

EXTENDED WELL-POSEDNESS FOR QUASIVARIATIONAL INEQUALITIES

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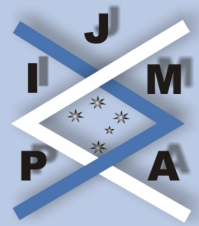
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Abstract: In this paper, we introduce the concepts of extended well-posedness for quasivariational inequalities and establish some characterizations. We show that the extended well-posedness is equivalent to the existence and uniqueness of solutions under suitable conditions. In addition, the corresponding concepts of extended well-posedness in the generalized sense are introduced and investigated for quasivariational inequalities having more than one solution.



Quasivariational Inequalities
Ke Zhang, Zhong-Quan He
and Da-Peng Gao
vol. 10, iss. 4, art. 107, 2009

[Title Page](#)

[Contents](#)



Page 1 of 24

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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Contents

| | | |
|---|---|----|
| 1 | Introduction | 3 |
| 2 | Preliminaries | 4 |
| 3 | Characterizations of Extended Well-Posedness | 8 |
| 4 | Characterizations of Extended Well-posedness in the Generalized Sense | 14 |
| 5 | Conditions for Extended Well-posedness | 18 |



Quasivariational Inequalities

Ke Zhang, Zhong-Qun He
and Da-Peng Gao

vol. 10, iss. 4, art. 107, 2009

Title Page

Contents



Page 2 of 24

Go Back

Full Screen

Close

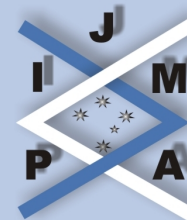
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1. Introduction

The importance of well-posedness is widely recognized in the theory of variational problems. Motivated by the study of numerical production optimization sequences, Tykhonov [18] introduced the concept of well-posedness for a minimization problem, which is known as Tykhonov well-posedness. Due to its importance in optimization problems, various concepts of well-posedness have been introduced and studied for minimization problems (see [18, 1, 5, 16, 19, 20]) in past decades. The concept of well-posedness has also been generalized to several related variational problems: saddle point problems [2], Nash equilibrium problems [11, 17, 15], inclusion problems [4, 7, 9], and fixed point problems [4, 7, 9]. A more general formulation for the above variational problems is the variational inequalities problems, which leads to the study of the well-posedness of variational inequalities. Lucchetti & Patrone [14], obtained a notion of well-posedness for a variational inequality. Lignola & Morgan [13] introduced the extended well-posedness for a family of variational inequalities and investigated its links with the extended well-posedness of corresponding minimization problems. Lignola [8] introduced the notion of well-posedness for quasivariational inequalities. Recently, Lalitha & Mehta [10] presented a class of variational inequalities defined by bifunctions. Fang & Hu [3], extended the notion of well-posedness of variational inequalities defined by bifunctions.

Inspired and motivated by above research works, in this paper, we study the well-posedness of quasivariational inequalities (in short, QVI) defined by bifunctions. We introduce the notion of extended well-posedness for QVI, and establish some of its characterizations. Under suitable conditions, we prove that the extended well-posedness is equivalent to the existence and uniqueness of solutions to QVI. With an additional compactness assumption, we also derive the equivalence between the extended well-posedness in the generalized sense and the existence of solutions to QVI.



Quasivariational Inequalities

Ke Zhang, Zhong-Qian He
and Da-Peng Gao

vol. 10, iss. 4, art. 107, 2009

Title Page

Contents



Page 3 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



2. Preliminaries

Throughout this paper, let E be a reflexive real Banach space and K be a nonempty closed convex subset of E , unless otherwise specified. Let $S : K \rightarrow 2^K$ be a set-valued mapping, and $h : K \times E \rightarrow \bar{R}$ be a bifunction, where $\bar{R} = R \cup \{+\infty\}$. The quasivariational inequality problem consists in finding a point $u_0 \in K$, such that

$$(QVI) \quad u_0 \in S(u_0) \quad \text{and} \quad h(u_0, u_0 - v) \leq 0, \quad \forall v \in S(u_0).$$

Note that QVI includes as a special case the quasivariational inequality. In this paper, we consider the parametric form of QVI which is formulated as follows:

$$(QVI)_p \quad u_0 \in S(u_0) \quad \text{and} \quad h(p, u_0, u_0 - v) \leq 0, \quad \forall v \in S(u_0),$$

where $h : P \times K \times E \rightarrow \bar{R}$ and P is a Banach space. Now we recall some concepts and results. Let $(X, \tau), (Y, \sigma)$ be topological spaces. The closure and interior of a nonempty set G of X are respectively denoted by $\text{cl}G$ and $\text{int}G$.

