EXTENDED WELL-POSEDNESS FOR QUASIVARIATIONAL INEQUALITIES

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Abstract: In this paper, we introduce the concepts of extended well-posedness for quasivariational inequalities and establish some characterizations. We show that the extended well-posedness is equivalent to the existence and uniqueness of solutions under suitable conditions. In addition, the corresponding concepts of extended well-posedness in the generalized sense are introduced and investigated for quasivariational inequalities having more than one solution.

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1. Introduction

The importance of well-posedness is widely recognized in the theory of variational problems. Motivated by the study of numerical production optimization sequences, Tykhonov [18] introduced the concept of well-posedness for a minimization problem, which is known as Tykhonov well-posedness. Due to its importance in optimization problems, various concepts of well-posedness have been introduced and studied for minimization problems (see [18, 1, 5, 16, 19, 20]) in past decades. The concept of well-posedness has also been generalized to several related variational problems: saddle point problems [2], Nash equilibrium problems [11, 17, 15], inclusion problems [4, 7, 9], and fixed point problems [4, 7, 9]. A more general formulation for the above variational problems is the variational inequalities problems, which leads to the study of the well-posedness of variational inequalities. Lucchetti & Patrone [14], obtained a notion of well-posedness for a variational inequality. Lignola & Morgan [13] introduced the extended well-posedness for a family of variational inequalities and investigated its links with the extended well-posedness of corresponding minimization problems. Lignola [8] introduced the notion of well-posedness for quasivariational inequalities. Recently, Lalitha & Mehta [10] presented a class of variational inequalities defined by bifunctions. Fang & Hu [3], extended the notion of well-posedness of variational inequalities defined by bifunctions.

Inspired and motivated by above research works, in this paper, we study the well-posedness of quasivariational inequalities (in short, QVI) defined by bifunctions. We introduce the notion of extended well-posedness for QVI, and establish some of its characterizations. Under suitable conditions, we prove that the extended well-posedness is equivalent to the existence and uniqueness of solutions to QVI. With an additional compactness assumption, we also derive the equivalence between the extended well-posedness in the generalized sense and the existence of solutions to QVI.

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2. Preliminaries

Throughout this paper, let E be a reflexive real Banach space and K be a nonempty closed convex subset of E, unless otherwise specified. Let $S : K \to 2^K$ be a set-valued mapping, and $h : K \times E \to \bar{R}$ be a bifunction, where $\bar{R} = R \cup \{+\infty\}$. The quasivariational inequality problem consists in finding a point $u_0 \in K$, such that

$$\text{(QVI)} \qquad u_0 \in S(u_0) \quad \text{and} \quad h(u_0, u_0 - v) \leq 0, \quad \forall v \in S(u_0).$$

Note that QVI includes as a special case the quasivariational inequality. In this paper, we consider the parametric form of QVI which is formulated as follows:

$$\text{(QVI)}_p \qquad u_0 \in S(u_0) \quad \text{and} \quad h(p, u_0, u_0 - v) \leq 0, \quad \forall v \in S(u_0),$$

where $h : P \times K \times E \to \bar{R}$ and P is a Banach space. Now we recall some concepts and results. Let $(X, \tau), (Y, \sigma)$ be topological spaces. The closure and interior of a nonempty set G of X are respectively denoted by cl$G$ and int$G$.

Definition 2.1 ([8]). *A set-valued mapping $F : (X, \tau) \to 2^{(Y,\sigma)}$ is called:*

(i) *closed-valued if the set $F(x)$ is nonempty and σ-closed, for every $x \in X$;*

(ii) *(τ, σ)-closed if the graph $G_F = \{(x, y) : y \in F(x)\}$ is closed in $\tau \times \sigma$;*

(iii) *(τ, σ)-lower semicontinuous if for every σ-open subset V of Y, the inverse image of the set V, $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is a τ-open subset of X;*

(iv) *(τ, σ)-subcontinuous on $H \subseteq E$ (E is a reflexive real Banach space) if for every net $\{x_a\}$ τ-converging in H, every net $\{y_a\}$, such that $y_a \in F(x_a)$, has a σ-convergent subset.*

