journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 3, Article 83, 5 pp.

ON EQUIVALENCE OF COEFFICIENT CONDITIONS. II

L. LEINDLER
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED, HUNGARY
leindler@math.u-szeged.hu

Received 28 August, 2008; accepted 16 September, 2008
Communicated by H. Bor

ABSTRACT. An additional theorem is proved pertaining to the equiconvergence of numerical series.

Key words and phrases: Numerical series, equiconvergence.

2000 *Mathematics Subject Classification.* 26D15, 40D15.

1. INTRODUCTION

In the papers [2], [3] and [4] we have studied the relations of the following sums:

$$S_1 := \sum_{n=1}^{\infty} c_n^q \mu_n,$$

$$S_2 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} c_k^q \right)^{\frac{p}{q}}, \qquad S_2^* := \sum_{n=1}^{\infty} \lambda_n \left(\mu_n^{-1} \sum_{k=1}^{n} \lambda_k \right)^{\frac{p}{q-p}},$$

$$S_3 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} c_k^q \right)^{\frac{p}{q}}, \qquad S_3^* := \sum_{n=1}^{\infty} \lambda_n \left(\mu_n^{-1} \sum_{k=n}^{\infty} \lambda_k \right)^{\frac{p}{q-p}},$$

$$S_4 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}}, \qquad S_4^* := \sum_{n=1}^{\infty} \lambda_n \left(\frac{\lambda_n}{\mu_{\nu_n}} \right)^{\frac{p}{q-p}},$$

where $0 < p < q$, $\lambda := \{\lambda_n\}$ and $\mathbf{c} := \{c_n\}$ are sequences of nonnegative numbers, $\nu := \{\nu_m\}$ is a subsequence of natural numbers, and $\mu := \{\mu_n\}$ is a certain nondecreasing sequence of positive numbers.

In [2] we verified that $S_2 < \infty$ if and only if there exists a μ satisfying the conditions $S_1 < \infty$ and $S_2^* < \infty$. Similarly $S_3 < \infty$ if and only if $S_1 < \infty$ and $S_3^* < \infty$.

In [3] we showed that $S_4 < \infty$ if and only if there exists a μ such that $S_1 < \infty$ and $S_4^* < \infty$.

242-08

Recently, in [4], we proved that if

$$\mu_n := \Lambda_n^{(1)}\, C_n^{p-q}, \quad \text{where} \quad C_n := \left(\sum_{k=n}^{\infty} c_k^q\right)^{1/q} \quad \text{and} \quad \Lambda_n^{(1)} := \sum_{k=1}^{n} \lambda_k,$$

then the sums S_1, S_2 and S_2^* are already equiconvergent.

Furthermore if

$$\mu_n := \Lambda_n^{(2)}\, \tilde{C}_n^{p-q}, \quad \text{where} \quad \tilde{C}_n := \left(\sum_{k=1}^{n} c_k^q\right)^{1/q} \quad \text{and} \quad \Lambda_n^{(2)} := \sum_{k=n}^{\infty} \lambda_k,$$

then the sums S_1, S_3 and S_3^* are equiconvergent.

Comparing the results proved in [4] and that of [2] and [3], we can observe that in the former one the explicit sequences $\{\mu_n\}$ are determined, herewith they state more than the outcomes of [2] and [3], where only the existence of a sequence $\{\mu_n\}$ is proved.

Furthermore, in [4] the equiconvergence of these concrete sums are guaranteed, too.

However the equiconvergence in [4] is proved only in connection with the sums S_2 and S_3, but not for S_4. This is a gap or shortcoming at these investigations.

The aim of this note is closing this gap. Unfortunately we cannot give a complete solution, namely our result to be verified requires an additional assumption on the sequence λ. In particular, λ should be quasi geometrically increasing, that is, we assume that there exist a natural number N and $K \geq 1$ such that $\lambda_{n+N} \geq 2\lambda_n$ and $\lambda_n \leq K\lambda_{n+1}$ hold for all n.

Then we can give an explicit sequence μ such that the sums S_1, S_4 and S_4^* are already equiconvergent. We also show that without some additional requirement on λ the equiconvergence does not hold. See the last part. Thus the following open problem can be raised: *What is the weakest additional assumption on sequence λ which ensures the equiconvergence of these sums*?