Definition 2.1 ([8]). A set-valued mapping $F : (X, \tau) \rightarrow 2^{(Y, \sigma)}$ is called:

- (i) closed-valued if the set $F(x)$ is nonempty and σ -closed, for every $x \in X$;
- (ii) (τ, σ) -closed if the graph $G_F = \{(x, y) : y \in F(x)\}$ is closed in $\tau \times \sigma$;
- (iii) (τ, σ) -lower semicontinuous if for every σ -open subset V of Y , the inverse image of the set V , $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is a τ -open subset of X ;
- (iv) (τ, σ) -subcontinuous on $H \subseteq E$ (E is a reflexive real Banach space) if for every net $\{x_\alpha\}$ τ -converging in H , every net $\{y_\alpha\}$, such that $y_\alpha \in F(x_\alpha)$, has a σ -convergent subnet.

Title Page

Contents



Page 4 of 24

Go Back

Full Screen

Close

journal of **inequalities**
 in pure and applied
 mathematics

issn: 1443-5756



Title Page

Contents



Page 5 of 24

Go Back

Full Screen

Close

Definition 2.2 ([8]). The Painleve-Kuratowski limits of sequence $\{H_n\}$, $H_n \subseteq Y$ are defined by:

$$\liminf_n H_n = \left\{ y \in Y : \exists y_n \in H_n, n \in N, \text{ with } \lim_n y_n = y \right\},$$

and

$$\limsup_n H_n = \left\{ y \in Y : \exists n_k \uparrow +\infty, n_k \in N, \exists y_{n_k} \in H_{n_k}, k \in N, \text{ with } \lim_k y_{n_k} = y \right\}.$$

Definition 2.3 ([3]). A bifunction $f : K \times E \rightarrow R$ is said to be:

(i) *monotone* if $f(x, y - x) + f(y, x - y) \leq 0, \forall x, y \in K$;

(ii) *strongly monotone* if there exists a constant $t > 0$ such that

$$f(x, y - x) + f(y, x - y) + t\|x - y\|^2 \leq 0, \forall x, y \in K;$$

(iii) *pseudomonotone* if for any $x, y \in K, f(x, y - x) \geq 0 \Rightarrow f(y, x - y) \leq 0$;

(iv) *hemicontinuous* if for every $x, y \in K$ and $t \in [0, 1]$, the function $t \mapsto f(x + t(y - x), y - x)$ is continuous at 0^+ .

In the sequel we introduce some notions of extended well-posedness for $(QVI)_p$.

Definition 2.4. Let $p \in P, \{p_n\} \in P$, with $p_n \rightarrow p$. A sequence $\{u_n\}$ is an approximation for $(QVI)_p$ corresponding to $\{p_n\}$ if:

(i) $u_n \in K, \forall n \in N$;

(ii) there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that $d(u_n, S(u_n)) \leq \varepsilon_n$ (i.e. $u_n \in B(S(u_n, \varepsilon_n))$), and $h(p_n, u_n, u_n - v) \leq \varepsilon_n, \forall v \in S(u_n), \forall n \in N$, where $B(S(u), \varepsilon) = \{y \in E : d(S(u), y) \leq \varepsilon\}$.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 24

Go Back

Full Screen

Close

Remark 1. When the set-valued mapping S is constant, say $S(u) = K$ for every $u \in K$, the parametric form of $(QVI)_p$ is a parametric form of a variational inequality. In this case, the class of approximating sequences coincides with the class defined in [13].

Definition 2.5.

- (i) $(QVI)_p$ is said to be extended well-posed if for every $p \in P$, $(QVI)_p$ has a unique solution u_p and every approximating sequence for $(QVI)_p$ corresponding to $p_n \rightarrow p$ converges to u_p .
- (ii) $(QVI)_p$ is said to be extended well-posed in the generalized sense if for every $p \in P$, $(QVI)_p$ has a nonempty solution set $T(p)$, and every approximating sequence for $(QVI)_p$ corresponding to $p_n \rightarrow p$ has a subsequence which converges to some point of $T(p)$.

Lemma 2.6 ([13]). Let K be a nonempty, closed, compact and convex subset of E , the set-valued mapping S is convex-valued and closed-valued. If the bifunction h is hemicontinuous and pseudomonotone, the following problems are equivalent:

- (i) find $u_0 \in K$, such that $u_0 \in S(u_0)$ and $h(u_0, u_0 - v) \leq 0, \forall v \in S(u_0)$;
- (ii) find $u_0 \in K$, such that $u_0 \in S(u_0)$ and $h(v, u_0 - v) \leq 0, \forall v \in S(u_0)$.