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Definition 2.2 ([8]). *The Painleve-Kuratouski limits of sequence* $\{H_n\}$, $H_n \subseteq Y$ *are defined by:*

$$\liminf_n H_n = \left\{ y \in Y : \exists y_n \in H_n, n \in N, \quad \textit{with} \quad \lim_n y_n = y \right\},$$

and

$$\limsup_n H_n = \left\{ y \in Y : \exists n_k \uparrow +\infty, n_k \in N, \exists y_{n_k} \in H_{n_k}, k \in N, \textit{ with } \lim_k y_{n_k} = y \right\}.$$

Definition 2.3 ([3]). *A bifunction* $f : K \times E \to R$ *is said to be:*

(i) *monotone if* $f(x, y - x) + f(y, x - y) \leq 0, \forall x, y \in K$*;*

(ii) *strongly monotone if there exists a constant* $t > 0$ *such that*

$$f(x, y - x) + f(y, x - y) + t\|x - y\|^2 \leq 0, \ \forall x, y \in K;$$

(iii) *pseudomonotone if for any* $x, y \in K$, $f(x, y - x) \geq 0 \Rightarrow f(y, x - y) \leq 0$*;*

(iv) *hemicontinuous if for every* $x, y \in K$ *and* $t \in [0, 1]$, *the function* $t \mapsto f(x + t(y - x), y - x)$ *is continuous at* 0^+.

In the sequel we introduce some notions of extended well-posedness for $(\mathrm{QVI})_p$.

Definition 2.4. *Let* $p \in P$, $\{p_n\} \in P$, *with* $p_n \to p$. *A sequence* $\{u_n\}$ *is an approximation for* $(\mathrm{QVI})_p$ *corresponding to* $\{p_n\}$ *if:*

(i) $u_n \in K, \forall n \in N$*;*

(ii) *there exists a sequence* $\{\varepsilon_n\} \downarrow 0$ *such that* $d(u_n, S(u_n)) \leq \varepsilon_n$ *(i.e.* $u_n \in B(S(u_n, \varepsilon_n))$*), and* $h(p_n, u_n, u_n - v) \leq \varepsilon_n$, $\forall v \in S(u_n)$, $\forall n \in N$, *where* $B(S(u), \varepsilon) = \{y \in E : d(S(u), y) \leq \varepsilon\}$.

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Remark 1. When the set-valued mapping S is constant, say $S(u) = K$ for every $u \in K$, the parametric form of (QVI)$_p$ is a parametric form of a variational inequality. In this case, the class of approximating sequences coincides with the class defined in [13].

Definition 2.5.

(i) *(QVI)$_p$ is said to be extended well-posed if for every $p \in P$, (QVI)$_p$ has a unique solution u_p and every approximating sequence for (QVI)$_p$ corresponding to $p_n \to p$ converges to u_p.*

(ii) *(QVI)$_p$ is said to be extended well-posed in the generalized sense if for every $p \in P$, (QVI)$_p$ has a nonempty solution set $T(p)$, and every approximating sequence for (QVI)$_p$ corresponding to $p_n \to p$ has a subsequence which converges to some point of $T(p)$.*

Lemma 2.6 ([13]). *Let K be a nonempty, closed, compact and convex subset of E, the set-valued mapping S is convex-valued and closed-valued. If the bifunction h is hemicontinuous and pseudomonotone, the following problems are equivalent:*

(i) *find* $u_0 \in K,$ *such that* $u_0 \in S(u_0)$ *and* $h(u_0, u_0 - v) \le 0, \ \forall v \in S(u_0)$;

(ii) *find* $u_0 \in K,$ *such that* $u_0 \in S(u_0)$ *and* $h(v, u_0 - v) \le 0, \ \forall v \in S(u_0)$.

Lemma 2.7 ([12]). *Let $\{H_n\}$ be a sequence of nonempty subsets of the space E such that:*

(i) *H_n is convex for every $n \in N$;*

(ii) *$H_0 \subseteq \liminf_n H_n$;*

(iii) *there exists $m \in N$ such that $\operatorname{int} \cap_{n \ge m} H_n \ne \emptyset$.*

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Then, for every $u_0 \in \operatorname{int} H_0$*, there exists a positive real number* δ *such that* $B(u_0, \delta) \subseteq H_n, \forall n \geq m$*.*

If E is a finite dimensional space, the assumption (iii) can be replaced by $\operatorname{int} H_0 \neq \emptyset$.