2. RESULT

Theorem 2.1. *If $0 < p < q$, $\mathbf{c} := \{c_n\}$ is a sequence of nonnegative numbers, $\nu := \{\nu_m\}$ is a subsequence of natural numbers, and $\lambda := \{\lambda_n\}$ is a quasi geometrically increasing sequence, and for $\nu_m \leq n < \nu_{m+1}$*

$$\mu_n := \lambda_m \left(\sum_{k=\nu_m}^{\infty} c_k^q\right)^{\frac{p}{q}-1}, \quad m = 0, 1, \ldots,$$

then the sums S_1, S_4 and S_4^* are equiconvergent.

3. LEMMA

In order to verify our theorem, first we shall prove a lemma regarding the equiconvergence of two special series.

Lemma 3.1. *Let $0 < \alpha < 1$, $\mathbf{a} := \{a_n\}$ be a sequence of nonnegative numbers, $\nu := \{\nu_m\}$ be a subsequence of natural numbers, and $\kappa := \{\kappa_m\}$ be a quasi geometrically increasing sequence. Furthermore let $A_k := \sum_{n=k}^{\infty} a_n$, and for $\nu_m \leq n < \nu_{m+1}$ let*

$$\mu_n := \kappa_m\, A_{\nu_m}^{\alpha-1}, \quad m = 0, 1, \ldots .$$

Then

$$\sigma_1 := \sum_{n=1}^{\infty} a_n\, \mu_n < \infty \tag{3.1}$$

holds if and only if

$$\sigma_2 := \sum_{m=1}^{\infty} \kappa_m A_{\nu_m}^{\alpha} < \infty. \tag{3.2}$$

Proof of Lemma 3.1. Before starting the proofs we note that the following inequality

$$\sum_{n=1}^{m} \kappa_n \le K \kappa_m \tag{3.3}$$

holds for all m, subsequent to the fact that κ is a quasi geometrically increasing sequence (see e.g. [1, Lemma 1]). Here and later on K denotes a constant that is independent of the parameters.

Furthermore we verify a useful inequality. If $0 \le a < b,\ 0 < \alpha < 1$ and

$$\frac{b^{\alpha} - a^{\alpha}}{b - a} = \alpha \xi^{\alpha - 1}, \tag{3.4}$$

then

$$\xi \ge \alpha^{1/(1-\alpha)} b =: \xi_0,$$

namely if $a = 0$ then $\xi = \xi_0$. Hence we get that

$$\alpha \xi^{\alpha-1} \le b^{\alpha-1}. \tag{3.5}$$

Now we show that (3.1) implies (3.2). Since $A_n \searrow 0$, thus, by (3.3),

$$\begin{aligned} \sum_{m=1}^{\infty} \kappa_m A_{\nu_m}^{\alpha} &= \sum_{m=1}^{\infty} \kappa_m \sum_{n=m}^{\infty} (A_{\nu_n}^{\alpha} - A_{\nu_{n+1}}^{\alpha}) \\ &= \sum_{n=1}^{\infty} (A_{\nu_n}^{\alpha} - A_{\nu_{n+1}}^{\alpha}) \sum_{m=1}^{n} \kappa_m \\ &\le K \sum_{n=1}^{\infty} \kappa_n (A_{\nu_n}^{\alpha} - A_{\nu_{n+1}}^{\alpha}). \end{aligned} \tag{3.6}$$

Using the relations (3.4) and (3.5) we obtain that

$$A_{\nu_n}^{\alpha} - A_{\nu_{n+1}}^{\alpha} = \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} a_k \right) \alpha \xi^{\alpha-1} \le \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} a_k \right) A_{\nu_n}^{\alpha-1}.$$

This and (3.6) yield that

$$\sum_{m=1}^{\infty} \kappa_m A_{\nu_m}^{\alpha} \le K \sum_{n=1}^{\infty} \kappa_n A_{\nu_n}^{\alpha-1} \sum_{k=\nu_n}^{\nu_{n+1}-1} a_k = K \sum_{n=1}^{\infty} \sum_{k=\nu_n}^{\nu_{n+1}-1} a_k \mu_k.$$

Herewith the implication (3.1) $\Rightarrow$ (3.2) is proved.