Lemma 2.7 ([12]). Let $\{H_n\}$ be a sequence of nonempty subsets of the space E such that:

- (i) H_n is convex for every $n \in N$;
- (ii) $H_0 \subseteq \liminf_n H_n$;
- (iii) there exists $m \in N$ such that $\text{int} \cap_{n \geq m} H_n \neq \emptyset$.

Then, for every $u_0 \in \text{int } H_0$, there exists a positive real number δ such that $B(u_0, \delta) \subseteq H_n, \forall n \geq m$.

If E is a finite dimensional space, the assumption (iii) can be replaced by $\text{int } H_0 \neq \emptyset$.



Quasivariational Inequalities

Ke Zhang, Zhong-Quan He
and Da-Peng Gao

vol. 10, iss. 4, art. 107, 2009

Title Page

Contents



Page 7 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



3. Characterizations of Extended Well-Posedness

In this section, we investigate some characterizations of extended well-posedness for quasivariational inequalities. For $(QVI)_p$, the set of approximating solutions is defined by

$$T(\delta, \varepsilon) = \bigcup_{\dot{p} \in B(p, \delta)} \{u \in K : u \in B(S(u), \varepsilon) \text{ and } h(\dot{p}, u, u-v) \leq \varepsilon, \quad \forall v \in S(u)\},$$

where $B(p, \delta)$ denotes the closed ball with radius δ and centered at p .

Theorem 3.1. *Let the following assumptions hold:*

- (i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω) -closed, (s, s) -lower semicontinuous, and (s, ω) -subcontinuous on K ;*
- (ii) *for every converging sequence $\{u_n\}$, there exists $m \in N$, such that $\text{int} \bigcap_{n \geq m} S_n \neq \emptyset$ (S_n is a sequence of mappings);*
- (iii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- (iv) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*
- (v) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the $(QVI)_p$ is extended well-posed if and only if for every $p \in P$, the solution set $T(p)$ is nonempty and

$$(3.1) \quad \text{diam } T(\delta, \varepsilon) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0),$$

where diam means the diameter of a set.

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 24

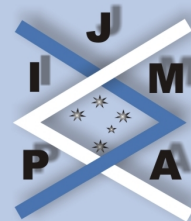
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 24

Go Back

Full Screen

Close

Proof. Suppose that $(QVI)_P$ is extended well-posed. Then it has a unique solution u_0 . If for some $p \in P$, $\text{diam } T(\delta, \varepsilon) \not\rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$, there exist a positive number l , and sequences $\delta_n > 0$ converging to 0, $\varepsilon_n > 0$ decreasing to 0, and $w_n, z_n \in K$, with $w_n \in T(\delta_n, \varepsilon_n)$, $z_n \in T(\delta_n, \varepsilon_n)$ such that

$$\|w_n - z_n\| > l, \quad \forall n \in N.$$

Since $w_n \in T(\delta_n, \varepsilon_n)$, $z_n \in T(\delta_n, \varepsilon_n)$ for each $n \in N$, there exists $p_n, \acute{p}_n \in B_n(p, \delta_n)$, such that

$$h(p_n, w_n, w_n - v) \leq \varepsilon_n,$$

and

$$h(\acute{p}_n, z_n, z_n - v) \leq \varepsilon_n,$$

where $\forall v \in S(u_0)$. This implies that $\{w_n\}$, $\{z_n\}$ are both approximating sequences for $(QVI)_p$ corresponding to $\{p_n\}$ and $\{\acute{p}_n\}$ respectively. Since $(QVI)_p$ is extended well-posed, they have to converge to the unique solution u_0 . This gives a contradiction. Thus condition (3.1) holds.

Conversely, assume that for every $p \in P$, $T(p)$ is nonempty and condition (3.1) holds. Let $p_n \rightarrow p \in P$ and $\{u_n\} \subset K$ be an approximating sequence for $(QVI)_p$ corresponding to $\{p_n\}$. There exists $\varepsilon_n > 0$ decreasing to 0, such that

$$d(u_n, S(u_n)) \leq \varepsilon_n,$$

and

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n,$$

where $\forall v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\|$. It follows from condition (3.1) that $\{u_n\}$ is a Cauchy sequence and strongly converges to a point $u_0 \in K$. To prove that u_0 solves $(QVI)_p$, we shall first show that

$$d(u_0, S(u_0)) \leq \liminf_n d(u_n, S(u_n)) \leq \lim \varepsilon_n = 0.$$



Assume that the left inequality does not hold. Then, there exists a positive number a such that