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3. Characterizations of Extended Well-Posedness

In this section, we investigate some characterizations of extended well-posedness for quasivariational inequalities. For (QVI)$_p$, the set of approximating solutions is defined by

$$T(\delta,\varepsilon)=\bigcup_{\acute{p}\in B(p,\delta)}\{u\in K:\ u\in B(S(u),\varepsilon)\quad\text{and}\quad h(\acute{p},u,u-v)\le\varepsilon,\quad\forall v\in S(u)\},$$

where $B(p,\delta)$ denotes the closed ball with radius δ and centered at p.

Theorem 3.1. *Let the following assumptions hold:*

- *(i) the set-valued mapping S is nonempty-valued and convex-valued, (s,ω)-closed, (s,s)-lower semicontinuous, and (s,ω)-subcontinuous on K;*
- *(ii) for every converging sequence $\{u_n\}$, there exists $m\in N$, such that $\operatorname{int}\bigcap_{n\ge m}S_n\ne\emptyset$ (S_n is a sequence of mappings);*
- *(iii) for every $p\in P$, $h(p,\cdot,\cdot)$ is monotone and hemicontinuous;*
- *(iv) for every $(p,u)\in P\times K$, $h(p,u,\cdot)$ is convex;*
- *(v) for every $u\in K$, $h(\cdot,u,\cdot)$ is lower semicontinuous;*

Then, the (QVI)$_p$ is extended well-posed if and only if for every $p\in P$, the solution set $T(p)$ is nonempty and

$$\operatorname{diam}T(\delta,\varepsilon)\to 0\quad as\quad(\delta,\varepsilon)\to(0,0),\tag{3.1}$$

where diam *means the diameter of a set.*

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Proof. Suppose that (QVI)$_P$ is extended well-posed. Then it has a unique solution u_0. If for some $p \in P$, $\operatorname{diam} T(\delta, \varepsilon) \not\to 0$ as $(\delta, \varepsilon) \to (0,0)$, there exist a positive number l, and sequences $\delta_n > 0$ converging to 0, $\varepsilon_n > 0$ decreasing to 0, and w_n, z_n $\in K$, with $w_n \in T(\delta_n, \varepsilon_n)$, $z_n \in T(\delta_n, \varepsilon_n)$ such that

$$\|w_n - z_n\| > l, \quad \forall n \in N.$$

Since $w_n \in T(\delta_n, \varepsilon_n)$, $z_n \in T(\delta_n, \varepsilon_n)$ for each $n \in N$, there exists $p_n, \acute{p}_n \in B_n(p, \delta_n)$, such that

$$h(p_n, w_n, w_n - v) \leq \varepsilon_n,$$

and

$$h(\acute{p}_n, z_n, z_n - v) \leq \varepsilon_n,$$

where $\forall v \in S(u_0)$. This implies that $\{w_n\}$, $\{z_n\}$ are both approximating sequences for (QVI)$_p$ corresponding to $\{p_n\}$ and $\{\acute{p}_n\}$ respectively. Since (QVI)$_p$ is extended well-posed, they have to converge to the unique solution u_0. This gives a contradiction. Thus condition (3.1) holds.

Conversely, assume that for every $p \in P$, $T(p)$ is nonempty and condition (3.1) holds. Let $p_n \to p \in P$ and $\{u_n\} \subset K$ be an approximating sequence for (QVI)$_p$ corresponding to $\{p_n\}$. There exists $\varepsilon_n > 0$ decreasing to 0, such that

$$d(u_n, S(u_n)) \leq \varepsilon_n,$$

and

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n,$$

where $\forall v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\|$. It follows from condition (3.1) that $\{u_n\}$ is a Cauchy sequence and strongly converges to a point $u_0 \in K$. To prove that u_0 solves (QVI)$_p$, we shall first show that

