-xv), involve $\mathcal{C}_{1;\lambda}=\mathcal{A}_\lambda$ and $\mathcal{C}_{0;\lambda}=\mathcal{H}_\lambda$. There are, however, no similar cases for $\mathcal{C}_{1/2;\lambda}$. Instead we have the following results for $\mathcal{G}_\lambda$:

$$\mathcal{G}^{\mathcal{G}(\lambda)}_{\frac{2\lambda-1}{\lambda}}=\Pi_1,\quad \Pi_2^{\mathcal{G}(1/3)}=\mathcal{G},\quad \mathcal{G}^{\mathcal{G}(\lambda)}_{\frac{3\lambda-1}{2\lambda}}=\mathcal{G},\quad \mathcal{G}^{\mathcal{G}(\lambda)}=\mathcal{G}_{\frac{\lambda}{2(1-\lambda)}},$$

but these are not Lehmer means.

Remark 5. It is easy to see that not all of the generalized means that appear in the above results are means. In such a case, the result given in Remark 1 can be negative. For example, in the case xv), if we consider

$$a_{n+1}=\mathcal{C}_1(a_n,b_n),\qquad b_{n+1}=\mathcal{C}_{1;3/2}(a_n,b_n),\qquad n\ge 0,$$

for $a_0=10$ and $b_0=1$, we get $a_2=a_0$ and $b_2=b_0$, thus the sequences are divergent. Also, in the case xii), if we take $\lambda=4/5$, the double sequence

$$a_{n+1}=\mathcal{C}_0(a_n,b_n),\qquad b_{n+1}=\mathcal{C}_{0;2}(a_n,b_n),\qquad n\ge 0,$$

has the limit zero for $a_0=10$ and $b_0=1$, which is different from $\mathcal{C}_{0;4/5}(10,1)$. This is because $\mathcal{C}_{0;4/5}$ is not defined in $(0,0)$, thus the proof of the Invariance Principle in [14] does not work.

Corollary 4.5. *For means we have*

$$\mathcal{C}_{p;\lambda}(\mathcal{C}_{r;\mu},\mathcal{C}_{u;\nu})=\mathcal{C}_{p;\lambda}$$

if we are in one of the following non-trivial cases:

$$\begin{array}{lll}
i) & \mathcal{C}_{1;\lambda}(\mathcal{C}_{1;(2\lambda-1)/\lambda},\mathcal{C}_{u;1})=\mathcal{C}_{1;\lambda}, & \lambda\in[1/2,1];\\
ii) & \mathcal{C}_{0;\lambda}(\mathcal{C}_{0;(2\lambda-1)/\lambda},\mathcal{C}_{u;1})=\mathcal{C}_{0;\lambda}, & \lambda\in[1/2,1];\\
iii) & \mathcal{C}_0(\mathcal{C}_{r;\mu},\mathcal{C}_{-r;1-\mu})=\mathcal{C}_0; & \\
iv) & \mathcal{C}_{1/2}(\mathcal{C}_{r;\mu},\mathcal{C}_{1-r;1-\mu})=\mathcal{C}_{1/2}; & \\
v) & \mathcal{C}_1(\mathcal{C}_{r;\mu},\mathcal{C}_{2-r;1-\mu})=\mathcal{C}_1; & \\
vi) & \mathcal{C}_{0;\lambda}(\mathcal{C}_{0;(3\lambda-1)/2\lambda},\mathcal{C}_{0;1/2})=\mathcal{C}_{0;\lambda}, & \lambda\in[1/3,1];\\
vii) & \mathcal{C}_{1;\lambda}(\mathcal{C}_{1;(3\lambda-1)/2\lambda},\mathcal{C}_1)=\mathcal{C}_{1;\lambda}, & \lambda\in[1/3,1];\\
viii) & \mathcal{C}_{0,1/3}(\mathcal{C}_{r;0},\mathcal{C}_0)=\mathcal{C}_{0;1/3}; & \\
ix) & \mathcal{C}_{1,1/3}(\mathcal{C}_{r;0},\mathcal{C}_1)=\mathcal{C}_{1;1/3}; & \\
x) & \mathcal{C}_{0;\lambda}(\mathcal{C}_0,\mathcal{C}_{0;\lambda/(2-2\lambda)})=\mathcal{C}_{0;\lambda}, & \lambda\in[0,2/3];\\
xi) & \mathcal{C}_{1;\lambda}(\mathcal{C}_1,\mathcal{C}_{1;\lambda/(2-2\lambda)})=\mathcal{C}_{1;\lambda}, & \lambda\in[0,2/3].
\end{array}$$