$$\liminf_n d(u_n, S(u_n)) < a < d(u_0, S(u_0)).$$

This means that there exists an increasing sequence $\{n_k\}$ and a sequence $\{z_k\}$, $z_k \in S(u_{n_k})$, such that

$$\|u_{n_k} - z_k\| < a, \quad \forall k \in N.$$

Since the set-valued mapping S is (s, ω) -subcontinuous and (s, ω) -closed, the sequence $\{z_k\}$ has a subsequence, still denoted by z_k , weakly converging to a point $z_0 \in S(u_0)$. Then, one gets

$$a < d(u_0, S(u_0)) \leq \|u_0 - z_0\| \leq \liminf_n \|u_{n_k} - z_k\| \leq a,$$

which gives a contradiction. So, $u_0 \in \text{cl}S(u_0) = S(u_0)$. Then consider a point $v \in S(u_0)$ and observe that, since the set-valued mapping S is (s, s) -lower semicontinuous, one has $S(u_0) \subseteq \liminf S(u_n)$. Also, observe that condition (ii), applied to the sequence $w_n = u_0$, for all $n \in N$, implies that $\text{int} S(u_0) \neq \emptyset$; from Lemma 2.7, it follows that, if $v \in \text{int} S(u_0)$, then $v \in S(u_n)$ for n sufficiently large. Condition (iv) and (v) give that

$$h(p, v, u_0 - v) = \lim_n h(p, v, u_n - v) \leq \liminf_n h(p, u_n, u_n - v) \leq \liminf_n \varepsilon_n = 0.$$

If $v \in S(u_0) - \text{int} S(u_0)$, let $\{v_n\}$ be a sequence to v , whose points belong to a segment contained in $\text{int} S(u_0)$. Since $v_n \in \text{int} S(u_0)$, for $n \in N$, one has

$$h(p, v_n, u_0 - v_n) \leq 0,$$

and in light of the hemicontinuity of the bifunction h ,

$$h(p, v, u_0 - v) \leq 0.$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 24

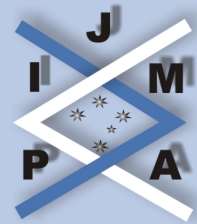
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 24

Go Back

Full Screen

Close

Then, the result follows from Lemma 2.6. Now it remains to prove that $(QVI)_p$ has a unique solution. If $(QVI)_p$ has two distinct solutions u_1, u_2 , it is easily seen that $u_1, u_2 \in T(\delta, \varepsilon)$ for all $\delta, \varepsilon > 0$. It follows that

$$0 < \|u_1 - u_2\| \leq \text{diam } T(\delta, \varepsilon) \rightarrow 0,$$

and we obtain a contradiction to (3.1). \square

Theorem 3.2. *Let the following assumptions hold:*

- (i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω) -closed, (s, s) -lower semicontinuous, and (s, ω) -subcontinuous on K ;*
- (ii) *for every converging sequence u_n , there exists $m \in N$, such that $\text{int } \bigcap_{n \geq m} S_n \neq \emptyset$;*
- (iii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- (iv) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*
- (v) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the $(QVI)_p$ is extended well-posed if and only if for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset, \forall \delta, \varepsilon > 0$,

$$(3.2) \quad \text{diam } T(\delta, \varepsilon) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0).$$

Proof. The necessity has been proved in Theorem 3.1. To prove the sufficiency, assume that for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset, \forall \delta, \varepsilon > 0$

$$\text{diam } T(\delta, \varepsilon) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0).$$



Let $p_n \rightarrow p \in P$ and $\{u_n\}$ be an approximating sequence for $(QVI)_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ decreasing to 0 such that

$$d(u_n, S(u_n)) \leq \varepsilon_n,$$

and

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n,$$

where $v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\|$. The rest of the proof follows on using similar arguments to those for Theorem 3.1. \square

We now present the following theorem in which assumption (ii) is dropped, while the continuity assumption on the bifunction h is strengthened.

Corollary 3.3. *Let the following assumptions hold:*

- (i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω) -closed, (s, s) -lower semicontinuous, and (s, ω) -subcontinuous on K ;*
- (ii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and (s, ω) -continuous;*
- (iii) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*
- (iv) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the $(QVI)_p$ is extended well-posed if and only if for every $p \in P$, $T(\delta, \varepsilon) \neq \emptyset$, $\forall \delta, \varepsilon > 0$

$$(3.3) \quad \text{diam}(\delta, \varepsilon) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0).$$

Proof. The conclusion follows by similar arguments to those for Theorem 3.1. \square

The following example is an application of characterizations of extended well-posedness.