$$d(u_0, S(u_0)) \leq \liminf_n d(u_n, S(u_n)) \leq \lim \varepsilon_n = 0.$$

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Assume that the left inequality does not hold. Then, there exists a positive number a such that

$$\liminf_n d(u_n, S(u_n)) < a < d(u_0, S(u_0)).$$

This means that there exists an increasing sequence $\{n_k\}$ and a sequence $\{z_k\}$, $z_k \in S(u_{n_k})$, such that

$$\|u_{n_k} - z_{n_k}\| < a, \qquad \forall k \in N.$$

Since the set-valued mapping S is (s,ω)-subcontinuous and (s,ω)-closed, the sequence $\{z_k\}$ has a subsequence, still denoted by z_k, weakly converging to a point $z_0 \in S(u_0)$. Then, one gets

$$a < d(u_0, S(u_0)) \le \|u_0 - z_0\| \le \liminf_n \|u_{n_k} - z_k\| \le a,$$

which gives a contradiction. So, $u_0 \in \text{cl}S(u_0) = S(u_0)$. Then consider a point $v \in S(u_0)$ and observe that, since the set-valued mapping S is (s,s)-lower semicontinuous, one has $S(u_0) \subseteq \liminf S(u_n)$. Also, observe that condition (ii), applied to the sequence $w_n = u_0$, for all $n \in N$, implies that $\text{int}\, S(u_0) \ne \emptyset$; from Lemma 2.7, it follows that, if $v \in \text{int}\, S(u_0)$, then $v \in S(u_n)$ for n sufficiently large. Condition (iv) and (v) give that

$$h(p, v, u_0 - v) = \lim_n h(p, v, u_n - v) \le \liminf_n h(p, u_n, u_n - v) \le \liminf_n \varepsilon_n = 0.$$

If $v \in S(u_0) - \text{int}\, S(u_0)$, let $\{v_n\}$ be a sequence to v, whose points belong to a segment contained in $\text{int}\, S(u_0)$. Since $v_n \in \text{int}\, S(u_0)$, for $n \in N$, one has

$$h(p, v_n, u_0 - v_n) \le 0,$$

and in light of the hemicontinuity of the bifunction h,

$$h(p, v, u_0 - v) \le 0.$$

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Then, the result follows from Lemma 2.6. Now it remains to prove that (QVI)$_p$ has a unique solution. If (QVI)$_p$ has two distinct solutions u_1, u_2, it is easily seen that $u_1, u_2 \in T(\delta, \varepsilon)$ for all $\delta, \varepsilon > 0$. It follows that

$$0 < \|u_1 - u_2\| \leq \operatorname{diam} T(\delta, \varepsilon) \to 0,$$

and we obtain a contradiction to (3.1). □

Theorem 3.2. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω)-closed, (s, s)-lower semicontinuous, and (s, ω)-subcontinuous on K;*

(ii) *for every converging sequence u_n, there exists $m \in N$, such that $\operatorname{int} \cap_{n \geq m} S_n \neq \emptyset$;*

(iii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*

(iv) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*

(v) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the (QVI)$_p$ is extended well-posed if and only if for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset$, $\forall \delta, \epsilon > 0$,

$$\operatorname{diam} T(\delta, \varepsilon) \to 0 \quad as \quad (\delta, \varepsilon) \to (0, 0). \tag{3.2}$$

Proof. The necessity has been proved in Theorem 3.1. To prove the sufficiency, assume that for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset$, $\forall \delta, \epsilon > 0$

$$\operatorname{diam} T(\delta, \varepsilon) \to 0 \quad \text{as} \quad (\delta, \varepsilon) \to (0, 0).$$

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Let $p_n \to p \in P$ and $\{u_n\}$ be an approximating sequence for (QVI)$_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ decreasing to 0 such that

$$d(u_n, S(u_n)) \leq \varepsilon_n,$$

and

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n,$$

where $v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\|$. The rest of the proof follows on using similar arguments to those for Theorem 3.1. ☐

We now present the following theorem in which assumption (ii) is dropped, while the continuity assumption on the bifunction h is strengthened.