The proof of (3.2) $\Rightarrow$ (3.1) is very easy. Namely

$$\begin{aligned} \sum_{n=\nu_1}^{\infty} a_n \mu_n &= \sum_{m=1}^{\infty} \sum_{n=\nu_m}^{\nu_{m+1}-1} a_n \mu_n \\ &= \sum_{m=1}^{\infty} \kappa_m A_{\nu_m}^{\alpha-1} \sum_{n=\nu_m}^{\nu_{m+1}-1} a_n \\ &\le \sum_{m=1}^{\infty} \kappa_m A_{\nu_m}^{\alpha}, \end{aligned}$$

that is, (3.2) $\Rightarrow$ (3.1) is verified.

Thus the proof is complete. $\square$

4. PROOF OF THEOREM 2.1

We shall use the result of Lemma 3.1 with $\alpha = \frac{p}{q}$, $a_n = c_n^q$ and $\kappa_m = \lambda_m$. Then $A_n = \sum_{k=n}^{\infty} c_k^q$ and for $\nu_m \le n < \nu_{m+1}$

$$\mu_n = \mu_{\nu_m} = \lambda_m \left(\sum_{k=\nu_m}^{\infty} c_k^q \right)^{\frac{p-q}{q}}. \tag{4.1}$$

Then $\sigma_1 = S_1$, thus by Lemma 3.1, $S_1 < \infty$ implies that $\sigma_2 < \infty$, that is,

$$S_4 = \sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=\nu_m}^{\nu_{m+1}-1} c_n^q \right)^{\frac{p}{q}} \le \sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=\nu_m}^{\infty} c_n^q \right)^{\frac{p}{q}} = \sigma_2. \tag{4.2}$$

Moreover, by (4.1),

$$S_4^* = \sum_{n=1}^{\infty} \lambda_n \left\{ \left(\sum_{k=\nu_n}^{\infty} c_k^q \right)^{\frac{q-p}{q}} \right\}^{\frac{p}{q-p}} = \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\infty} c_k^q \right)^{\frac{p}{q}} = \sigma_2,$$

thus $S_1 < \infty$ implies that both $S_4 < \infty$ and $S_4^* < \infty$ hold.

Conversely, if $S_4 < \infty$, then it suffices to show that $\sigma_2 = S_4^* < \infty$ also holds.

Applying the inequality

$$\left(\sum a_k \right)^{\alpha} \le \sum a_k^{\alpha}, \quad 0 < \alpha \le 1, \ a_k \ge 0,$$

and (3.3), we obtain that

$$\begin{aligned} \sigma_2 &= \sum_{m=1}^{\infty} \lambda_m A_{\nu_m}^{p/q} \le \sum_{m=1}^{\infty} \lambda_m \sum_{n=m}^{\infty} \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \sum_{m=1}^{n} \lambda_m \\ &\le K \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \\ &= K S_4 < \infty. \end{aligned}$$

This, (4.2) and, by Lemma 3.1, the implication $\sigma_2 < \infty \Rightarrow \sigma_1 = S_1 < \infty$ complete the proof of Theorem 2.1.

Proof of the necessity of some additional assumption on λ. Let $p = 1,\ q = 2,\ \lambda_n = \log n,\ \nu_n = n$ and

$$c_n := \begin{matrix} m^{-3} & \text{if} \quad n = 2^m, \\ 0 & \text{otherwise.} \end{matrix}$$

Then

$$S_4 = \sum_{m=2}^{\infty} \frac{\log 2^m}{m^3} < \infty,$$

but $S_1 < \infty$ and $S_4^* < \infty$ cannot be fulfilled simultaneously. Namely, then with a nondecreasing sequence $\{\mu_n\}$ the conditions

$$S_1 = \sum_{m=1}^{\infty} m^{-6}\, \mu_{2^m} < \infty$$

and

$$S_4^* = \sum_{m=2}^{\infty} \frac{\log^2 m}{\mu_m} < \infty$$

yield a trivial contradiction. □

REFERENCES

[1] L. LEINDLER, On equivalence of coefficient conditions with applications, *Acta Sci. Math. (Szeged)*, **60** (1995), 495–514.

[2] L. LEINDLER, On equivalence of coefficient conditions and application, *Math. Inequal. Appl. (Zagreb)*, **1** (1998), 41–51.

[3] L. LEINDLER, On a new equivalence of coefficient conditions and applications, *Math. Inequal. Appl.*, **10**(2) (1999), 195–202.

[4] L. LEINDLER, On equivalence of coefficient conditions, *J. Inequal. Pure and Appl. Math.*, **8**(1) (2007), Art. 8. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=821`].