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 24

Go Back

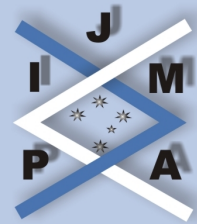
Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Example 3.1. Let $E = \mathbb{R}$, $K = [0, +\infty)$, $h(p, u, v) = u^2 - v^2$, and consider the set-valued function S defined by $S(u) = [0, \frac{u}{2}]$. It is easily seen that $T(p) = \{0\}$, and $T(\delta, \varepsilon) = [0, \sqrt{\varepsilon}]$. It follows that $\text{diam} T(\delta, \varepsilon) \rightarrow 0$, as $(\delta, \varepsilon) \rightarrow (0, 0)$. By Theorem 3.1, the $(\text{QVI})_p$ is extended well-posed.



Quasivariational Inequalities

Ke Zhang, Zhong-Qian He
and Da-Peng Gao

vol. 10, iss. 4, art. 107, 2009

Title Page

Contents



Page 13 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



4. Characterizations of Extended Well-posedness in the Generalized Sense

The aim of this section is to investigate some characterizations of extended well-posedness in the generalized sense for $(QVI)_p$. First, we recall two useful definitions.

Definition 4.1 ([6]). Let H be a nonempty subset of a metric space (X, d) . The measure of noncompactness μ of the set H is defined by

$$\mu(H) = \inf\{\varepsilon > 0 : H \subseteq \cup_{i=1}^n H_i, \text{diam } H_i < \varepsilon, i = 1, \dots, n\}.$$

Definition 4.2 ([6]). The Hausdorff distance between two nonempty bounded subsets H and K of a metric space (X, d) is

$$H(H, K) = \max \left\{ \sup_{u \in H} d(u, K), \sup_{w \in K} d(H, w) \right\}.$$

Theorem 4.3. Let the following assumptions hold:

- (i) the set-valued mapping S is nonempty-valued and convex-valued, (s, ω) -closed, (s, s) -lower semicontinuous, and (s, ω) -subcontinuous on K ;
- (ii) for every converging sequence u_n , there exists $m \in N$, such that $\text{int } \cap_{n \geq m} S_n \neq \emptyset$;
- (iii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;
- (iv) for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;
- (v) for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;

Title Page

Contents



Page 14 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Then, the $(QVI)_p$ is extended well-posed in the generalized sense if and only if for every $p \in P$, the solution set $T(p)$ is nonempty compact and

$$(4.1) \quad H(T(\delta, \varepsilon), T(p)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0).$$

Proof. Assume that $(QVI)_p$ is extended well-posed in the generalized sense. Then, $T(p) \neq \emptyset$ for all $p \in P$. To show that $T(p)$ is compact, let $\{u_n\}$ be a sequence for $(QVI)_p$. Since $(QVI)_p$ is extended well-posed in a generalized sense, $\{u_n\}$ has a subsequence converging to some point of $T(p)$. Thus, $T(p)$ is compact. Now, we prove that $H(T(\delta, \varepsilon), T(p)) \rightarrow 0$, $H(T(\delta, \varepsilon), T(p)) = \sup_{u \in T(\delta, \varepsilon)} d(u, T(p)) \rightarrow 0$. Suppose by contradiction that $H(T(\delta, \varepsilon), T(p)) \not\rightarrow 0$, as $(\delta, \varepsilon) \rightarrow (0, 0)$. Then there exists $\tau > 0$ converging to 0, $\varepsilon_n > 0$ decreasing to 0, and $u_n \in K$ with $u_n \in T(\delta_n, \varepsilon_n)$ such that

$$(4.2) \quad u_n \notin T(p) + B(0, \tau).$$

Since $u_n \in T(\delta_n, \varepsilon_n)$, $\{u_n\}$ is an approximating sequence for $(QVI)_p$. As $(QVI)_p$ is extended well-posed in the generalized sense, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to some point of $T(p)$. This contradicts (4.2) and so condition (4.1) holds.

For the converse, assume that $T(p)$ is nonempty compact for all $p \in P$ and condition (4.1) holds. Let $p_n \rightarrow p \in P$ and $\{u_n\}$ be an approximating sequence for $(QVI)_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ decreasing to 0 such that

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n,$$

where $v \in S(u_n)$, $\forall n \in N$. This yields $u_n \in T(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\|$. From condition (4.1), there exists a sequence $\{v_n\}$ in $T(p)$ such that $d(u_n, T(p)) \leq H(T(\delta, \varepsilon), T(p)) \rightarrow 0$

$$\|u_n - v_n\| = d(u_n, T(P)) \rightarrow 0, \quad \forall n \in N.$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 15 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 24

Go Back

Full Screen

Close

Since $T(p)$ is compact, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converging to $v \in T(p)$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to v . Thus $(QVI)_p$ is extended well-posed in the generalized sense. \square

The follow theorem presents the characterization of extended well-posedness in the generalized sense by considering the measure of noncompactness of the approximating solution sets.