Corollary 3.3. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω)-closed, (s, s)-lower semicontinuous, and (s, ω)-subcontinuous on K;*

(ii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and (s, ω)-continuous;*

(iii) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*

(iv) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the (QVI)$_p$ is extended well-posed if and only if for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset$, $\forall \delta, \epsilon > 0$

$$\operatorname{diam}(\delta, \varepsilon) \to 0 \quad \textit{as} \quad (\delta, \varepsilon) \to (0, 0). \tag{3.3}$$

Proof. The conclusion follows by similar arguments to those for Theorem 3.1. ☐

The following example is an application of characterizations of extended well-posedness.

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Example 3.1. Let $E = R$, $K = [0, +\infty)$, $h(p, u, v) = u^2 - v^2$, and consider the set-valued function S defined by $S(u) = [0, \frac{u}{2}]$. It is easily seen that $T(p) = \{0\}$, and $T(\delta, \varepsilon) = [0, \sqrt{\varepsilon})$. It follows that $\operatorname{diam} T(\delta, \varepsilon) \to 0$, as $(\delta, \varepsilon) \to (0, 0)$. By Theorem 3.1, the $(\mathrm{QVI})_p$ is extended well-posed.

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4. Characterizations of Extended Well-posedness in the Generalized Sense

The aim of this section is to investigate some characterizations of extended well-posedness in the generalized sense for $(\mathrm{QVI})_p$. First, we recall two useful definitions.

Definition 4.1 ([6]). *Let H be a nonempty subset of a metric space (X, d). The measure of noncompactness μ of the set H is defined by*

$$\mu(H) = \inf\{\varepsilon > 0 : H \subseteq \cup_{i=1}^{n} H_i,\ \operatorname{diam} H_i < \varepsilon, i = 1, \dots, n\}.$$

Definition 4.2 ([6]). *The Hausdorff distance between two nonempty bounded subsets H and K of a metric space (X, d) is*

$$H(H, K) = \max\left\{\sup_{u\in H} d(u, K), \sup_{w\in K} d(H, w)\right\}.$$

Theorem 4.3. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω)-closed, (s, s)-lower semicontinuous, and (s, ω)-subcontinuous on K;*

(ii) *for every converging sequence u_n, there exists $m \in N$, such that $\operatorname{int} \cap_{n\geq m} S_n \neq \emptyset$;*

(iii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*

(iv) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*

(v) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

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Then, the (QVI)$_p$ is extended well-posed in the generalized sense if and only if for every $p \in P$, the solution set $T(p)$ is nonempty compact and

$$H(T(\delta,\varepsilon),T(p)) \to 0 \quad \textit{as} \quad (\delta,\varepsilon) \to (0,0). \tag{4.1}$$

Proof. Assume that (QVI)$_p$ is extended well-posed in the generalized sense. Then, $T(p) \neq \emptyset$ for all $p \in P$. To show that $T(p)$ is compact, let $\{u_n\}$ be a sequence for (QVI)$_p$. Since (QVI)$_p$ is extended well-posed in a generalized sense, $\{u_n\}$ has a subsequence converging to some point of $T(p)$. Thus, $T(p)$ is compact. Now, we prove that $H(T(\delta,\varepsilon),T(p)) \to 0$, $H(T(\delta,\varepsilon),T(p)) = \sup_{u\in T(\delta,\varepsilon)} d(u,T(p)) \to 0$. Suppose by contradiction that $H(T(\delta,\varepsilon),T(p)) \not\to 0$, as $(\delta,\varepsilon) \to (0,0)$. Then there exists $\tau > 0$ converging to 0, $\varepsilon_n > 0$ decreasing to 0, and $u_n \in K$ with $u_n \in T(\delta_n,\varepsilon_n))$ such that

$$u_n \neq T(p) + B(0,\tau). \tag{4.2}$$

Since $u_n \in T(\delta_n,\varepsilon_n)$, $\{u_n\}$ is an approximating sequence for (QVI)$_p$. As (QVI)$_p$ is extended well-posed in the generalized sense, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to some point of $T(p)$. This contradicts (4.2) and so condition (4.1) holds.