Remark 6. Each of the above results allows us to define a double sequence of Gauss type with known limit.

Corollary 4.6. *For symmetric means, we have*

$$\mathcal{C}_p(\mathcal{C}_r, \mathcal{C}_u) = \mathcal{C}_p$$

if and only if we are in the following non-trivial cases:

$$i) \qquad \mathcal{C}_0(\mathcal{C}_r, \mathcal{C}_{-r}) = \mathcal{C}_0;$$

$$ii) \qquad \mathcal{C}_{1/2}(\mathcal{C}_r, \mathcal{C}_{1-r}) = \mathcal{C}_{1/2};$$

$$iii) \qquad \mathcal{C}_1(\mathcal{C}_r, \mathcal{C}_{2-r}) = \mathcal{C}_1.$$

REFERENCES

[1] J.M. BORWEIN AND P.B. BORWEIN, *Pi and the AGM - a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1986.

[2] P.S. BULLEN, *Handbook of Means and their Inequalities*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 2003.

[3] I. COSTIN AND G. TOADER, A weighted Gini mean, *Proceedings of the International Symposium Specialization, Integration and Development, Section Quantitative Economics*, Babeş-Bolyai University Cluj-Napoca, Romania, 137–142, 2003.

[4] Z. DARÓCZY AND Zs. PÁLES, Gauss-composition of means and the solution of the solution of the Matkowski-Sutô problem, *Publ. Math. Debrecen*, **61**(1-2) (2002), 157–218.

[5] H.W. GOULD, Coefficient identities for powers of Taylor and Dirichlet series, *Amer. Math. Monthly*, **81** (1974), 3–14.

[6] J. JARCZYK AND J. MATKOWSKI, Invariance in the class of weighted quasi-arithmetic means, *Ann. Polon. Math.*, **88**(1) (2006), 39–51.

[7] D.H. LEHMER, On the compounding of certain means, *J. Math. Anal. Appl.*, **36** (1971), 183–200.

[8] J. MATKOWSKI, Invariant and complementary quasi-arithmetic means, *Aequationes Math.*, **57** (1999), 87–107.

[9] J. MATKOWSKI, On invariant generalized Beckenbach-Gini means, Functional Equations - Results and Advances (Z. Daróczy and Zs. Páles, eds.), *Advances in Mathematics*, Vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, 219-230.

[10] J. MATKOWSKI, On iteration of means and functional equations, in *Iteration Theory* (ECIT '4), W. Förg-Rob, L. Gardini, D. Gronau, L. Reich, J. Smital (Eds.), Grazer Math. Ber., ISSN 1016-7692, Bericht 350(2006), 184-201.

[11] O. SUTÔ, Studies on some functional equations I, *Tôhoku Math. J.*, **6** (1914), 1–15; II, *Tôhoku Math. J.*, **6** (1914), 82–101.

[12] G. TOADER, Integral generalized means, *Math. Inequal. Appl.*, **5**(3) (2002), 511–516.

[13] G. TOADER AND S. TOADER, *Greek Means and the Arithmetic-Geometric Mean*, RGMIA Monographs, Victoria University, 2005. [ONLINE: `http://www.staff.vu.edu.au/rgmia/monographs.asp`].

[14] G. TOADER AND S. TOADER, Means and generalized means, *J. Inequal. Pure Appl. Math.*, **8**(2) (2007), Art. 45. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=850`].