Theorem 4.4. *Let the following assumptions hold:*

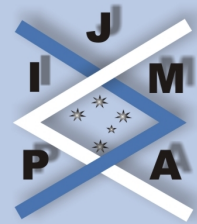
- (i) *the set-valued mapping S is nonempty-valued and convex-valued, (s, ω) -closed, (s, s) -lower semicontinuous, and (s, ω) -subcontinuous on K ;*
- (ii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is (s, ω) -continuous;*
- (iii) *for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;*
- (iv) *for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;*

Then, the $(QVI)_p$ is extended well-posed in the generalized sense if and only if for every $p \in P$,

$$(4.3) \quad T(\delta, \varepsilon) \neq \emptyset, \quad \forall \delta, \varepsilon > 0, \quad \text{and} \quad \mu(T(\delta, \varepsilon)) \rightarrow 0 \quad \text{as} \quad (\delta, \varepsilon) \rightarrow (0, 0).$$

Proof. Assume that $(QVI)_p$ is extended well-posed in the generalized sense. Then, $T(p) \neq \emptyset$ and $T(p) \subset T(\delta, \varepsilon) \neq \emptyset$, for all $p \in P$, $\delta, \varepsilon > 0$, and $T(p)$ is compact. Observe that for every $\delta, \varepsilon > 0$, we have

$$\begin{aligned} H(T(\delta, \varepsilon), T(p)) &= \max \left\{ \sup_{u \in T(\delta, \varepsilon)} d(u, T(p)), \sup_{v \in T(p)} d(T(\delta, \varepsilon), v) \right\} \\ &= \sup_{u \in T(\delta, \varepsilon)} d(u, T(p)). \end{aligned}$$



Title Page

Contents



Page 17 of 24

Go Back

Full Screen

Close

In order to prove that $\mu(T(\delta, \varepsilon)) \rightarrow 0$, consider $\delta_n > 0$ converging to 0, and $\varepsilon_n > 0$ decreasing to 0 such that

$$\mu(T(\delta, \varepsilon), T(p)) \leq H(T(\delta, \varepsilon), T(p)) + \mu(T(p)).$$

Since, by the assumptions, the set $T(p)$ is compact, $\mu(T(p)) = 0$. So we need only to prove that

$$\lim_n H(T(\delta, \varepsilon), T(p)) = \sup_{u \in T(\delta_n, \varepsilon_n)} d(u, T(p)) \rightarrow 0.$$

By Theorem 4.3, we have the desired result.

For the converse, we start by proving that $T(\delta, \varepsilon)$ is closed for $\delta, \varepsilon > 0$. Letting $z_n \in T(\delta, \varepsilon)$ for $n \in N$, the sequence $\{z_n\}$ converges to z_0 . Reasoning as in Theorem 3.1, one first proves that $d(z_0, S(z_0)) \leq \varepsilon$. Since the set-valued mapping S is (s, s) -lower semicontinuous, for every $w \in S(z_0)$ there exists a sequence $\{w_n\}$ converging to w such that $w_n \in S(z_n)$ for $n \in N$; and for $p_n \in B(p, \delta)$, one gets $h(p_n, z_n, z_n - w_n) \leq \varepsilon$. Without loss of generalization we suppose that $p_n \rightarrow \acute{p} \in B(p, \delta)$. In light of the assumption (iii), we have

$$h(\acute{p}, z_0, z_0 - w) \leq \varepsilon.$$

This yields $z_0 \in T(\delta, \varepsilon)$, and so $T(\delta, \varepsilon)$ is nonempty and closed. Observe now that

$$T(p) = \bigcap_{\delta > 0, \varepsilon > 0} T(\delta, \varepsilon),$$

since the set-valued mapping S is closed-valued. Then, since $\mu(T(\delta, \varepsilon)) \rightarrow 0$, the theorem on p. 412 in [6] can be applied and one concludes that the set $T(p)$ is nonempty, compact, and $H(T(\delta, \varepsilon), T(p)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$. The rest of the proof follows from the same arguments in Theorem 4.3. \square



5. Conditions for Extended Well-posedness

The following theorem shows that under suitable conditions, the extended well-posedness of $(QVI)_p$ is equivalent to the existence and uniqueness of solutions.