For the converse, assume that $T(p)$ is nonempty compact for all $p \in P$ and condition (4.1) holds. Let $p_n \to p \in P$ and $\{u_n\}$ be an approximating sequence for (QVI)$_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ decreasing to 0 such that

$$h(p_n,u_n,u_n - v) \leq \varepsilon_n,$$

where $v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n,\varepsilon_n)$ with $\delta_n = \|p_n - p\|$. From condition (4.1), there exists a sequence $\{v_n\}$ in $T(p)$ such that $d(u_n,T(p)) \leq H(T(\delta,\varepsilon),T(p)) \to 0$

$$\|u_n - v_n\| = d(u_n,T(P)) \to 0, \quad \forall n \in N.$$

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Since $T(p)$ is compact, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converging to $v \in T(p)$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to v. Thus (QVI)$_p$ is extended well-posed in the generalized sense. $\square$

The follow theorem presents the characterization of extended well-posedness in the generalized sense by considering the measure of noncompactness of the approximating solution sets.

Theorem 4.4. *Let the following assumptions hold:*

(i) the set-valued mapping S is nonempty-valued and convex-valued, (s,ω)-closed, (s,s)-lower semicontinuous, and (s,ω)-subcontinuous on K;

(ii) for every $p \in P$, $h(p,\cdot,\cdot)$ is (s,ω)-continuous;

(iii) for every $(p,u) \in P \times K$, $h(p,u,\cdot)$ is convex;

(iv) for every $u \in K$, $h(\cdot,u,\cdot)$ is lower semicontinuous;

Then, the (QVI)$_p$ is extended well-posed in the generalized sense if and only if for every $p \in P$,

$$T(\delta,\varepsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \textit{and} \quad \mu(T(\delta,\varepsilon)) \to 0 \quad \textit{as} \quad (\delta,\varepsilon) \to (0,0). \tag{4.3}$$

Proof. Assume that (QVI)$_p$ is extended well-posed in the generalized sense. Then, $T(p) \neq \emptyset$ and $T(p) \subset T(\delta,\varepsilon) \neq \emptyset$, for all $p \in P$, $\delta, \epsilon > 0$, and $T(p)$ is compact. Observe that for every $\delta, \epsilon > 0$, we have

$$\begin{aligned} H(T(\delta,\varepsilon),T(p)) &= \max\left\{ \sup_{u \in T(\delta,\varepsilon)} d(u,T(p)), \sup_{v \in T(p)} d(T(\delta,\varepsilon),v) \right\} \\ &= \sup_{u \in T(\delta,\varepsilon)} d(u,T(p)). \end{aligned}$$

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In order to prove that $\mu(T(\delta,\varepsilon)) \to 0$, consider $\delta_n > 0$ converging to 0, and $\varepsilon_n > 0$ decreasing to 0 such that

$$\mu(T(\delta,\varepsilon), T(p)) \leq H(T(\delta,\varepsilon), T(p)) + \mu(T(p)).$$

Since, by the assumptions, the set $T(p)$ is compact, $\mu(T(p)) = 0$. So we need only to prove that

$$\lim_n H(T(\delta,\varepsilon), T(p)) = \sup_{u\in T(\delta_n,\varepsilon_n)} d(u, T(p)) \to 0.$$

By Theorem 4.3, we have the desired result.

For the converse, we start by proving that $T(\delta,\varepsilon)$ is closed for $\delta, \epsilon > 0$. Letting $z_n \in T(\delta,\varepsilon)$ for $n \in N$, the sequence $\{z_n\}$ converges to z_0. Reasoning as in Theorem 3.1, one first proves that $d(z_0, S(z_0)) \leq \varepsilon$. Since the set-valued mapping S is (s,s)-lower semicontinuous, for every $w \in S(z_0)$ there exists a sequence $\{w_n\}$ converging to w such that $w_n \in S(z_n)$ for $n \in N$; and for $p_n \in B(p,\delta)$, one gets $h(p_n, z_n, z_n - w_n) \leq \varepsilon$. Without loss of generalization we suppose that $p_n \to \acute{p} \in B(p,\delta)$. In light of the assumption (iii), we have

$$h(\acute{p}, z_0, z_0 - w) \leq \varepsilon.$$

This yields $z_0 \in T(\delta,\varepsilon)$, and so $T(\delta,\varepsilon)$ is nonempty and closed. Observe now that

$$T(p) = \cap_{\delta>0,\varepsilon>0} T(\delta,\varepsilon),$$

since the set-valued mapping S is closed-valued. Then, since $\mu(T(\delta,\varepsilon)) \to 0$, the theorem on p. 412 in [6] can be applied and one concludes that the set $T(p)$ is nonempty, compact, and $H(T(\delta,\varepsilon), T(p)) \to 0$ as $(\delta,\varepsilon) \to (0,0)$. The rest of the proof follows from the same arguments in Theorem 4.3. □

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5. Conditions for Extended Well-posedness

The following theorem shows that under suitable conditions, the extended well-posedness of (QVI)$_p$ is equivalent to the existence and uniqueness of solutions.

Theorem 5.1. *Let $E = R^n$ and K be a nonempty, compact, and convex subset of E. Let the following assumptions hold:*

- *(i) the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K;*
- *(ii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- *(iii) for every $p \in P$ and $x \in K$, $h(p, x, \cdot)$ is positively homogeneous and sublinear, and $h(p, x, 0) = 0$;*
- *(iv) for every $u \in K$, $h(\cdot, u, \cdot)$ is continuous.*

Then, the (QVI)$_p$ is extended well-posed if and only if for every $p \in P$, (QVI)$_p$ has a unique solution.

Proof. The necessity holds trivially. For the sufficiency, assume that $(QVI)_p$ has a unique solution u_0 for all $p \in P$. If (QVI)$_p$ is not extended well-posed, there exist some $p \in P$, $p_n \to p$, and an approximating sequence $\{u_n\}$ for (QVI)$_p$ corresponding to $\{p_n\}$ such that $u_n \not\to u_0$. Set $t_n = \frac{1}{\|u_n - u_0\|}$ and $z_n = u_0 + t_n(u_n - u_0)$. We assert that $\{u_n\}$ is bounded. Indeed, if $\{u_n\}$ is not bounded, then without loss of generality we suppose that $\|u_n\| \to +\infty$, $z_n \in K$ and $z_n \to z \neq u_0$. By using the conditions (iii) and (iv), we have

$$\begin{aligned}
&h(p_n, v, z - v)\\
&\quad \leq h(p_n, v, z - z_n) + h(p_n, v, z_n - v)\\
&\quad \leq h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + h(p_n, v, z_n - u_0)
\end{aligned}$$

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$$\begin{aligned} &= h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n h(p_n, v, u_n - u_0) \\ &\le h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n h(p_n, v, u_n - v) + t_n h(p_n, v, v - u_0), \\ &\quad \forall v \in S(u_0). \end{aligned}$$

Since $\{u_n\}$ is an approximating sequence for (QVI)$_p$ corresponding to $\{p_n\}$, we can find $\varepsilon_n > 0$ decreasing to 0, such that $h(p_n, u_n, u_n - v) \le \varepsilon_n, \quad \forall v \in S(u_0)$. In light of the assumption (ii), we get $h(p_n, v, u_n - v) \le \varepsilon_n, \quad \forall v \in S(u_0)$. From the assumptions (ii) and (iv),

$$\begin{aligned} h(p, v, z - v) &= \lim_n h(p_n, v, z_n - v) \\ &\le \lim_n \{h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n \varepsilon_n + h(p_n, v, v - u_0)\} \\ &= h(p, v, u_0 - v) \le 0, \quad \forall v \in S(u_0). \end{aligned}$$

From Lemma 2.6, z is a solution of (QVI)$_p$. This is a contradiction to the uniqueness of the solution. Thus $\{u_n\}$ is bounded. Since the set K is compact, the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which converges to a point $z_0 \in K$, which is a fixed point for S, and $h(p, v, z_0 - v) \le 0, \quad \forall v \in S(u_0)$. Then, applying Lemma 2.6, z_0 solves (QVI)$_p$. So it coincides with u_0. The uniqueness of the solution also implies that the whole sequence $\{u_n\}$ converges to u_0. Therefore, (QVI)$_p$ is extended well-posed. $\square$

For extended well-posedness in the generalized sense, we have the following results.