Theorem 5.1. *Let $E = R^n$ and K be a nonempty, compact, and convex subset of E . Let the following assumptions hold:*

- (i) *the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K ;*
- (ii) *for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- (iii) *for every $p \in P$ and $x \in K$, $h(p, x, \cdot)$ is positively homogeneous and sublinear, and $h(p, x, 0) = 0$;*
- (iv) *for every $u \in K$, $h(\cdot, u, \cdot)$ is continuous.*

Then, the $(QVI)_p$ is extended well-posed if and only if for every $p \in P$, $(QVI)_p$ has a unique solution.

Proof. The necessity holds trivially. For the sufficiency, assume that $(QVI)_p$ has a unique solution u_0 for all $p \in P$. If $(QVI)_p$ is not extended well-posed, there exist some $p \in P$, $p_n \rightarrow p$, and an approximating sequence $\{u_n\}$ for $(QVI)_p$ corresponding to $\{p_n\}$ such that $u_n \not\rightarrow u_0$. Set $t_n = \frac{1}{\|u_n - u_0\|}$ and $z_n = u_0 + t_n(u_n - u_0)$. We assert that $\{u_n\}$ is bounded. Indeed, if $\{u_n\}$ is not bounded, then without loss of generality we suppose that $\|u_n\| \rightarrow +\infty$, $z_n \in K$ and $z_n \rightarrow z \neq u_0$. By using the conditions (iii) and (iv), we have

$$\begin{aligned} & h(p_n, v, z - v) \\ & \leq h(p_n, v, z - z_n) + h(p_n, v, z_n - v) \\ & \leq h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + h(p_n, v, z_n - u_0) \end{aligned}$$

Title Page

Contents



Page 18 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 19 of 24

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\begin{aligned} &= h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n h(p_n, v, u_n - u_0) \\ &\leq h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n h(p_n, v, u_n - v) + t_n h(p_n, v, v - u_0), \\ &\quad \forall v \in S(u_0). \end{aligned}$$

Since $\{u_n\}$ is an approximating sequence for $(QVI)_p$ corresponding to $\{p_n\}$, we can find $\varepsilon_n > 0$ decreasing to 0, such that $h(p_n, u_n, u_n - v) \leq \varepsilon_n$, $\forall v \in S(u_0)$. In light of the assumption (ii), we get $h(p_n, v, u_n - v) \leq \varepsilon_n$, $\forall v \in S(u_0)$. From the assumptions (ii) and (iv),

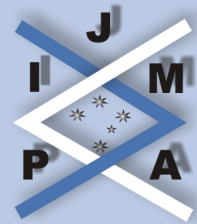
$$\begin{aligned} h(p, v, z - v) &= \lim_n h(p_n, v, z_n - v) \\ &\leq \lim_n \{h(p_n, v, z - z_n) + h(p_n, v, u_0 - v) + t_n \varepsilon_n + h(p_n, v, v - u_0)\} \\ &= h(p, v, u_0 - v) \leq 0, \quad \forall v \in S(u_0). \end{aligned}$$

From Lemma 2.6, z is a solution of $(QVI)_p$. This is a contradiction to the uniqueness of the solution. Thus $\{u_n\}$ is bounded. Since the set K is compact, the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which converges to a point $z_0 \in K$, which is a fixed point for S , and $h(p, v, z_0 - v) \leq 0$, $\forall v \in S(u_0)$. Then, applying Lemma 2.6, z_0 solves $(QVI)_p$. So it coincides with u_0 . The uniqueness of the solution also implies that the whole sequence $\{u_n\}$ converges to u_0 . Therefore, $(QVI)_p$ is extended well-posed. \square

For extended well-posedness in the generalized sense, we have the following results.

Theorem 5.2. *Let the following assumptions hold:*

- (i) *the set K is bounded;*
- (ii) *the set-valued mapping S is nonempty-valued and convex-valued, (ω, ω) -closed, (ω, s) -lower semicontinuous on K ;*



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 20 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

(iii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and (s, s) -continuous;

(iv) for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;

(v) for every $u \in K$, $h(\cdot, u, \cdot)$ is lower semicontinuous;

Then, the $(QVI)_p$ is extended well-posed in the generalized sense with respect to weak convergence.