Theorem 5.2. *Let the following assumptions hold:*

(i) the set K is bounded;

(ii) the set-valued mapping S is nonempty-valued and convex-valued, (ω, ω)-closed, (ω, s)-lower semicontinuous on K;

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(iii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and (s, s)-continuous;

(iv) for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;

(v) for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;

Then, the (QVI)$_p$ is extended well-posed in the generalized sense with respect to weak convergence.

Proof. Let $p_n \to p \in P$ and $\{u_n\}$ be an approximating sequence corresponding to $\{p_n\}$, that is

$$d(u_n, S(u_n)) \leq \varepsilon_n, \quad \text{and} \quad h(p_n, u_n, u_n - v) \leq \varepsilon_n, \quad \forall v \in S(u_n), \quad \forall n \in N,$$

where $\varepsilon_n > 0$ decreases to 0. Since the set K is bounded, the sequence $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, which weakly converges to a point $u_0 \in K$. As in Theorem 3.1, one proves that

$$d(u_0, S(u_0)) \leq \liminf_n d(u_n, S(u_n)) \leq \lim_n \varepsilon_n = 0.$$

Indeed, if the left inequality does not hold, there exists a positive number a such that

$$\liminf_n d(u_n, S(u_n)) < a < d(u_0, S(u_0)).$$

Consequently, there exist an increasing sequence $\{n_k\}$ and a sequence $\{z_k\}$, $z_k \in S(u_{n_k})$, $\forall k \in N$, such that $\|u_k - z_k\| < a$. Since the set K is bounded, and the set-valued mapping S is (ω, ω)-closed, the sequence $\{z_k\}$ has a subsequence, still denoted by $\{z_k\}$, weakly converging to a point $z_0 \in S(u_0)$. Then, one gets

$$a < d(u_0, S(u_0)) \leq \|u_0 - z_0\| \leq \liminf_n \|u_{n_k} - z_{n_k}\| \leq a,$$

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which gives a contradiction. So $u_0 \in \mathrm{cl}S(u_0) = S(u_0)$ and u_0 is a fixed point for the set mapping S. To complete the proof, let $v \in S(u_0)$ and $\{v_n\}$ be a sequence converging to v such that $v_n \in S(u_n)$, $\forall n \in N$. By using the assumption (iii), we have $h(p, u_0, u_0 - v) \leq 0$. This yields u_0 as a solution of (QVI)$_p$, and so (QVI)$_p$ is extended well-posed in the generalized sense. ☐

Theorem 5.3. *Let $E = R^n$ and K be bounded. Let the following assumptions hold:*

- *(i) the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K;*
- *(ii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- *(iii) for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*
- *(iv) for every $u \in K$, $h(\cdot, u, \cdot)$ is continuous;*

If for each $p \in P$, there exists some $\varepsilon > 0$ such that $T(\epsilon, \epsilon)$ is nonempty and bounded, then the (QVI)$_p$ is extended well-posed in the generalized sense.

Proof. Let $p_n \to p \in P$ and $\{u_n\}$ be an approximating sequence for (QVI)$_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n, \forall v \in S(u_n), \quad \forall n \in N.$$

Let $\varepsilon > 0$ such that $T(\varepsilon, \varepsilon)$ is nonempty bounded, then there exists n_0 such that $u_n \in T(\varepsilon, \varepsilon)$ for all $n > n_0$, and so $\{u_n\}$ is bounded. There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$, as $k \to \infty$. Using the same arguments as for Theorem 5.1, u_0 solves (QVI)$_p$. Then (QVI)$_p$ is extended well-posed in the generalized sense. ☐

Corollary 5.4. *Let $E = R^n$ and K be bounded. Let the following assumptions hold:*

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(i) *the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K;*

(ii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*

(iii) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*

(iv) *for every $u \in K$, $h(\cdot, u, \cdot)$ is continuous;*

then the (QVI)$_p$ is extended well-posed in the generalized sense. In addition, if $h(p, \cdot, \cdot)$ is strictly monotone for all $p \in P$, then the (QVI)$_p$ is extended well-posed.

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