Proof. Let $p_n \rightarrow p \in P$ and $\{u_n\}$ be an approximating sequence corresponding to $\{p_n\}$, that is

$$d(u_n, S(u_n)) \leq \varepsilon_n, \quad \text{and} \quad h(p_n, u_n, u_n - v) \leq \varepsilon_n, \quad \forall v \in S(u_n), \quad \forall n \in N,$$

where $\varepsilon_n > 0$ decreases to 0. Since the set K is bounded, the sequence $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, which weakly converges to a point $u_0 \in K$. As in Theorem 3.1, one proves that

$$d(u_0, S(u_0)) \leq \liminf_n d(u_n, S(u_n)) \leq \lim_n \varepsilon_n = 0.$$

Indeed, if the left inequality does not hold, there exists a positive number a such that

$$\liminf_n d(u_n, S(u_n)) < a < d(u_0, S(u_0)).$$

Consequently, there exist an increasing sequence $\{n_k\}$ and a sequence $\{z_k\}$, $z_k \in S(u_{n_k})$, $\forall k \in N$, such that $\|u_{n_k} - z_k\| < a$. Since the set K is bounded, and the set-valued mapping S is (ω, ω) -closed, the sequence $\{z_k\}$ has a subsequence, still denoted by $\{z_k\}$, weakly converging to a point $z_0 \in S(u_0)$. Then, one gets

$$a < d(u_0, S(u_0)) \leq \|u_0 - z_0\| \leq \liminf_n \|u_{n_k} - z_{n_k}\| \leq a,$$



| |
|---------------------------------------|
| Title Page |
| Contents |
| ◀◀ ▶▶ |
| ◀ ▶ |
| Page 21 of 24 |
| Go Back |
| Full Screen |
| Close |

which gives a contradiction. So $u_0 \in \text{cl}S(u_0) = S(u_0)$ and u_0 is a fixed point for the set mapping S . To complete the proof, let $v \in S(u_0)$ and $\{v_n\}$ be a sequence converging to v such that $v_n \in S(u_n), \forall n \in N$. By using the assumption (iii), we have $h(p, u_0, u_0 - v) \leq 0$. This yields u_0 as a solution of $(QVI)_p$, and so $(QVI)_p$ is extended well-posed in the generalized sense. \square

Theorem 5.3. *Let $E = R^n$ and K be bounded. Let the following assumptions hold:*

- (i) *the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K ;*
- (ii) *for every $p \in P, h(p, \cdot, \cdot)$ is monotone and hemicontinuous;*
- (iii) *for every $(p, u) \in P \times K, h(p, u, \cdot)$ is convex;*
- (iv) *for every $u \in K, h(\cdot, u, \cdot)$ is continuous;*

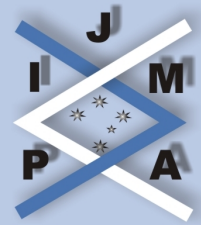
If for each $p \in P$, there exists some $\varepsilon > 0$ such that $T(\varepsilon, \varepsilon)$ is nonempty and bounded, then the $(QVI)_p$ is extended well-posed in the generalized sense.

Proof. Let $p_n \rightarrow p \in P$ and $\{u_n\}$ be an approximating sequence for $(QVI)_p$ corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$ such that

$$h(p_n, u_n, u_n - v) \leq \varepsilon_n, \forall v \in S(u_n), \quad \forall n \in N.$$

Let $\varepsilon > 0$ such that $T(\varepsilon, \varepsilon)$ is nonempty bounded, then there exists n_0 such that $u_n \in T(\varepsilon, \varepsilon)$ for all $n > n_0$, and so $\{u_n\}$ is bounded. There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightarrow u_0$, as $k \rightarrow \infty$. Using the same arguments as for Theorem 5.1, u_0 solves $(QVI)_p$. Then $(QVI)_p$ is extended well-posed in the generalized sense. \square

Corollary 5.4. *Let $E = R^n$ and K be bounded. Let the following assumptions hold:*



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 22 of 24

Go Back

Full Screen

Close

(i) the set-valued mapping S is nonempty-valued and convex-valued, closed, lower semicontinuous on K ;

(ii) for every $p \in P$, $h(p, \cdot, \cdot)$ is monotone and hemicontinuous;

(iii) for every $(p, u) \in P \times K$, $h(p, u, \cdot)$ is convex;

(iv) for every $u \in K$, $h(\cdot, u, \cdot)$ is continuous;

then the $(QVI)_p$ is extended well-posed in the generalized sense. In addition, if $h(p, \cdot, \cdot)$ is strictly monotone for all $p \in P$, then the $(QVI)_p$ is extended well-posed.

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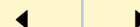
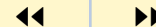
Quasivariational Inequalities

Ke Zhang, Zhong-Qian He
and Da-Peng Gao

vol. 10, iss. 4, art. 107, 2009

Title Page

Contents



Page 23 of 24

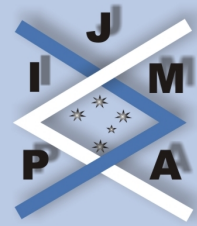
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



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Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 24 of 24

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756